THE OBSERVABILITY OF CASCADE CONNECTED NONLINEAR SYSTEMS

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Abstract. This paper studies the observability of cascade connected nonlinear systems. The analysis of nonlinear observability is carried out following the differential geometric approach developed by Hermann and Krener and makes large use of concepts related to the notion of (f,g) invariant distribution introduced by the authors themselves. The basic result is a characterization of the unobservability of the composite system in terms of a partial "matching" between the dynamics of one system and that of the other system modified by suitable state feedback. This caracterization, in the case of linear systems, reduces to the geometric equivalent of a well known observability criterion for cascade connected linear systems.

Keywords. Observability; nonlinear systems; cascade control; state-space methods; Lie-algebra.

GEOMETRIC PROPERTIES OF CASCADE CONNECTED LINEAR SYSTEMS

In this section we show that the notion of (A,B)-invariant subspace (see Wonham, 1979) plays a natural role in the study of observability of cascade connected linear systems. Our comments will be of some interest in interpreting the results that will be proved in the following sections about nonlinear systems.

We consider a pair of systems S_1, S_2 described by

$$\dot{x}_1 = A_1x_1 + B_1u_1$$
$$y_1 = C_1x_1$$

with $x_1 \in X_1$ (an n_1-dimensional vector space), and respectively by

$$\dot{x}_2 = A_2x_2 + B_2u_2$$
$$y_2 = C_2x_2$$

with $x_2 \in X_2$ (an n_2-dimensional vector space). The two systems are cascaded connected, i.e.

$$u_2 = y_1$$

We assume that both S_1 and S_2 are observable and we wish to characterize the (possible) loss of observability in the composite system $S_2 \circ S_1$, which is described by

$$\dot{x}_1 = A_1x_1 + B_1u_1$$
$$\dot{x}_2 = A_2x_2 + B_2C_1x_1$$
$$y_2 = C_2x_2$$

with $(x_1,x_2) \in X_1 \times X_2$.

Since S_1 and S_2 are both observable, a loss of observability may happen if and only if a nonzero initial state x_1^o of S_1 produces a (zero-input)-response which is annihilated by S_2. This may happen with the initial state x_2^o of S_2 being either zero or nonzero. In what follows we shall make the additional assumption that x_2^o is necessarily nonzero, i.e., in other terms, we shall make the assumption that the unobservables of $S_2 \circ S_1$ are a subspace v of $X_1 \times X_2$ with the property that

$$V \cap (V_1 \times \{0\}) = \{(0,0)\} \qquad \text{(H)}$$

This makes it possible to give a nice geometric characterization of unobservability in $S_2 \circ S_1$. For, observe that the subspace V must satisfy

$$\begin{pmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{pmatrix} V \subseteq V$$

and

$$(0 \quad C_2)V = \{0\}$$

The subspace V can be described in the form

$$V = \{\pi_1 x, \pi_2 x) \in X_1 \times X_2 : x \in \mathbb{R}^m\}$$

where m = dimV. Note that, the observability of S_2 implies $\pi_1: \mathbb{R}^m \to X_1$ to be injective, whereas the hypothesis (H) implies $\pi_2: \mathbb{R}^m \to X_2$ to be injective.

In this case, the first condition can be rephrased as follows: for all $x \in \mathbb{R}^m$ there exists $z \in \mathbb{R}^m$ such that

$$A_1\pi_1 x = \pi_1 z$$

$$(B_2C_1\pi_1 + A_2\pi_2)x = \pi_2 z$$

One easily realizes that this z depends linearly on the x, i.e. there exists a unique matrix G such that in the above equalities we can put $z = Gx$. Thus the above equalities are replaced by the following ones

$$A_1\pi_1 = \pi_1 G$$

$$B_2C_1\pi_1 + A_2\pi_2 = \pi_2 G$$

The second one can be further manipulated if we introduce an F such that

$$C_1\pi_1 = F\pi_2$$

thus arriving at

$$(B_2F + A_2)\pi_2 = \pi_2 G$$

Starting from this, it is easy to characterize the loss of observability of $S_2 \circ S_1$ in the following terms

Proposition 1. Let S_1 and S_2 be observable and let V denote the unobservables of $S_2 \circ S_1$. Assume also that V satisfies condition (H). Then, the following statements are equivalent

