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A GENERALIZATION OF CHOW'S THEOREM AND THE BANG-BANG THEOREM TO NONLINEAR CONTROL PROBLEMS*

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Abstract. The main results of this paper are two-fold. The first, Theorem 1, is a generalization of the work of Chow and others concerning the set of locally accessible points of a nonlinear control system. It is shown that under quite general conditions, this set lies on a surface in state space and has a nonempty interior in the relative topology of that surface.

The second result, Theorem 3, generalizes the bang-bang theorem to nonlinear control systems using higher order control variations as developed by Kelley and others. As a corollary we obtain Halkin's bang-bang theorem for a linear piecewise analytic control system.

1. Introduction. Consider the control system

$$\dot{x} = f(x(t), u(t)), \qquad x(0) = x^0, \qquad u(t) \in \Omega, \tag{1}$$

where $x = (x_1, \cdots, x_n)$ are coordinates of the state space, M is a paracompact n-dimensional manifold, $u = (u_1, \cdots, u_k)$ is the control, $x^0 = (x_1^0, \cdots, x_n^0)$ is the initial state, $\Omega \subseteq \mathbb{R}^k$ is the set of admissible controls, and f is an n-vector-valued function. We assume that $x_1 = t$ so that the first coordinate of f is identically 1; also we assume that f is C^∞ with respect to $x_2, \cdots, x_n, u_1, \cdots, u_k$ and piecewise C^∞ with respect to x_1. We require that $u(t)$ be a piecewise C^∞-function of $t = x_1$. The requirement of C^∞ differentiability is not essential, it is only to avoid counting the degree of differentiability required in any argument. The tangent space to M at x is denoted by M_x. A control $u(t)$ defines a vector field, $f_u(x) = f(x, u(x_1))$ on M; and given two controls $u(t)$, $v(t)$, we can define a new vector field by means of the Lie bracket,

$$[f_u, f_v](x) = \frac{\partial f_v}{\partial x}(x) f_u(x) - \frac{\partial f_u}{\partial x}(x) f_v(x), \tag{2}$$

where $(\partial f_v/\partial x)(x)$ is an $n \times n$ matrix of partial derivatives at x.

A slight problem arises since f_u, f_v are only piecewise C^∞-functions of x_1, but at those values of x_1 we can consider (2) as either undefined or as double-valued by taking left and right limits. Since the difficulties that arise because of this can be dealt with by simple but lengthy arguments, we shall ignore them.

2. Integrability and semi-integrability. The set, $V(M)$, of all C^∞-vector fields on M is a module over the ring, $C(M)$, of all C^∞-real-valued functions with domain M, with addition and multiplication defined pointwise. With the definition of the bracket (2), $V(M)$ becomes a Lie algebra of infinite dimension over the field, $\mathbb{R}$. Suppose H is a submodule of $V(M)$. We define $H_x = \{f(x) : f \in H\}$. Let U be an open subset of M and L a submanifold of U. L is an *integral manifold* of H in U if L is connected and $H_x = L_x$ for all $x \in L$ (L_x is the tangent space to L at x). An integral manifold of H in U is always contained in a maximal integral manifold of H in U. H is *integrable on* U if there exists a partition of U by maximal integral

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manifolds of H in U. For H to be integrable on U a necessary condition is that H restricted to U be a subalgebra of $V(U)$. If, in addition, H satisfies one of the following then H is integrable on U;

(i) Frobenius. The dimension of H_x is constant for all $x \in U$.

(ii) Hermann [9]. H is *locally finitely generated*, that is, for all $x \in U$, there exists a neighborhood $V \subseteq U$ of x such that H restricted to V is the $C(V)$ span of a finite number of vector fields of H. (Lobry [14] has a slightly weaker form of this condition.)

(iii) Nagano [16]. M is a real analytic manifold and H is a subalgebra of the Lie algebra of real analytic vector fields on U.

If H is a submodule but not a subalgebra, then there exists a smallest subalgebra containing H, which we denote by DH. We can construct DH as follows. We define $D^0H = H$ and $D^kH = D^{k-1}H + [H, D^{k-1}H]$. For example, D^1H is the submodule of all linear combinations of vector fields of H and Lie brackets of vector fields of H with coefficients from $C(M)$. DH is the union of this ascending sequence of submodules.

