THE HIGH ORDER MAXIMAL PRINCIPLE

Arthur J. Krener

University of California, Davis

1. INTRODUCTION

First order necessary conditions for optimality are the Pontryagin Maximal Principle (PMP) or the Euler-Lagrange and Legendre-Clebsch conditions. In many problems of practical importance, particularly those when the control enters linearly, these conditions are inconclusive for determining the optimal control. In 1964 Kelley [1] discovered a generalization of the Legendre-Clebsch condition and this was extended by Robbins [2,3], Tait [4], Goh [5,6], Kopp and Moyer [7], and Kelley, Kopp and Moyer [8]. An excellent survey of these papers and related results is Gabasov and Kirillova [9].

This condition, now known as the Generalized Legendre-Clebsch Condition (GLC), was developed by studying the high order effect of special control variations on the cost functional of the system. In 1969 Jacobson [10] used similar techniques to develop a different necessary condition. Many of the derivations of the GLC explicitly or implicitly assume that the problem is normal, i.e., there exists sufficient control variations to enable one to cancel out undesirable lower order effects of the special control variations. When there are terminal constraints, this assumption is necessary to insure they can be satisfied, and even without constraints the assumption is necessary to insure the correct form of the test. There generally has been no discussion of when such an assumption is valid or what to do when it is invalid.

In the PMP no assumption of normality is necessary; it is taken care of by the transversality condition on the adjoint variable and the additivity of first order variations. The purpose of this paper is to develop a High Order Maximal Principle (HMP) which extends the PMP, includes the GLC and Jacobson's condition when they apply and also includes numerous other conditions which can be specifically constructed for the problem at hand without assuming normality. We shall only sketch the proof to indicate why the hypotheses are necessary; for a rigorous proof see Krener [12].

2. FIRST ORDER VARIATIONS

The problem we wish to consider is to minimize

(2.1) $$y_o(x(t_e))$$

where the system equation is

(2.2) $$\dot{x} = f(x,u)$$

subject to the constraints

(2.3) $$x(t_o) = x^o$$

(2.4) $$y_i(x(t_e)) = 0, \quad i = 1,\dots,m$$

(2.5) $$u(t) \in \Omega, \quad t \in [t_o,t_e].$$

For convenience $x = (x_o,x_1,\dots,x_n)$ with $x_o = t$ and $u = (u_1,\dots,u_n)$. We assume that f in C^∞ with respect to $x_1,\dots,x_n$, $u_1,\dots,u_k$, and a piecewise C^∞ with respect to x^o. Infinite differentiability is not essential, it is only to avoid counting the degree of differentiability needed in a particular setting. Piecewise differentiability means that left and right limits always exist and there are only a finite number of jumps in any compact interval. To speed the exposition we shall ignore the jumps, they are easily handled using standard techniques. We restrict to piecewise C^∞ controls, $u(t)$. The functions $y_o(x),\dots,y_m(x)$ are assumed to be C^∞ and the matrix $(\partial y_i/\partial x_j(x))$ $i = 0,\dots,m$, $j = 0,\dots,n$ is assumed to be of rank $m + 1$. Suppose u^o satisfies (2.5) and the corresponding solution $t \to x(t)$ of (2.2) satisfies (2.3) and (2.4). To develop necessary conditions for u^o we choose some point $t_1 \in (t_o,t_e)$ and modify u^o by replacing it with another admissible control, u^1, in a neighborhood of t_1.

To be more precise, let $s \to \gamma_i(s)x$ be the family of solutions of the differential equation $d/ds\ \gamma_i(s)x = f(\gamma_i(s)x,u^i)$ with initial conditions $\gamma_i(0)x = x$ for $i = 0,1$. Consider the family of trajectories whose locus of endpoints is

$$(2.6)\quad \gamma_o(t_e-t_1)\gamma_1(s)x(t_1-s) = \gamma_o(t_e-t_1)\gamma_1(s)\gamma_o(-s)x(t_1).$$

This locus is differentiable with respect to s and its effect on y_o and y_i studied, yielding first order necessary conditions.

The above control variation was made before t_1; it could have been made after t_1 or around t_1 as follows:

$$(2.7)\quad \gamma_o(t_e-t_1-s)\gamma_1(s)x(t_1) = \gamma_o(t_e-t_1)\gamma_o(-s)\gamma_1(s)x(t_1)$$

or

$$(2.8)\quad \gamma_o(t_e-t_1-s/2)\gamma_1(s)x(t_1-s/2)$$
$$= \gamma(t_e-t_1)\gamma_o(-s/2)\gamma_1(s)\gamma_o(-s/2)x(t_1)$$

and the same necessary conditions result.

