SMOOTHING OF STATIONARY CYCLIC PROCESSES

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Abstract

A deficiency of the standard state space foundation of the fixed interval smoothing problem is the presumption that only information is available about the initial state a priori. This rules out cyclic process, where $x(0)$ is known to equal $x(T)$. We derive the formulas for the smoothing of stationary cyclic processes arising from a linear state space model.

1. INTRODUCTION

In linear estimation problems such as Kalman-Bucy filtering, one models the state process by a linear stochastic differential equation driven by white noise with independent initial conditions. One assumes that the matrices of the model and the first and second moments of the driving noise and initial conditions are known apriori. No apriori assumptions are made regarding the state at other times except those that follow implicitly from the assumption that the model fits.

It many situations one has a priori information of the state at various times during the life of the process which one would like to incoporate directly into the model. As an example consider the estimation of a cyclic process observed in white noise. The apriori information is that that the state at time 0 equals the state at time T. In general such processes are not Markov and hence cannot be generated by a model of Kalman-Bucy type. In this paper we discuss a class of linear non-Markov models, necessary and sufficient conditions for such models to generate a stationary process, and the smoothing of stationary cyclic processes.

2. NON-MARKOV LINEAR MODELS

Consider the linar system

$$\dot{x} = Ax + Bu \qquad (2.1a)$$

$$v = V^{o}x(0) + V^{T}x(T) \qquad (2.1b)$$

$$y = Cx + Dw \qquad (3.1c)$$

where x and v are n dimensional, u is m dimensional, y is p dimensional, and A,B,C,D,V^{o} and V^{T} are constant matrices of compatible dimensions. Such models are called *acausal linear systems* and are treated in [1],[2] and[3]. Henceforth we shall assume that the boundary value problem (2.1a,b) is well posed, i.e., that there exists a unique solution $x(t)$ for each vector $v \in \mathbb{R}^{n}$ and integrable function $u(t) \in L_1^m[0,T]$. This is equivalent to the invertibility of

$$F = V^{o} + V^{T} e^{AT} \qquad (2.2)$$

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The solution of (2.1a,b) is given by

$$x(t) = e^{At}F^{-1}v + \int_0^t G(t,s)Bu(s)ds \qquad (2.3)$$

where the *Green's function* $G(t,s)$ is given by

$$G(t,s) = \begin{cases} e^{At}F^{-1}V^o e^{-As} & t > s \\ -e^{At}F^{-1}V^T e^{A(T-s)} & t < s \end{cases} \qquad (2.4)$$

It is convenient to make a change of coordinates in the space of boundary values v, so that $F = I$. Henceforth we assume that this has been done.

If $v = 0$ and $w(t) = 0$ then

$$y(t) = \int_0^T W(t,s)u(s)ds$$

where $W(t,s) = CG(t,s)B$ is the *impluse response* or *weighing function* of the system. A function of two variable such as $W(t,s)$ is said to be *stationary* if it only depends on the difference of t and s; in abuse of notation $W(t,s) = W(t-s)$. We quote several lemmas from [3].

Lemma 2.1 Suppose (A,B) is controllable and (C,A) is observable then $W(t,s)$ is stationary iff $G(t,s)$ is,

Lemma 2.2 $G(t,s)$ is stationary iff

$$[V^o,A] = 0 \text{ and } [V^T,A] = 0$$

where bracket denotes the commutator, i.e.

$$[V^o,A] = V^oA-AV^o \ .$$

Suppose we assume that $u(t)$ is a standard white Gaussian noise and v is an independent Gaussian random vector of zero mean and covariance P, then the solution of (2.1a,b) is a stochastic process given by (2.3) where the integral is interpreted in the sense of Wiener. This is a zero mean process with covariance $R_x(t,s)$ given by

$$R_x(t,s) = e^{At}Pe^{A's} + \int_0^T G(t,\tau)BB'G'(s,\tau)d\tau \ . \qquad (2.5)$$

Again from [3] we have

Lemma 2.3 The model (2.1a,b) generates a stationary process $x(t)$ with $R_x(t,t) = R$ iff

(i) $[A,V^o] = [A,V^T] = 0$

(ii) $AR+RA' = V^Te^{AT}BB'e^{A'T}V^{T'} - V^oBB'V^o$

(iii) $R-\int_0^T V^Te^{A(T-\tau)}BB'e^{A'(T-\tau)}V^{T'}d\tau = P \geq 0$.

The process $x(t)$ is *cyclic* if $x(0) = x(T)$ or in other words $V^o = -V^T$ and $P = 0$.