Suppose U is an open neighborhood of $x^0 = (x_1^0, \cdots, x_n^0)$. Then we split U into two open halves,

$$U^+ = \{x \in U : x_1 > x_1^0\} \quad \text{and} \quad U^- = \{x \in U : x_1 < x_1^0\}.$$

The control system (1) is *locally semi-integrable* if for all $x^0 \in M$, there exists an open neighborhood U of x^0 and submodules H^+, H^- of $V(M)$ such that

(i) $H_x^+ = \text{span}\,\{f(x, u) : u \in \Omega\} \subseteq M_x$ for all $x \in U^+$,
$H_x^- = \text{span}\,\{f(x, u) : u \in \Omega\} \subseteq M_x$ for all $x \in U^-$.

(ii) DH^+ and DH^- are integrable on U with maximal integrable manifolds L^+ and L^- in U containing x^0.

Suppose $u(t)$ is an admissible control and $\gamma_u(s)x$ is the family of integral curves of the vector field $f_u(x)$, that is, $\gamma_u(0)x = x$ and $(d/ds)\gamma_u(s)x = f_u(\gamma_u(s)x)$. We define the *set* $\mathscr{A}(x^0, U)$ *of all points accessible from* x^0 *in* U as $\{\gamma_u(s)x^0 : s \geqq 0, u(t) \in \Omega$, and $\gamma_u(r)x^0 \in U$ for all $r \in [0, s]\}$ and the *set* $\mathscr{C}(x^0, U)$ *of all points controllable to* x^0 *in* U as $\{\gamma_u(s)x^0 : s \leqq 0, u(t) \in \Omega$, and $\gamma_u(r)x^0 \in U$ for all $r \in [s, 0]\}$.

If (1) is locally semi-integrable in U, then it is easily shown that $\mathscr{A}(x^0, U) \subseteq L^+$ and $\mathscr{C}(x^0, U) \subseteq L^-$.

We now raise the question whether $\mathscr{A}(x^0, U)$ is "thick" in L^+, i.e., whether $\mathscr{A}(x^0, U)$ has any interior as a subset of L^+. The answer is affirmative as the following generalization of the work of Chow [2], Lobry [14], and Sussmann and Jurdjevic [19] shows.

THEOREM 1. *Assume* (1) *is a locally semi-integrable control system and* x^0, U, L^+ *and* L^- *are as above. Then the* L^+*-interior of* $\mathscr{A}(x^0, U)$ *and the* L^-*-interior of* $\mathscr{C}(x^0, U)$ *are nonempty.*

Proof. In theorems of this type we shall only prove one assertion since the proof of the other is identical. We construct inductively a sequence of maps $\varphi_j : V^j \to \mathscr{A}(x^0, U) \subseteq L^+$ defined on a sequence of open sets $V^j \subseteq \mathbb{R}^j$ such that the image $N^j = \varphi_j(V^j)$ is a submanifold of dimension j. We continue until j equals the dimension of L^+.

Choose any control, say $u^1(t) = (u_1^1(t), \cdots, u_k^1(t))$, and let $f_1(x)$ be the vector field $f_1(x) = f(x, u^1(x_1))$. Let $\delta > 0$ such that the integral curve $s_1 \mapsto \gamma_1(s_1)x^0$

of f_1 is C^∞ for $s_1 \in (0, \delta)$. Let $V^1 = (0, \delta)$ and $\varphi_1(s_1) = \gamma_1(s_1)x^0$. Since the first coordinate of f_1 is identically 1, the image $N^1 = \varphi_1(V^1)$ is a one-dimensional submanifold of L^+.

Suppose we have constructed $N^{j-1} = \varphi_{j-1}(V^{j-1})$ and $j \leqq$ dimension of L^+. Choose $x \in N^{j-1}$ and a control u^j such that $f_j(x) = f(x, u^j(x_1)) \notin N_x^{j-1}$, the tangent space to N^{j-1} at x. This can always be done, for if not, then for all $x \in N^{j-1}$ and for all $u \in \Omega$, $f_u(x) \in N_x^{j-1}$. This implies that $H_x \subseteq N_x^{j-1}$ for all $x \in N^{j-1}$, and the set of vector fields on N^{j-1}, $V(N^{j-1})$, is an algebra; therefore DH restricted to N^{j-1} is contained in $V(N^{j-1})$. But this implies that $j \leqq$ dimension of $L^+ =$ dimension $DH_x \leqq$ dimension of $N_x^{j-1} = j - 1$.