Another type of control variation (2.9) is to stop short of t_e or to continue past t_e using the same control. This is equivalent to abbreviating or lengthening the trajectory at any intermediate point, t_1.

$$(2.9)\qquad \gamma_o(\pm s)x(t_e) = \gamma_o(t_e-t_1)\gamma_o(\pm s)x(t_1).$$

A standard proof of the Maximal Principle is to consider the convex cone K^1 of vectors generated by the derivatives with respect to s of all expressions of the form (2.6) through (2.9). This cone is a measure of the controllability at x^e. This is true because the magnitude of control variation can be changed by multiplying s by a nonnegative constant, the derivative of two control variations made jointly at two different times is the sum of their individual derivatives, and for two variations of type (2.6) made at the same time with controls u^1 and u^2, the derivative of (2.10) is the sum of their individual derivatives.

$$(2.10)\qquad \gamma_o(t_e-t_1)\gamma_2(s)\gamma_1(s)\gamma_o(-2s)x(t_1).$$

Since we are interested in the effect that these variations have on $y_o,\ldots,y_m$ we define another convex cone L^1 of $m+1$ dimensional vectors by

$$(2.11)\qquad L^1 = \left[\frac{\partial y_i}{\partial x_j}(x^e)\right] K^1.$$

Using a fixed point argument (Halkin [11]) it can be shown that if

u^o is optimal then L^1 can be separated from the m + 1 vector (-1,0,...,0) by linear functional $\nu = (\nu_o,\ldots,\nu_m)$ where $\nu_o \leq 0$. This linear functional on m + 1 vectors defines a linear functional $\lambda(t_e) = \nu[\partial y_i/\partial x_j(x^e)]$ on n + 1 vectors which can be pulled back along x(t) using the adjoint differential equation and the result is the Pontryagin Maximal Principle:

If u^o is optimal then there is nontrivial $\lambda(t)$ such that if $H(\lambda,x,u) \triangleq \lambda f(x,u)$ then

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x}(\lambda(t),x(t),u^o(t)), \tag{2.12}$$

$$\lambda(t_e) = \nu\left[\frac{\partial y_i}{\partial x_j}(x^e)\right] \quad \text{with } \nu_o \leq 0, \tag{2.13}$$

$$0 = H(\lambda(t),x(t),u^o(t)) \geq H(\lambda(t),x(t),u) \quad \forall\, t \in [t_o,t_e] \text{ and } \forall\, u \in \Omega. \tag{2.14}$$

3. THE HIGH ORDER MAXIMAL PRINCIPLE WITH TERMINAL CONSTRAINTS

A first order control variation is inconclusive if

$$\frac{d}{ds}\gamma_o(t_e-t_1)\gamma_1(s)\gamma_o(-s)x(t_1) = 0 \quad \text{at } s = 0. \tag{3.1}$$

Let

$$\beta(s)x = \gamma_1(s)\gamma_o(-s)x. \tag{3.2}$$

We consider the effect that the second derivative has on the cost and terminal constraints. In fact if we could include the vector

$$\frac{d^2}{ds^2}\gamma_o(t_e-t_1)\beta(0)x(t_1) \tag{3.3}$$

in the cone K^1 we would obtain a new necessary condition. This is not possible since the vectors of K^1 are first order effects. However if we replace the parameter s by $s^2/2$ in any one of the variations (2.6) through (2.10) then the first derivative of the new variation vanishes and the second derivative is the old first derivative. One might hope to construct a cone of second derivatives since the magnitude of a second order control can be changed by multiplying s by a nonnegative constant. However in general the joint second derivative of two variations satisfying (3.1) made at different times is not equal to the sum of their individual

We define K^k as the convex cone generated by all vectors of the form

$$\text{(3.16)} \qquad \frac{d^k}{ds^k} \gamma_o(t_e - t_1)\beta(s)x^1$$

where β is of order k at x^1. By replacing s by s^h a control variation of order k is shifted to one of order $k \cdot h$ therefore $K^k \subseteq K^{k \cdot h}$. This allows us to define the convex cones $K = \bigcup_{k \geq 1} K^k$ and $L = \left[\frac{\partial y_i}{\partial x_j}(x^e)\right]K$. The rest of the development of the HMP proceeds as before using a fixed point argument. The result is the High Order Maximal Principle with terminal constraints.

