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LINEARIZATION AND BILINEARIZATION OF CONTROL SYSTEMS

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ABSTRACT

The standard autonomous and time dependent control systems are first order approximations to what are generally nonlinear phenomena in neighborhoods of a point and reference trajectory respectively. This helps to explain mathematically the wide applicability of these models. In this paper we show that higher order approximations around a point and a reference trajectory naturally result in autonomous and time dependent bilinear control systems respectively. This augurs well for the theoretical and applied usefulness of these models which are now being extensively studied.

1. LINEARIZATION AROUND A POINT

The time evolution of many engineering processes can be effectively modelled by an autonomous control system of nonlinear type (1.1),

$$\begin{aligned} \dot{x} &= f(x,u), \\ y &= g(x,u), \\ x(0) = x^{o} \quad \text{and} \quad & |u_i - u_i^{o}| \leq c_i \ . \end{aligned} \tag{1.1}$$

Here the n vector x represents the state of the system, the h vector u the input or control, the k vector y the output, and x^{o} and u^{o} are initial values of the state and control.

In general the functions f and g, which are assumed to be as smooth as needed, may be extremely complex in their global dependence on x and u but the actual system may operate in a relatively narrow range of x and u values around x^{o} and u^{o}. For convenience we change x and u coordinates so that $x^{o} = 0$, $u^{o} = 0$ and $c_i = 1$ then (1.1) can be approximated by

$$\begin{aligned} \dot{x} &= f(0,0) + \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial u}(0,0)u \ , \\ y &= g(0,0) + \frac{\partial g}{\partial x}(0,0)x + \frac{\partial g}{\partial u}(0,0)u \ , \\ x(0) &= 0 \quad \text{and} \quad |u| \leq 1 \end{aligned} \tag{1.2}$$

For any admissible input u(t) let x(t), y(t) and $\overline{x}(t)$, $\overline{y}(t)$ denote the solution of (1.1) and (1.2) respectively. In general for small t we expect $|x(t)|$ and $|\overline{x}(t)|$ to grow like t and therefore, ignoring errors due to the truncation of terms in u, we expect that the right sides of (1.1) and (1.2) will differ by a term of order t^2 from which it follows that $|x(t)-\overline{x}(t)|$ will grow like t^3 and $|y(t)-\overline{y}(t)|$ like t^2.

If we introduce a new coordinate x_o which is identically one then (1.2) can be transformed into a linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \ , \\ y &= Cx + Du \ , \\ x(0) = (1,0,\ldots,0), & \quad |u| \leq 1 \ , \end{aligned} \tag{1.3}$$

where

$$A = \left[\begin{array}{c|c} 0 & 0 \quad \cdots \quad 0 \\ f(0,0) & \frac{\partial f}{\partial x}(0,0) \end{array}\right] \qquad B = \left[\begin{array}{c} 0 \quad \cdots \quad 0 \\ \frac{\partial f}{\partial u}(0,0) \end{array}\right]$$

$$C = \left[\begin{array}{c|c} g(0,0) & \frac{\partial g}{\partial x}(0,0) \end{array}\right] \qquad D = \left[\frac{\partial g}{\partial u}(0,0)\right]$$

We have an impressive body of theory regarding such systems and their practical usefulness has been proven in a wide variety of applications. This seems to contradict our rough error analysis made above since it led us to expect errors that grow rather fast, like t^3. However that argument was predicated on the assumption that $|x(t)|$ is growing like t. In many real situations the processes being modeled are inherently stable so that (1.2) is asymptotically stable and any errors caused by the approximation decay rather than grow.

2. BILINEARIZATION AROUND A POINT

In order to generate approximations to (1.1) of higher accuracy we expand the right side in a series containing higher terms. We restrict to systems where the control u enters linearly and is a scalar. The generalization to vector controls is straightforward but if the dependence on u is nonlinear other techniques must be employed. With these restrictions (1.1) becomes

$$\begin{aligned} \dot{x} &= f(x,u) = f_o(x) + uf_1(x) \quad , \\ y &= g(x,u) = g_o(x) + ug_1(x) \quad , \\ x(0) &= 0 \quad \text{and} \quad |u| \leq 1 \quad . \end{aligned} \tag{2.1}$$

If we truncate after the second order terms we obtain

$$\begin{aligned} \dot{x} &= f_o(0) + \frac{\partial f_o(0)}{\partial x} x + \frac{1}{2} x^T \frac{\partial^2 f_o(0)}{\partial x^2} x + uf_1(0) + u \frac{\partial f_1(0)}{\partial x} x + ux^T \frac{\partial^2 f_1(0)}{\partial x^2} x \\ y &= g_o(0) + \frac{\partial g_o(0)}{\partial x} x + \frac{1}{2} x^T \frac{\partial^2 g_o(0)}{\partial x^2} x + ug_1(0) + u \frac{\partial g_1(0)}{\partial x} x + ux^T \frac{\partial g_1(0)}{\partial x} x \\ x(0) &= 0 \quad \text{and} \quad |u| \leq 1 \end{aligned} \tag{2.2}$$