By passing to a smaller V^{j-1} and N^{j-1} we can assume that $f_j(x) \notin N_x^{j-1}$ for all $x \in N^{j-1}$ and also for some $\delta > 0$, the integral curve of f_j starting at x satisfies $\gamma_j(s_j)x \in U$ for all $x \in N^{j-1}$ and $0 < s_j < \delta$. We define $\varphi_j(s_1, \cdots, s_j) = \gamma_j(s_j)\varphi_{j-1}(s_1, \cdots, s_{j-1})$, $V^j = V^{j-1} \times (0, \delta)$ and $N^j = \varphi_j(V^j) \subseteq \mathscr{A}(x^0, U)$. The Jacobian $(\partial\varphi_j/\partial s)(s_1, \cdots, s_{j-1}, 0)$ is nonsingular for every $(s_1, \cdots, s_{j-1}) \in V^{j-1}$ and hence for δ sufficiently small $\varphi_j : V^j \to N^j$ is a diffeomorphism. Q.E.D.

Example 1. Suppose $M = \mathbb{R}^2$ and consider the control system $\dot{x}_1 = 1$, $\dot{x}_2 = u \cdot g(x_1)$, $|u| \leqq 1$, where $g(x_1)$ is a C^∞-(or pwC^∞)-function satisfying $g(x_1) = 0$ if $x_1 \leqq 0$ and $g(x_1) > 0$ if $x_1 > 0$. The system is locally semi-integrable; for example, if $x^0 = (0, 0)$, then we take $U = M, H^+ = DH^+ = V(M), L^+ = M$,

$$\mathscr{A}(x^0, U) = \left\{(x_1, x_2) : x_1 \geqq 0, |x_2| \leqq \int_0^{x_1} g(x_1)\, dx_1\right\},$$

$$H^- = DH^- = \left\{\begin{pmatrix} h(x) \\ 0 \end{pmatrix} : h(x) \in C(M)\right\},$$

$$L^- = \{(x_1, 0) : x_1 \in \mathbb{R}\} \quad \text{and} \quad \mathscr{C}(x^0, U) = \{(x_1, 0) : x_1 \leqq 0\}.$$

The system $\dot{x}_1 = 1, \dot{x}_2 = u \cdot g(x_2)$ is not locally semi-integrable.

Example 2. Suppose $M = \mathbb{R}^2$ and $\dot{x}_1 = 1, \dot{x}_2 = ux_2, |u| \leqq 1$. The submodule $H = \{(h_1(x), x_2h_2(x)) : h_i(x) \in C(M)\}$ is an integrable subalgebra and carries the system everywhere; that is, for each $x \in M$, $H_x = \operatorname{span}\{f(x, u) : |x| \leqq 1\}$. It partitions M into 3 integral manifolds:

$$L^1 = \{(x_1, x_2) : x_2 > 0\}, \qquad L^2 = \{(x_1, x_2) : x_2 = 0\}$$

and

$$L^3 = \{(x_1, x_2) : x_2 < 0\}.$$

Suppose $x^0 = (0, 1) \in L^1$. Then $U = M, H^+ = DH^+ = H^- = DH^- = H, L^+ = L^- = L^1, \mathscr{A}(x^0, U) = \{(x_1, x_2) : x_1 \geqq 0, e^{-x_1} \leqq x_2 \leqq e^{x_1}\}$ and $\mathscr{C}(x^0, U) = \{(x_1, x_2) : x_1 \leqq 0, e^{x_1} \leqq x_2 \leqq e^{-x_1}\}$. If $x^0 = (0, 0) \in L^2$, then $U = M$, $H^+ = DH^+ = H^- = DH^- = H$, $L^+ = L^- = L^2$, $\mathscr{A}(x^0, U) = \{(x_1, 0) : x_1 \geqq 0\}$ and $C(x^0, U) = \{(x_1, 0) : x_1 \leqq 0\}$.

Example 3. Suppose $M = \mathbb{R}^3$ and $\dot{x}_1 = 1$, $\dot{x}_2 = u$, $\dot{x}_3 = ux_1$, $|u| \leqq 1$. Let $f_1(x), f_{-1}(x)$ be the vector fields corresponding to the constant controls $u = \pm 1$.

Let H be the submodule which is the $C(M)$ span of f_1 and f_{-1}. There is one linearly independent bracket

$$[f_1, f_{-1}](x) = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix},$$

so DH is spanned by f_1, f_{-1} and $[f_1, f_{-1}]$. The integrable manifold of DH through any point x^0 is M. The sets $\mathscr{A}(x^0, M) = \{\gamma_1(s_3)\gamma_{-1}(s_2)\gamma_1(s_1)x^0 : s_i \geqq 0\}$ and $\mathscr{C}(x^0, M) = \{\gamma_1(s_3)\gamma_{-1}(s_2)\gamma_1(s_1)x^0 : s_i \leqq 0\}$ both have nonempty interior.