If u^o is optimal then there is a nontrivial $\lambda(t)$ such that

$$\text{(3.17)} \qquad \dot{\lambda}(t) = -\frac{\partial H}{\partial x}(\lambda(t), x(t), u^o(t)),$$

$$\text{(3.18)} \qquad \lambda(t_e) = \nu\left[\frac{\partial y_i}{\partial x_j}(x^e)\right] \text{ with } \nu_o \leq 0,$$

$$\text{(3.19)} \qquad 0 = H(\lambda(t), x(t), u^o(t)) \geq H(\lambda(t), x(t), u) \quad \forall\, t \in [t_o, t_e] \text{ and } \forall\, u \in \Omega,$$

(3.20) if $\beta(s)x$ is a control variation of order k at x^1 then

$$\lambda(t_1)\frac{d^k}{ds^k}\beta(0)x^1 \leq 0.$$

If the terminal constraints are absent then additivity of variations is not needed and the situation is much simpler. Condition (3.18) reduces to $\lambda(t_e) = -dy_o(x^e)$ and $\lambda(t)$ is defined by (3.17). Condition (3.19) remain the same but (3.20) changes. The first nonzero derivative of the form

$$\text{(3.21)} \qquad \frac{d^k}{ds^k} y_o(\gamma_o(t_e - t_1)\beta(0)x^1)$$

must be nonpositive.

4. EXAMPLES

Consider the problem of minimizing $y(x(t_e))$ subject to

$$\text{(4.1)} \qquad \dot{x} = a_o(x) + u_1(t)a_1(x),$$

$$\text{(4.2)} \qquad |u| \leq 1,\ x(t_o) = x^o,\ y_i(x(t_e)) = 0,\ i = 1,\ldots,m.$$

Let $u^o(t)$ be the candidate for the optimal control and $\lambda(t)$ satisfy (3.17) and (3.18).

Using a first order variation, $\beta_o(s)x = \gamma_o(\pm s)x$, of type (2.9) and condition (3.20) we obtain

$$H(\lambda(t),x(t),u^o(t)) = 0 \tag{4.3}$$

and using a different variation, $\beta_1(s)x = \gamma_1(s)\gamma_o(-s)x$ of type (2.6) where u^1 = constant and (3.20) we obtain

$$H(\lambda(t),x(t),u^o(t)) \geq H(\lambda(t),x(t),u^1). \tag{4.4}$$

We see that (3.20) implies (3.19).

If there are no terminal constraints, by (4.3) we can apply (3.21) to β_o at t_e,

$$\lambda(t_e)\frac{\partial a_o}{\partial x}(x^e) + a_o(x^e)\frac{\partial^2 y_o}{\partial x^2} a_o(x^e) \leq 0.$$

If $|u^o(t)| < 1 \ \forall \ t \in [t_o,t_e]$, then (4.4) implies that

$$\frac{\partial H}{\partial u} (\lambda(t),x(t),u^o(t)) = \lambda(t)a_1(x(t)) = 0, \ \forall \ t \in [t_o,t_e]. \tag{4.5}$$

To differentiate (4.5) with respect to t we note that if $b(x)$ is a vector valued function of x then

$$\frac{d}{dt} \lambda(t)b(x(t)) = \lambda(t)[a_o,b](x(t)) + u^o(t)\lambda(t)[a_1,b](x(t)) \tag{4.6}$$

where the Lie bracket is defined by

$$[b_1,b_2](x) = \frac{\partial b_2}{\partial x} (x)b_1(x) - \frac{\partial b_1}{\partial x} (x)b_2(x) \tag{4.7}$$

and satisfies

$$[b_1,b_2] = -[b_2,b_1] \tag{4.8}$$

$$[b_1[b_2,b_3]] = [[b_1,b_2]b_3] + [b_2[b_1,b_3]]. \tag{4.9}$$

Differentiating (4.5) we obtain

$$\frac{d}{dt}\frac{\partial H}{\partial u} = \lambda(t)[a_o,a_1](x(t)) = 0 \tag{4.10}$$

$$\frac{d^2}{dt^2}\frac{\partial H}{\partial u} = \lambda(t)[a_o[a_o,a_1]](x(t)) + u^o(t)\lambda(t)[a_1[a_o,a_1]](x(t)) = 0. \tag{4.11}$$

If the coefficient of u^o is not zero then u^o is determined by (4.11). If it is zero then we can continue differentiating until, perhaps, some higher derivative determines u. It can be shown using (4.8) and (4.9) that the first k, such that $\partial/\partial u(d^k/dt^k(\partial H/\partial u))$ is not zero for all t, will be even.

Henceforth for convenience we assume that $u^o \equiv 0$. Jacobson's condition [10] is based on the fact that if there are no terminal constraints then (4.5) implies (3.21) is zero for β_1 with $k = 1$. Applying (3.21) with $u^1 = \pm 1$ and $k = 2$ we obtain

$$\lambda(t)\frac{\partial a_1}{\partial x}(x(t))a_1(x(t)) + a_1(x(t))\frac{\partial^2}{\partial x^2}y_o(\gamma_o(t_e-t)x(t))a_1(x(t)) \leq 0. \tag{4.12}$$