(a) dim $V = m \geq 1$

(b) There exist two injective linear mapping $\pi_1: \mathbb{R}^m \to X_1$, $\pi_2: \mathbb{R}^m \to X_2$, a matrix G and a matrix F such that

(b1) $A_1\pi_1 = \pi_1 G$

(b2) $(A_2 + B_2F)\pi_2 = \pi_2 G$

(b3) $C_1\pi_1 = F\pi_2$

(b4) $C_2\pi_2 = 0$

(c) There exist two injective linear mappings $\pi_1: \mathbb{R}^m \to X_1$, $\pi_2: \mathbb{R}^m \to X_2$, a matrix G and a matrix F such that

(c1) $A_1(\mathrm{Im}\pi_1) \subseteq \mathrm{Im}\pi_1$

(c2) $(A_2 + B_2F)(\mathrm{Im}\pi_2) \subseteq \mathrm{Im}\pi_2$

(c3) $C_1\{\exp(A_1t)\}\pi_1 = F\{\exp(A_2 + B_2F)t\}\pi_2$ for all t

(c4) $\mathrm{Im}\pi_2 \subseteq \ker C_2$

(d) There exist a subspace W_1 of X_1, a subspace W_2 of X_2, with dim W_1 = dim W_2 = m

(d1) W_1 is A_1-invariant

(d2) W_2 is (A_2,B_2)-invariant

and there exists an $F \in F(W_2)$ such that[1]

(d3) $A_1|_{W_1}$ is similar to $A_2 + B_2F|_{W_2}$

(d4) $C_1W_1 = FW_2$

Note, in particular, that (d3) implies the coincidence of the eigenvalues of S_1 with the transmission zeroes of S_2 (see Macfarlane and Karcanias, 1976).

SOME BACKGROUND MATERIAL ON NONLINEAR SYSTEMS

In what follows we consider nonlinear systems described by differential equations of the form

$$\dot{x} = f(x) + g(x)u \tag{1}$$

$$y = h(x) \tag{2}$$

where $u \in R$, $x \in M$, a C^∞ connected manifold of dimension n, $y \in R$, f and g are C^∞ complete vector fields on M and h is a C^∞ function from M to R.

We approach nonlinear observability along the lines developed by Hermann and Krener (1977) and we make use of some properties of feedback controlled nonlinear systems developed by the authors themselves (1979). Nonlinear observability will be characterized in terms of distributions on the state space M. We recall that a *distribution* Δ on M is a mapping which assigns to each $x \in M$ a subspace $\Delta(x)$ of the tanget space T_xM of M at x. A distribution has *constant rank* m on M if dim $\Delta(x) = m$ for all $x \in M$ and is *involutive* if it is closed under the Lie bracket [2]. It is well known (Hermann and Krener, 1977) that an involutive distribution Δ of constant rank m induces on M *a partition into maximal integral submanifolds*, all of

1 $F(W)$ stands for the set of all the "friends" of W (see Wonham, 1979).

2 The Lie bracket $[\tau_1,\tau_2]$ of two vector fields on M bot belonging to Δ, i.e. such that $\tau_i(x) \in \Delta(x)$ for all $x \in M$ and $i = 1,2$, belongs to Δ.

dimension m [3]. A distribution Δ is *invariant* with respect to the dynamics (1) if, for all $\tau \in \Delta$

$$[f,\tau] \in \Delta$$

$$[g,\tau] \in \Delta$$

Following an approach of Hermann and Krener, nonlinear observability can be characterized in terms of a distribution on M which is invariant with respect to the dynamics (1). We recall the following definitions. A state x^1 is indistinguishable from x^o (denoted x^oIx^1) if for any admissible input the corresponding outputs are identical. A state x^o is *strongly indistinguishable* from x^1 (denoted x^oSIx^1) if there exists a continuous curve α: $[0,1] \to M$ such that $\alpha(0) = x^o$, $\alpha(1) = x^1$ and $x^oI\alpha(s)$ for all $s \in [0,1]$. Moreover, define an involutive distribution Δ_{SI} on M as follows. Let Γ denote the subalgebra of the algebra $C^\infty(M)$ of C^∞ functions from M to $\mathbb{R}$ generated by $h, L_{(f+gu')}h, L_{(f+gu'')}, L_{(f+gu''')}h, \ldots$ [4], and let [5]

$$\Delta_{SI}: x \mapsto \{v \in T_xM: (d\gamma)_x v = 0 \text{ for all } \gamma \in \Gamma\} \tag{3}$$

Then we have the following result (Hermann and Krener, 1977).