3. The bang-bang theorem. Henceforth we shall consider the system

$$\dot{x} = \sum_{i=0}^{k} u_i(t)a_i(x), \tag{3}$$
$$x(0) = x^0, \quad u_i(t) \geqq 0, \quad \sum u_i(t) = 1,$$

where $a_0, \cdots, a_k$ are vector-valued functions C^∞ with respect to $x_2, \cdots, x_n$ and piecewise C^∞ with respect to $x_1 = t$. The controls, $u(t)$, are piecewise C^∞-functions of $t = x_1$, lying in the compact convex set $\Omega = \{u : u_i \geqq 0, \sum u_i = 1\}$. We let E denote the set of extreme points of Ω. E is the set of unit vectors, $(0, \cdots, 0, 1, 0, \cdots, 0)$, in $\mathbb{R}^k$. We call Ω the set of *admissible* controls and E the set of *bang-bang* controls. We alter our notation to distinguish between the set of points, $\mathscr{A}(x^0, U, \Omega)$, accessible in U from x^0 by admissible controls, and the set of points, $\mathscr{A}(x^0, U, E)$, accessible in U from x^0 by bang-bang controls. We adopt a similar convention regarding $\mathscr{C}(x^0, U, \Omega)$ and $\mathscr{C}(x^0, U, E)$. The bang-bang question is, under what conditions is it true that $\mathscr{A}(x^0, U, E) = \mathscr{A}(x^0, U, \Omega)$ and $\mathscr{C}(x^0, U, E) = \mathscr{C}(x^0, U, \Omega)$. It is well known that $\mathscr{A}(x^0, U, E) \subseteq \mathscr{A}(x^0, U, \Omega) \subseteq$ closure $\mathscr{A}(x^0, U, E)$ and $\mathscr{C}(x^0, U, E) \subseteq \mathscr{C}(x^0, u, \Omega) \subseteq$ closure $\mathscr{C}(x^0, U, E)$.

THEOREM 2. *Suppose* (3) *is locally semi-integrable and* U, L^+, L^- *are as above. Then* L^+-interior $\mathscr{A}(x^0, U, E) = L^+$-interior $\mathscr{A}(x^0, U, \Omega)$ *and* L^--interior $\mathscr{C}(x^0, U, E) = L^-$-interior $\mathscr{C}(x^0, U, \Omega)$.

Proof. To simplify the proof we restrict (3) to a control system on the manifold L^+, in other words we take $M = L^+$. Clearly interior $\mathscr{A}(x^0, U, E) \subseteq$ interior $\mathscr{A}(x^0, U, \Omega)$. To show the opposite inclusion we let $x \in$ interior $\mathscr{A}(x^0, U, \Omega)$. We choose an open connected neighborhood V of x such that $V \subseteq$ interior $\mathscr{A}(x^0, U, \Omega)$. The set of vector fields $\{f_u : u \in \Omega\}$ and $\{f_u : u \in E\}$ generate the same submodule H and hence by Theorem 1, $\mathscr{C}(x, V, E)$ has a nonempty interior. Let $y \in$ interior $\mathscr{C}(x, V, E) \subseteq V \subseteq \mathscr{A}(x^0, U, \Omega) \subseteq$ closure $\mathscr{A}(x^0, U, E)$. Then there is a sequence $y^m \in \mathscr{A}(x^0, U, E)$, such that y^m converges to y. For m sufficiently large, $y^m \in$ interior $\mathscr{C}(x, V, E)$, so y^m is bang-bang accessible from x^0 and bang-bang controllable to x. This implies $x \in \mathscr{A}(x^0, U, E)$. Q.E.D.

From Theorem 2 it is clear that $\mathscr{A}(x^0, U, E)$ will equal $\mathscr{A}(x^0, U, \Omega)$ if every admissible trajectory which does not come from a bang-bang control goes to an interior point of $\mathscr{A}(x^0, U, \Omega)$. To decide when this will happen we study the effect of control variations.