Kelley constructed a control variation of order 2 from a pair of variations of order one which cancel each other out. Let $\beta_2(s)x = \gamma_2(s)\gamma_1(s)\gamma_o(-2s)x$ where $u^1 = 1$ and $u^2 = -1$ then $d/ds\ \beta_2(0)x = 0$ and $d^2/ds^2\ \beta_2(0)x = [a_o,a_1](x)$. Reversing u^1 and u^2 gives another variation of order 2 with the opposite second derivative, but together they yield (4.10). By the technique (3.15) of adding control variations made at the same time we obtain $\beta_3(s)x = \gamma_1(s)\gamma_2(2s)\gamma_1(s)\gamma_o(-4s)x$ a variation of order 3 and the condition

$$\lambda(t)\frac{d^3}{ds^3}\beta_3(0)x(t) = 12\lambda(t)[a_o[a_o,a_1]](x(t)) - 4\lambda(t)[a_1[a_o,a_1]](x(t)) \leq 0. \tag{4.13}$$

By (4.11) and $u^o \equiv 0$ this becomes Kelley's condition

$$\frac{\partial}{\partial u}\frac{d^2}{dt^2}\frac{\partial H}{\partial u} = \lambda(t)[a_1[a_o,a_1]](x(t)) \geq 0. \tag{4.14}$$

If $d^3/ds^3\ \beta_3(0)(x(t)) = 0$, then β_3 is a control variation of order 4 and its fourth derivative yields a new condition.

If u^1 and u^2 are reversed in β_3 then another variation of order 3 is obtained with third derivative, $-12[a_o[a_o,a_1]](x(t)) - 4[a_1[a_o,a_1]](x(t))$. If $[a_1[a_o,a_1]](x(t)) = 0$, then by adding these two variations of order 3 a variation, β_4, of order 4 is obtained. The fourth derivative of this new variation can be applied to (3.20) to yield a new test, if the fourth derivative is zero then β_4 is a variation of order 5 and its fifth derivative can be applied to (3.20). Another possible way to construct a variation of order 5 is to reverse u^1 and u^2 in β_4 to obtain a different variation of order 4 and then add them. Kelley, Kopp and Moyer [8] used this technique to obtain the GLC for this problem. The first nonzero derivative of the form

$$(4.15) \qquad (-1)^h \frac{\partial}{\partial u} \frac{d^{2h}}{dt^{2h}} \frac{\partial H}{\partial u} (\lambda(t), x(t), u^o(t))$$

must be nonpositive. This appears with the opposite inequality if $\nu_o \geq 0$. To derive this test an assumption of normality is necessary even if there are no terminal constraints. No explicit definition of normality was given by Kelley, Kopp and Moyer; therefore the usefulness of this test is severely limited.

An alternate approach is possible if $[a_1[a_o,a_1]](x(t))$ is a linear combination of $a_o(x(t))$, $a_1(x(t))$ and $[a_o,a_1](x(t))$ it is possible to reparametrize β_3 by s^2, β_2 by s^3, β_1 and β_o by s^6, and combine them into a variation of order > 6 where $[a_1[a_o,a_1]](x(t))$ is cancelled out. The general method for constructing new variations is to take two (or more) variations which cancel each other, adjust their parameters so they are of the same order and combine them. It is shown in [9] that Goh's [6] generalization of (4.15) can be obtained in this way for the problem with several control variables entering linearly.

REFERENCES

1. H. J. Kelley, A second variation test for singular extremals, AIAA J., 2(1964), pp. 1380-1382.

2. H. M. Robbins, Optimality of intermediate-thrust arcs of rocket trajectories, AIAA J. 3(1965), pp. 1094-1098.

3. H. M. Robbins, A generalized Legendre-Clebsch condition for the singular cases of optimal control, IBM J. Res. Develop., 11(1967), pp. 361-372.

4. K. Tait, Singular problems in optimal control, Ph.D. thesis, Harvard, 1965.

5. B. S. Goh, The second variation for the singular Bolza problem, SIAM J. Control, 4(1966), pp. 309-325.

6. B. S. Goh, Necessary conditions for singular extremals involving multiple control variables, SIAM J. Control, 4(1966), pp. 716-731.

7. R. E. Kopp and H. G. Moyer, Necessary conditions for singular extremals, AIAA J., 3(1965), pp. 1439-1444.

8. H. J. Kelley, R. E. Kopp and H. G. Moyer, Singular Extremals, Topics in Optimization, Academic Press, N.Y., 1967, pp. 63-101.

9. R. Gabasov and F. M. Kirillova, High order necessary conditions for optimality, SIAM J. Control, 10(1972), pp. 127-168.

10. D. H. Jacobson, A new necessary condition of optimality for singular control problems, SIAM J. Control, 7(1969), pp. 578-595.

11. H. Halkin, A maximum principle of the Pontryagin type for systems described by nonlinear difference equations, SIAM J. Control, 4(1966), pp. 90-111.

12. A. J. Krener, The high order maximum principle for singular extremals, to appear.