Theorem 1. If Δ_{SI} has constant rank on M, than the partition of M into maximal integral submanifolds induced by Δ_{SI} coincides with the partition of M into equivalence classes induced by the relation SI.

The distribution Δ_{SI} is the maximal distribution on M invariant with respect to the dynamics (1) and contained in the distribution

$$\mathrm{Ker}(dh): x \mapsto \{v \in T_xM: (dh)_x v = 0\} \tag{4}$$

We note, moreover, that if

$$\dim \Delta_{SI}(x) = 0 \tag{5}$$

for all $x \in M$, then the system is *locally weakly observable*. Conversely, if

$$\dim \Delta_{SI}(x) = m > 0$$

for all $x \in M$, there exist a nontrivial partition of M into equivalence classes of strongly indistinguishable states, each element of the partition being an immersed submanifold of M, of dimension m. We refer to that as to *strong unobservability*.

Finally, we recall the definition of an (f,g)-invariant distribution (Isidori and alii, 1979), which will play a central role in the sequel. A distribution Δ on M is (f,g)-invariant if there exist C^∞ functions α: $M \to \mathbb{R}$ and β: $M \to \mathbb{R}$ such that Δ is invariant with respect to the dynamics

$$\dot{x} = \tilde{f}(x) + \tilde{g}(x)u \tag{6}$$

where

$$\tilde{f}(x) = f(x) + g(x)\alpha(x) \tag{7}$$

$$\tilde{g}(x) = g(x)\beta(x) \tag{8}$$

or, what is the same, for all $\tau \in \Delta$

$$[\tilde{f},\tau] \in \Delta$$

$$[\tilde{g},\tau] \in \Delta$$

In particular, we shall be concerned with (f,g)-invariant distributions contained in the distribution ker(dh), and we shall make use of the following result (Isidori and alii, 1979). Let ρ denote the largest integer such that for all $k < \rho$ and all $x \in M$

$$L_gL_f^kh(x) = 0$$

If no such integer exists, we take $\rho = \infty$. Then we have

Theorem 2. Let $\rho < \infty$ and let

$$L_gL_f^\rho h(x) \neq 0 \tag{9}$$

for all $x \in M$. Then the distribution

$$\Delta^*: x \mapsto \{v \in T_xM: (dh)_x v = 0, (dL_fh)_x v = 0, \ldots, (dL_f^\rho h)_x v = 0\} \tag{10}$$

is the unique maximal (f,g)-invariant distribution contained in the distribution ker(dh). Δ^* is invariant with respect to the dynamics (5) with

$$\alpha(x) = -\frac{1}{L_gL_f^\rho h(x)} L_f^{\rho+1}h(x) \tag{11}$$

$$\beta(x) = \frac{1}{L_gL_f^\rho h(x)} \tag{12}$$

[3] An integral submanifold M' of Δ is a connected submanifold od M with the property that $T_xM' = \Delta(x)$ for all $x \in M'$. M' is a maximal integral submanifold of Δ if it is not property contained in any other integral submanifold of Δ.

[4] $(L_\tau h)(x)$ denotes the differentiation of h along the direction of τ at $x \in M$.

[5] $(d\gamma)_x$ denotes the differential of γ at $x \in M$.