Let $u^j(x_1)$ be an admissible control, $f_j(x) = \sum u_i^j(x_1)a_i(x)$ and $\gamma_j(s)x$ be the family of integral curves of $f_j(x)$ for $j = 0, 1$. Suppose as we approach $x = \gamma_0(s)x^0 \in \mathscr{A}(x^0, U, \Omega)$ using the control u^0, we replace u^0 with u^1 for r units of time. The result is a trajectory whose endpoint is $\gamma_1(r)\gamma_0(s-r)x^0 = \gamma_1(r)\gamma_0(-r)x$. If we vary r through small nonnegative values, we obtain a C^∞-curve $q(r) = \gamma_1(r)\gamma_0(-r)x$ satisfying $q(0) = x$. To compute the derivative from the right at 0, we define $q(r_0, r_1) = \gamma_1(r_1)\gamma_0(-r_0)x$. Then

$$\frac{dq(0_+)}{dr} = \frac{\partial q(0)}{\partial r_1} - \frac{\partial q(0)}{\partial r_0} = f_1(x) - f_0(x).$$

If we continue to $x^1 = \gamma_0(s_1)x^0 \in \mathscr{A}(x^0, U, \Omega)$ using the control u^0, we can define a new curve $q(r) = \gamma_0(s_1 - s)\gamma_1(r)\gamma_0(-r)x$. This is also C^∞ for small nonnegative r and $q(0) = x^1$. The derivative from the right at 0 is

$$\frac{dq(0_+)}{dr} = \gamma_0(s_1 - s)_*(f_1(x) - f_0(x)), \tag{4}$$

where $\gamma_0(s_1 - s)_*$ is the tangent space map induced by the map $x \mapsto \gamma_0(s_1 - s)x$. If $f_0(x)$ and $f_1(x)$ are C^∞ in a neighborhood of the trajectory joining x and x^1, then (4) can be expressed in a Taylor series,

$$\frac{dq(0_+)}{dr} = \sum_{m=0}^{h} \frac{(s - s_1)^m}{m!} ad^m(f_0)(f_1 - f_0)(x^1) + \mathscr{O}(s - s_1)^{h+1}, \tag{5}$$

where

$$ad^0(f_0)(f_1 - f_0)(x^1) = f_1(x^1) - f_0(x^1),$$
$$ad^m(f_0)(f_1 - f_0)(x^1) = [f_0, ad^{m-1}(f_0)(f_1 - f_0)](x^1)$$

and $\mathscr{O}(s - s_1)^{h+1}$ is an error term of order $(s - s_1)^{h+1}$.

The second type of control variation is similar to the one introduced by Kelley [11].

Suppose u^0, u^1, u^2, u^3 are admissible controls such that $u^0 = (2u^1 + u^2 + u^3)/4$. Then $f_0(x) = (2f_1(x) + f_2(x) + f_3(x))/4$. For ease of notation we introduce another control $u^4 = u^1$ so $u^0 = (\sum_{i=1}^4 u_i)/4$, $f_0(x) = (\sum_{i=1}^4 f_i(x))/4$. Consider the control modification $p(r)$ made at $x \pm \gamma(s)x^0$, where $p(r) = \gamma_4(r) \cdot \gamma_3(r)\gamma_2(r)\gamma_1(r)\gamma_0(-4r)x$.

To compute the first two derivatives of this curve, we introduce new variables $r_0 = -4r, r_1 = r_2 = r_3 = r_4 = r$ and use the chain rule

$$\frac{dp(0_+)}{dr} = \sum_{i=0}^{4} \frac{dr_i}{dr}\frac{\partial p(0)}{\partial r_i} = f_1(x) + f_2(x) + f_3(x) + f_4(x) - 4f_0(x) = 0,$$

$$\frac{d^2p(0_+)}{dr^2} = \sum_{i=0}^{4} \left(\frac{dr_i}{dr}\right)^2 \frac{\partial^2 p(0)}{\partial r_i^2} + 2 \sum_{0 \leq i < j \leq 4} \frac{dr_i}{dr}\frac{dr_j}{dr}\frac{\partial^2 p(0)}{\partial r_i \partial r_j}$$

$$= \sum_{i=0}^{4} \left(\frac{dr_i}{dr}\right)^2 \frac{\partial f_i(x)}{\partial x} f_i(x) + 2 \sum_{0 \leq i < j \leq 4} \frac{dr_i}{dr}\frac{dr_j}{dr}\frac{\partial f_j(x)}{\partial x} f_i(x)$$

$$= [f_2, f_3](x).$$

Since $dp(0_+)/dr = 0$, the curve $q(r)$ defined for small nonnegative r by $q(r^2/2) = p(r)$ is C^1 and $dq(0_+)/dr = d^2p(0_+)/dr^2$. We can pull this control modification along to $x^1 = \gamma_0(s_1)x^0$ as before and obtain

$$(6)\quad \gamma_0(s_1 - s)_*[f_2, f_3](x) = \sum_{m=0}^{h} \frac{(s - s_1)^m}{m!} ad^m(f_0)[f_2, f_3](x^1) + \mathcal{O}(s - s_1)^{h+1}.$$