3. SMOOTHING

We assume that the model (2.1a,b) generates a stationary cyclic process $x(t)$. The observation process $y(t)$ is given by (2.1c) where $w(t)$ is a standard white Gaussian noise independent of $u(t)$ and v. D is an invertible matrix. We seek the optimal smoothed estimate $\hat{x}(t)$ of $x(t)$ given the full observation history $y(s)$, $0 \leq s \leq T$, where optimal is to be interpreted as minimizing the covariance of the error $\tilde{x}(t) = x(t) - \hat{x}(t)$.

It is well known [4] that the Kalman-Bucy filter can be derived by solving a family of linear quadratic regulators. This technique generalizes to a much wider class of linear Gaussian estimation problems [3]. From these assumptions, we know that $\hat{x}(t)$ is a linear function of the observations, i.e., there exists an $n\times p$ matrix valued function $K(t,s)$ such that

$$\hat{x}(t) = \int_0^T K(t,s)y(s)ds$$

Given $K(t,s)$ for $0 \leq s, t \leq T$, let $H(t,s)$ be the piecewise differentiable $n\times n$ matrix valued function satisfying

$$\frac{\partial}{\partial s} H(t,s) = -H(t,s)A + K(t,s)C \quad t \neq s \qquad (3.2a)$$

$$H(t,t^-) - H(t,t^+) = I \qquad (3.2b)$$

$$H(t,0) = L(t)V^o \quad H(t,T) = -L(t)V^T \qquad (3.2c)$$

where $L(t)$ is arbitrary. Since $V^o = V^T$ (3.2c) is equivalment to

$$H(t,0) = H(t,T)$$

The arguments t^- and t^+ denote left and right limits. Some care must be exercised in interpreting (3.2b,c) particularly if $t = 0$ or T where limiting values are to be taken. For example $H(0,0^-)$ and $H(0,0)$ are $H(0^+,0)$ and $H(T,T^+)$ and $H(T,T)$ are $H(T^-,T)$. For the sake of brevity we do not consider the questions of existence and uniqueness of H. Using intergration by parts (with particular care at $t = 0$ or T) we have

$$\hat{x}(t) = \int_0^T K(t,s)(Cx(s) + D\,w(s))ds$$

$$= \int_0^{t^-} + \int_{t^+}^T \left\{\frac{\partial}{\partial s} H(t,s) + H(t,s)A\right\} x(s) + K(t,s)D\,w(s)ds$$

$$= H(t,s)x(s)\Big]_0^{t^-} + \Big]_{t^+}^T + \int_0^T H(t,s))Ax(s)-\dot{x}(s)) + K(t,s)D\,w(s)ds$$

$$= (H(t,t^-) - H(t,t^+))x(t) + H(t,T)x(T) - H(t,0)x(0) - \int_0^T H(t,s)Bu(s) - K(t,s)D\,w(s)ds$$

$$= x(t) - L(t)(V^T x(t) + V^o x(0)) - \int_0^T H(t,s)Bu(s) - K(t,s)D\,w(s)ds$$

$$= x(t) - \int_0^T H(t,s)Bu(s) - K(t,s)D\,w(s)ds$$

Therefore we can express the error $\tilde{x}(t)=x(t)-\hat{x}(t)$ as $\tilde{x}(t) = \int_0^T H(t,s)Bu(s) - K(t,s)D\,w(s)ds$

and its covariance

$$E(\tilde{x}(t)\tilde{x}(t)') = \int_0^T H(t,s)BB'H'(t,s) + K(t,s)DD'K'(t,s)ds. \quad (3.3)$$

For each t, $K(t,s)$ is the optimal control which minimizes (3.3) where the state $H(t,s)$ satisfies (3.2).

Suppose for $t = 0$ we have found the solutions $K(0,s)$ and $H(0,s)$ to (3.2) for $0 \le s \le T$. In this case the boundary conditions (3.2b,c) on $H(0,s)$ reduce to

$$H(0,T) - H(0,0^+) = I \quad (3.4)$$

We extend these periodically to all $s \in \mathbb{R}$ by the relations

$$K(0,s+T) = K(0,s) \quad (3.5a)$$

$$H(0,s+T) = H(0,s) \quad (3.5b)$$

From (3.4) we see that this implies a jump in $H(0,s)$ at multiples of T,

$$H(0,jT^-) - H(0,jT^+) = I .$$

Then we extend K and H to stationary functions for all t and s.