OBSERVABILITY OF CASCADE CONNECTED NONLINEAR SYSTEMS

We consides a pair of systems S_1, S_2 described by

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)u_1 \qquad y_1 = h_1(x_1) \tag{13}$$

with $x_1 \in M_1$, $u_1 \in \mathbb{R}$, $y_1 \in \mathbb{R}$, and respectively by

$$\dot{x}_2 = f_2(x_2) + g_2(x_2)u_2 \qquad y_2 = h_2(x_2) \tag{14}$$

with $x_2 \in M_2$, $u_2 \in \mathbb{R}$, $y_2 \in \mathbb{R}$. The two systems are cascade connected, i.e.

$$u_2 = y_1$$

In this paper we want to study the (possible) *loss of observability in the composite system* $S_2 \circ S_1$

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)u_1 \tag{15a}$$

$$\dot{x}_2 = f_2(x_2) + g_2(x_2)h_1(x_1) \tag{15b}$$

$$y_2 = h_2(x_2)$$

with input $u = u_1 \in \mathbb{R}$, output $y = y_2 \in \mathbb{R}$ and state $x = (x_1,x_2) \in M_1 \times M_2$. More precisely, we assume that S_1 and S_2 both satisfy the condition (5), i.e. that

$$\Delta_{SI,1}(x_1) = \{0\} \text{ for all } x_1 \in M_1 \tag{H1}$$

$$\Delta_{SI,2}(x_2) = \{0\} \text{ for all } x_2 \in M_2 \tag{H2}$$

and we want to give an interpretation of the conditions under which there exist a non-trivial partition of $M_1 \times M_2$ into equivalence classes with respect to strong indistinguishability. We do this under the additional assumptions

$$L_{g_2}L_{f_2}^{\rho_2}h_2(x_2) \neq 0 \text{ for all } x_2 \in M_2 \tag{H3}$$

$$\Delta_2^* \text{ has constant rank on } M_2 \tag{H4}$$

The interpretation we want to give is based on a comparison of the output of S_1 with the output of a suitable system S_2^* related to S_2. The auxiliary system S_2^* is described as follows

$$\dot{x}_2 = f_2(x_2) + g_2(x_2)\alpha(x_2) + g_2(x_2)\beta(x_2)v \tag{17}$$

$$w = \alpha(x_2) + \beta(x_2)v \tag{18a}$$

$$y_2 = h_2(x_2) \tag{18b}$$

with $x_2 \in M_2$, $v \in \mathbb{R}$, $w \in \mathbb{R}$ and

$$\alpha(x_2) = -\frac{1}{L_{g_2}L_{f_2}^{\rho_2}h_2(x_2)} L_{f_2}^{\rho_2+1}h_2(x_2) \tag{19}$$

$$\beta(x_2) = \frac{1}{L_{g_2}L_{f_2}^{\rho_2}h_2(x_2)} \tag{20}$$

One can easily see, on the basis of Theorem 2, that the dynamics (17) of S_2^* is obtained from that of S_2 by means of a feedback control law which makes Δ_2^* invariant (see Fig.1)

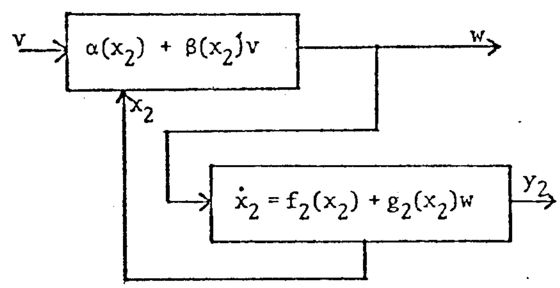

Fig. 1

The output y_1 at time t of S_1 in the initial condition x_1^o under the input u_1 will be denoted by

$$y_1(t,x_1^o,u_1)$$

whereas the output w at time t of S_2^* in the initial condition x_2^o under the input v will be denoted by

$$w(t,x_2^o,v)$$

The main result of the paper is the following

Theorem 3. Assume the hypotheses (H1), (H2), (H3), (H4) are satisfied. Then, the composite system $S_2 \circ S_1$ is strongly unobservable if and only if there exist an involutive distribution Δ on $M_1 \times M_2$, of constant rank $m > 0$ with the properties

(i) $(d\pi_2)_x\Delta(x) \subseteq \Delta_2^*(\pi_2(x))$ for all $x \in M_1 \times M_2$ [6]

(ii) for each input u_1 to system S_1, and for each element N_a of the partition $\{N_a\}_{a\in A}$ induced on $M_1 \times M_2$ by Δ, there exists an input v to system S_2^* such that

$$y_1(t,x_1^o,u_1) = w(t,x_2^o,v)$$

for all $t \geq 0$ and for all $(x_1^o,x_2^o) \in N_a$.