Notice that if we reverse u^2 and u^3 in defining $p(r)$, we obtain $\gamma_0(s_1 - s)_*[f_3, f_2](x) = -\gamma_0(s_1 - s)_*[f_2, f_3](x)$.

The last type of control modification which we consider is to stop short of x^1 or continue on past x^1. These lead to curves $q(r) = \gamma_0(x_1 \pm r)x^0 = \gamma_0(\pm r)x^1$, whose derivatives are

$$(7)\qquad \frac{dq(0_+)}{dr} = \pm f_0(x^1).$$

Let K_{x^1} be the convex cone in $L^+_{x^1}$ generated by the vectors of the form (4), (6) and (7), for all $0 < s \leqq s_1$ and admissible controls $u^i(t)$, $i = 1, 2, 3$. We say the trajectory of u^0 between x^0 and x^1 is *singular* if K_{x^1} is a proper subset of $L^+_{x^1}$. This definition is different from the usual one stated in terms of the maximal principle (see Gabasov and Kirillova [6] and Hermes [20]). Since $t = x_1$, the usual one is equivalent to the following: the trajectory is singular if the cone generated by the vectors of the form (4) and (7) is a proper subset of M_{x^1}. There are of course less singular controls under our definition. It can be shown, using the standard methods (implicit function or fixed point theorem), that if $K_{x^1} = L^+_{x^1}$, then $x^2 \in L^+$-interior $\mathscr{A}(x^0, U, \Omega)$ and so is bang-bang accessible. It follows then that $\mathscr{A}(x^0, U, E)$ will equal $\mathscr{A}(x^0, U, \Omega)$ if the only singular trajectories are bang-bang. Consider the following examples.

Example 4. Let $M = \mathbb{R}^3$ and $\dot{x} = ua_1 + (1 - u)a_2$, $0 \leqq u \leqq 1$, where

$$a_1(x) = \begin{pmatrix} 1 \\ 1 \\ x_2 \end{pmatrix} \quad \text{and} \quad a_2(x) = \begin{pmatrix} 1 \\ -1 \\ -x_2 \end{pmatrix}.$$

H is the $C(M)$ span of a_1 and a_2 and since $[a_1, a_2] = 0$, $DH = H$. The integral manifold of H through $x^0 = (0, 0, 0)$ is $L = \{(x_1, x_2, x_3): x_3 = (x_2)^2\}$. Let $u^0(t) = 1/2$ and $x^1 = \gamma_0(s_1)x^0 = (s_1, 0, 0)$. The cone K_{x^1}, generated by $\pm(1/2)(a_1(x^1) + a_2(x^1))$, $(a_1(x^1) - a_0(x^1))$, and $(a_2(x^1) - a_0(x^1)) = (a_0(x^1) - a_1(x^1))$ equals L_{x^1} so the trajectory is not singular in our sense. However, K_{x^1} is a proper subset of M_{x^1} and so the trajectory is singular in the usual sense. Notice that x^1 is bang-bang accessible, $x^1 = \gamma_2(s_1/2)\gamma_1(s_1/2)x^0$ or any other bang-bang trajectory that uses a_1 and a_2 each a total of $s_1/2$ units of time.

Example 5. Let $M = \mathbb{R}^4$ and $\dot{x} = u_1a_1(x) + u_2a_2(x) + u_3a_3(x)$, $u_i \geqq 0$, $\sum u_i = 1$, where

$$a_1(x) = \begin{pmatrix} 1 \\ -1/2 \\ -1/2 \\ -x_2/2 \end{pmatrix}, \quad a_2(x) = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad a_3(x) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ x_2 \end{pmatrix},$$

$$[a_1, a_2](x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/2 \end{pmatrix}, \quad [a_1, a_3](x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1/2 \end{pmatrix}, \quad [a_2, a_3](x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

and all other brackets are zero.