$$K(t,s) = K(0,s-t) \quad (3.6a)$$

$$H(t,s) = H(0,s-t) \quad (3.6b)$$

It is straightforeard to verify that $K(t,s)$ and $H(t,s)$ are solutions to (3.2) for any $0 \le t \le T$. Moreover let $\sigma = s-t$ then

$$\int_0^T H(t,s)BB'H'(t,s) + K(t,s)DD'K'(t,s)ds$$

$$= \int_{-t}^{T-t} H(0,\sigma)BB'H'(0,\sigma) + K(0,\sigma)DD'K'(\sigma,\sigma)d\sigma$$

$$= \int_0^T H(0,\sigma)BB'H'(0,\sigma) + K(0,\sigma)DD'K'(0,\sigma)d\sigma$$

by periodicity (3.5). Therefore if $K(0,s)$ and $H(0,s)$ are the optimal solutions for $t = 0$ then their periodic and stationary extensions defined by (3.5) and (3.6) are the solutions for any t. As in Kalman-Bucy filtering we need only solve one linear quadratic regulator. We seek $K(0,s)$ and $H(0,s)$ for $0 \le s \le T$ which minimizes the cost (3.3) subject to the differential equations (3.2a) and the boundary conditions (3.4).

Let $Q(s)$, $R(s)$ and $S(s)$ be $n\times n$ matrices such that Q is symmetric.

$$\dot{Q} = AQ + QA' + BB' - QC'(DD')^{-1}CQ \quad (3.7a)$$

$$Q(0) = Q(T) \quad (3.7b)$$

$$\dot{R} = (A-QC'(DD')^{-1}C)R \quad (3.7c)$$

$$R(0) - R(T) = Q(0) \quad (3.7d)$$

$$\dot{S} = R'C'(DD')^{-1}CR . \quad (3.7e)$$

If we add the zero quantity

$$\Big(H(0,s)Q(s)H'(0,s)+H(0,s)R(s)+R'(s)H'(0,s)+S(s)\Big)\Big]_0^T - \int_0^T \frac{d}{ds}(H(0,s)Q(s)H'(0,s)+H(0,s)R(s)+R'(s)H'(0,s)+S(s))ds$$

to the cost we obtain

$$\int_0^T (-H(0,s)Q(s)C'D'^{-1}+K(0,s)D-R(s)'C'D'^{-1})(-H(0,s)Q(s)C'D'^{-1}+K(0,s)D-R'(s)C'D'^{-1})'ds + Q(0) + R(T) + R'(T) + S(T) - S(0). \quad (3.8)$$

Clearly the optimal solution is given by

$$K(0,s)=H(0,s)Q(s)C'(DD')^{-1}+R'(s)C'(DD')^{-1} \quad (3.9)$$

which we plug into (3.2a) and solve subject to the boundary condition (3.6). We shall not discuss the question of existence of solutions to this equation or to (3.7). We note that Hermann and Martin [5] have discussed the existence of periodic solutions for the matrix Riccati equation (3.7a,b).

We reformulate (3.9) as

$$K(0,s) = H(0,s)\tilde{Q}(s)C'(DD')^{-1}$$

where $\tilde{Q}(s)$ is the not necessarily symmetric matrix

$$\tilde{Q}(s) = Q(s) + H(0,s)^{-1}R'(s) . \quad (3.10)$$

It is straightforward to verify $\tilde{Q}(s)$ also satisfies the matrix Riccati differential equation (3.7a) but it is not so easy to compute the boundary conditions which it satisfies.

If we let $\tilde{Q}(t,s)$ be the periodic and stationary extension of $\tilde{Q}(s)$ then the optimal estimate is given by

$$\hat{x}(t) = \int_0^T H(t,s)\tilde{Q}(t,s)C'(DD')^{-1}y(s)ds$$

where H(t,s) satisfies

$$\frac{\partial}{\partial s} H(t,s) = -H(t,s)(A-\tilde{Q}(t,s)C'(DD')^{-1}C)$$

subject to (3.2b) and (3.2c) .

If $\tilde{Q}(s)$ is constant then so is $\tilde{Q}(t,s)$ and $H(t,s)$ is the Green's function of the boundary value system

$$\dot{z} = \tilde{A}z + f$$
$$0 = V^o z(0) + U^T z(T)$$

where

$$\tilde{A} = A - \tilde{Q}C'(DD')^{-1}C \quad .$$

In this case $\hat{x}(t)$ is the solution of

$$\frac{d}{dt}\hat{x} = \tilde{A}\hat{x} - \tilde{Q}C'(DD')^{-1}y$$

subject to

$$0 = V^o\hat{x}(0) + V^T\hat{x}(T) \ .$$

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