[6] The mapping π_2: $M_1 \times M_2 \rightarrow$ is defined as $\pi_2(x_1,x_2) = x_2$ and $d\pi_2$ is the corresponding differential.

Proof. Necessity. Assume $S_2 \circ S_1$ is strongly unobservable, i.e. there exists an involutive distribution Δ_{SI} of constant rank $m > 0$, invariant with respect to the composite dynamics (15) and contained in ker(dh), where h is the mapping $(x_1,x_2) \mapsto h_2(x_2)$. Let τ be a vector field belonging to Δ and

$$\hat{f}(x) = (f_1(x_1), f_2(x_2) + g_2(x_2)h_1(x_1))$$

$$\hat{g}(x) = (g_1(x_1), 0)$$

Then, by Theorem 1 we have

$$(dh)\tau = 0$$

$$(dL_{(\hat{f}+\hat{g}u')}h)\tau = 0$$

$$(dL_{(\hat{f}+\hat{g}u'')}L_{(\hat{f}+\hat{g}u')}h)\tau = 0, \ldots$$

on $M_1 \times M_2$. From the definition of ρ_2 the first ρ_2+1 of such equalities become

$$(dh_2)\tau_2 = 0$$

$$(dL_{f_2}h_2)\tau_2 = 0$$

$$\cdots$$

$$(dL_{f_2}^{\rho_2}h_2)\tau_2 = 0.$$

where $\tau_2 = (d\pi_2)\tau$. The other ones become

$$d\left(\frac{h_1-\alpha}{\beta}\right)\tau = 0$$

$$dL_{(\hat{f}+\hat{g}u')}\left(\frac{h_1-\alpha}{\beta}\right)\tau = 0$$

$$dL_{(\hat{f}+\hat{g}u'')}L_{(\hat{f}+\hat{g}u')}\left(\frac{h_1-\alpha}{\beta}\right)\alpha = 0, \ldots$$

The first ρ_2+1 equalities, according to the result expressed by (10), show that (i) is true. From the other ones and Theorem 2 we can deduce that the distribution Δ induces a partition into equivalence classes with respect to strong unobservability in the state space $M_1 \times M_2$ of the system

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)u_1$$

$$\dot{x}_2 = f_2(x_2) + g_2(x_2)h_1(x_1)$$

$$v = \frac{h_1(x_1) - \alpha(x_2)}{\beta(x_2)}$$

Thus, for each fixed u_1, the output v of this system is the same for all initial states (x_1^0, x_2^0) belonging to the same maximal integral submanifold N_a of Δ. This function v depends uniquely on the input u_1 and the submanifold N_a. If this function is used as an input to S_2^*, in the initial condition x_2^0, then the output w of S_2^*, coincides with the output y_1 of S_1, in the initial condition x_1^0, subject to the input u_1. This, being true for all (x_1^0,x_2^0) on N_a, shows that (ii) holds.
Sufficiency. Assume there exists an involutive distribution Δ of constant rank satisfying (ii). Then, for each fixed u_1 and N_a, the output y_2 of the composite system $S_2 \circ S_1$, in the initial condition $(x_1^0,x_2^0) \in N_a$ coincides with the output y_2 of the auxiliary system S_2^*, in the initial condition x_2^0, subject to the input v. Because of (i), the subset $\pi_2(N_a)$ is contained into a maximal integral submanifold of Δ_2^*. On the other hand, for each fixed v, any state on the same maximal integral submanifold of Δ_2^* produces the same output y_2, because Δ_2^* is invariant under the dynamics (17) and contained into ker(dh_2). Thus, for each fixed u_1 any state $(x_1^0,x_2^0) \in N_a$ produces the same output y_2 in the composite system $S_2 \circ S_1$ and this is strongly unobservable.

We stress that this Theorem should not be viewed as an observability criterion for cascade connected nonlinear system but, rather, as an interpretation of the conditions under which a loss of observability may occurr. Namely, we have shown that the composite system is unobservable if and only if the dynamics of S_1 is (partially) "matched" by that of S_2^*, i.e. that of S_2 modified by suitable state feedback.

The condition (i) and (ii) appear to be nonlinear equivalent of the conditions that have been discussed in the introductory Section.

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