DH is of dimension 4 everywhere so the integral manifold of DH through $x^0 = (0, 0, 0, 0)$ is exactly M. The control $u^0 = (1/2, 1/4, 1/4)$ gives rise to the vector field

$$f_0(x) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and $[f_0, a_i] = 0$. If $x^1 = \gamma_0(s_1)x^0 = (s_1, 0, 0, 0)$, then the cone generated by control variations of type (4) and (7) is a linear space of dimension 3, since the trajectory is singular in the usual sense. If we add the variations of type (6), we see $K_{x^1} = L_{x^1} = M_{x^1}$ and so is not singular in our sense. Notice that x^1 is bang-bang accessible,

$$x^1 = \gamma_3(s_1/8)\gamma_2(s_1/8)\gamma_1(s_1/2)\gamma_2(s_1/8)\gamma_3(s_1/8)x^0.$$

A *subsystem* of (3) is a system obtained by restricting the control $u(t)$ to lie on one of the faces of Ω, that is, if I is a subset of $\{1, \cdots, k\}$ the subsystem specified by I is given by requiring $u_i(t) = 0$ for $i \notin I$. We consider Ω a face of Ω, so that (3) is a subsystem of itself.

THEOREM 3. *Suppose for every subset I of $\{1, \cdots, k\}$, the subsystem specified by I is locally semi-integrable. Let U be a neighborhood of x^0 and H^+ and L^+ be the submodule and integral manifold which carry the subsystem specified by I on U^+. If there exists $h > 0$ such that*

(i) *$D^h H_x^+ = DH_x^+$ for all $x \in L^+$,*

(ii) *given any $i_j \in I, j = 1, \cdots, 4$, and any m, $1 \leqq m < h$, there exists a function $\mu(x) \geqq 0$ such that for all $x \in L^+$ either*

$$ad^m(a_{i_3})[a_{i_1}, a_{i_2}](x) \equiv \mu(x) ad^m(a_{i_4})[a_{i_1}, a_{i_2}](x) \bmod D^m H_x^+$$

or

$$ad^m(a_{i_3})[a_{i_1}, a_{i_2}](x) \equiv \mu(x) ad^m(a_{i_4})[a_{i_1}, a_{i_2}](x) \bmod D^m H_x^+,$$

then $\mathscr{A}(x^0, U, E) = \mathscr{A}(x^0, U, \Omega)$.

Proof. Let $I = \{1, \cdots, k\}$ and $u^1(t), \cdots, u^m(t)$, $v^1(t), \cdots, v^m(t)$ be controls lying in the interior of Ω, that is, $0 < u_i^j(t) < 1$ and $0 < v_i^j(t) < 1$ for $j = 1, \cdots, m$, $i = 1, \cdots, k$. Let $f_j(x) = \sum_{i=1}^k u_i^j(x_1)a_i(x)$ and $g_j(x) = \sum_{i=1}^k v_i^j(x_1)a_i(x)$. By induction on $m < h$, we show there exists a $\lambda(x) > 0$ such that

$$[f_1 \cdots [f_m[a_{i_1}, a_{i_2}]] \cdots](x) \equiv \lambda(x)[g_1 \cdots [g_m[a_{i_1}, a_{i_2}]] \cdots](x) \bmod D^m H_x^+.$$

It is trivial for $m = 0$ and it follows immediately from (ii) for $1 \leqq m < h$.

Therefore if $u^0(t)$ lies in the interior of Ω in some neighborhood of $x^1 = \gamma_0(s_1)x^0$, then $D^hH^+_{x^1}$ is spanned by the vectors

$$a_i(x^1), [a_i, a_j](x^1), ad^1(f_0)[a_i, a_j](x^1), \cdots, ad^{h-1}(f_0)[a_i, a_j](x^1),$$

for $1 \leqq i, j \leqq k$. But the cone K_{x^1} contains $\pm f_0(x^1), a_i(x^1) - f_0(x^1)$, and

$$\sum_{m=0}^{h-1} \frac{(s - s_1)^m}{m!} ad^m(f_0)[a_i, a_j](x^1) + \mathcal{O}(s - s_1)^h$$

for all $1 \leqq i, j \leqq k$ and small $s - s_1 \leqq 0$. Hence, K_{x^1} equals $D^hH^+_{x^1} = DH^+_{x^1}$. This implies $x^1 \in \mathscr{A}(x^0, U, E)$.

If $u^0(t)$ is not interior to Ω at x^1 but is interior to some face of Ω of dimension $\geqq 1$, then we repeat the above argument for the subsystem generated by that face. The controls that lie on faces of dimension 0 are bang-bang controls. Q.E.D.

There is a bang-bang controllability version of Theorem 3 that assumes the same hypothesis except H^- and L^- replace H^+ and L^+ in (i) and (ii). Together they yield a global result.

COROLLARY 4. *Suppose for all $x \in M$, there exists a neighborhood U of x such that $\mathscr{A}(x, U, E) = \mathscr{A}(x, U, \Omega)$ and $\mathscr{C}(x, U, E) = \mathscr{C}(x, U, \Omega)$. Then $\mathscr{A}(x^0, M, E) = \mathscr{A}(x^0, M, \Omega)$ and $\mathscr{C}(x^0, M, E) = \mathscr{C}(x^0, M, \Omega)$.*

Proof. Suppose $u^0(t)$ is an admissible control. We must show $\gamma_0(s)x^0 \in \mathscr{A}(x^0, M, E)$ for all $s \geqq 0$. Let $s = \inf\{r \geqq 0 : \gamma_0(s)x^0 \notin \mathscr{A}(x^0, M, E)\}$ and $x = \gamma_0(s)x^0$. If $x \in \mathscr{A}(x^0, M, E)$, then by hypothesis there exists an $\varepsilon > 0$ such that for all $r \in [0, \varepsilon)$, $\gamma_0(r)x = \gamma_0(r + s)x^0 \in \mathscr{A}(x, M, E) \subseteq \mathscr{A}(x^0, M, E)$. This contradicts the definition of s. If $x \notin \mathscr{A}(x^0, M, E)$, then there exists $\varepsilon > 0$ such that for all $r \in (-\varepsilon, 0]$, $\gamma_0(r)x = \gamma_0(r + s)x^0 \in \mathscr{C}(x, M, E)$. By the definition of s, for small r, $\gamma_0(r + s)x^0 \in \mathscr{A}(x^0, M, E)$ so $x \in \mathscr{A}(x^0, M, E)$. This is a contradiction.

COROLLARY 5 (Halkin–Levinson). *Consider the linear control system defined on $M = \mathbb{R}^n$ by*

$$\dot{x} = F(t)x + G(t)v + h(t), \tag{8}$$

where $F(t)$, $G(t)$ are matrices, $h(t)$ is a vector of piecewise analytic functions and the control $v(t) = (v_1(t), \cdots, v_n(t))$ is a piecewise analytic function satisfying $|v_i(t)| \leqq 1$. If x is accessible from x^0 by an admissible control, then x is accessible from x^0 by a piecewise analytic bang-bang control $v(t)$, where $|v_i(t)| = 1$.

Proof. Let $a_1(x), \cdots, a_k(x)$ be the right side of (8) for the finite number of constant controls satisfying $|v_i(t)| = 1$. Then (8) can be put in the form (3) and each of the $a_i(x)$ is piecewise analytic. It follows that every subsystem of (3) is locally semi-integrable. By direct computation it is easy to show

$$ad^m(a_{i_3})[a_{i_1}, a_{i_2}](x) = ad^m(a_{i_4})[a_{i_1}, a_{i_2}](x) \quad \text{for all } m = 1, 2, 3, \cdots,$$

and for any x^0 there always exists a neighborhood, U, of x^0 and $h > 0$ such that $D^hH^+_x = DH^+_x$ for all $x \in U^+$ and $D^hH^-_{x^0}$ for all $x \in U^-$, so the result follows from Theorem 3 and Corollary 4. Q.E.D.

Notice that Examples 4 and 5 satisfy the conditions of Theorem 3. As a counterexample, consider this one taken from Filippov [5] as modified by Lobry [14].

Example 6. Let $M = \mathbb{R}^3$, $\dot{x} = ua_1 + (1 - u)a_2$ and $0 \leqq u \leqq 1$, where

$$a_1(x) = \begin{pmatrix} 1 \\ 1 - x_3^2 \\ 1 \end{pmatrix}, \qquad a_2(x) = \begin{pmatrix} 1 \\ 1 - x_3^2 \\ -1 \end{pmatrix},$$

$$[a_1, a_2](x) = \begin{pmatrix} 0 \\ -4x_3 \\ 0 \end{pmatrix}, \qquad [a_1[a_1, a_2]](x) = [a_2[a_2, a_1]](x) = \begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix}.$$

Condition (ii) of Theorem 3 fails and the point $(1, 1, 0)$ is accessible from $(0, 0, 0)$ by the singular control $u = 1/2$, but is not bang-bang accessible (see Filippov [5]).

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