These equations cannot be linearized, however by introducing several new state variables in a technique due to Carleman ([1] and [2]) we can put (2.2) into bilinear form

$$\begin{aligned} \dot{x} &= Ax + uBx \quad , \\ y &= Cx + uDx \quad , \\ x(0) &= (1,0,\ldots,0) \quad , \quad |u| \leq 1 \quad . \end{aligned}$$

To show this we proceed as follows, let $x^{[p]}$ denote the vectors whose components consists of all linearly independent monomials of degree p in the variables $x_1,\ldots,x_n$, ordered lexographically. (A similar approach is found in Brockett, [3] and [4].) For example

$$x^{[2]} = (x_1^2, x_1x_2, \ldots, x_1x_n, x_2^2, x_2x_3, \ldots, x_n^2) \quad \text{and}$$

$$x^{[3]} = (x_1^3, x_1^2x_2, \ldots, x_1^2x_n, x_1x_2^2, \ldots, x_n^3) \quad .$$

Let $x^{[0]}$ be identically the scalar 1. Then $\dot{x}^{[0]} = 0$ and from (2.2)

$$\dot{x}^{[1]} = A_o^1x^{[0]} + A_1^1x^{[1]} + A_2^1x^{[2]} + uB_o^1x^{[0]} + uB_1^1x^{[1]} + uB_2^1x^{[2]} \tag{2.4}$$

where A_i^1 and B_i^1 are matrices of the appropriate dimensions easily computable from $\frac{\partial^i f_o(0)}{\partial x^i}$ and $\frac{\partial^i f_1(0)}{\partial x^i}$ respectively.

To develop a differential equation for $x^{[2]}$ consider a particular component x_ix_j. By the product rule, $d/dt(x_ix_j) = \dot{x}_ix_j + x_i\dot{x}_j$ and since $\dot{x}_i$ and $\dot{x}_j$ depend bilinearly on $x^{[0]}$, $x^{[1]}$, $x^{[2]}$ and u it follows that $d/dt(x_ix_j)$ depends bilinearly on $x^{[1]}$, $x^{[2]}$, $x^{[3]}$ and u,

$$\dot{x}^{[2]} = A_1^2x^{[1]} + A_2^2x^{[2]} + A_3^2x^{[3]} + uB_1^2x^{[1]} + uB_2^2x^{[2]} + uB_3^2x^{[3]} \quad . \tag{2.5}$$

The matrices A_i^2 and B_i^2 are computable from A_i^1 and B_i^1. In general $x^{[p]}$ satisfies a bilinear differential equation

$$\dot{x}^{[p]} = A_{p-1}^p x^{[p-1]} + A_p^p x^{[p]} + A_{p+1}^p x^{[p+1]} + uB_{p-1}^p x^{[p-1]} + uB_p^p x^{[p]} + uB_{p+1}^p x^{[p+1]} . \tag{2.6}$$

Truncating at terms of order 2 in the transition from (2.1) to (2.2) is equivalent to setting $x^{[p]} = 0$ for $p \geq 3$ so this is assumed in (2.5) and (2.6).

In a similar fashion the output map is expanded

$$y = C_o x^{[0]} + C_1x^{[1]} + C_2x^{[2]} + uB_1x^{[1]} + uB_2x^{[2]} + uB_3x^{[3]} \quad , \tag{2.7}$$

and the result is (2.3), a finite dimensional bilinear system.

In general the nonlinear system (2.1) gives rise to an infinite dimensional bilinear system

$$\dot{x}^{[p]} = \sum_{i=p-1}^{\infty} A_i^p x^{[i]} + uB_i^p x^{[i]} \quad ,$$

$$y = \sum_{i=0}^{\infty} C_i x^{[i]} + uD_i x^{[i]} \quad , \tag{2.8}$$

$$x(0) = (1,0,0,\ldots) \quad \text{and} \quad |u| \leq 1 \quad ,$$

of the type studied by Brockett and Fuhrmann [5]. If (2.8) is truncated by setting $x^{[p]} = 0$ for $p \geq q$ the result is a finite dimensional bilinear system

$$\dot{x}^{[p]} = \sum_{i=p-1}^{q-1} A_i^p x^{[i]} + uB_i^p x^{[i]} \quad ,$$
$$y = \sum_{i=0}^{q-1} C_i x^{[i]} + uD_i x^{[i]} \tag{2.9}$$

where $x = x^{[0]}, x^{[1]}, \ldots, x^{[q-1]}$,

$$x(0) = (1,0,\ldots,0) \quad \text{and} \quad |u| \leq 1 \quad ,$$

which is used in the proof of the following.

Theorem 1 (Bilinearization around a point): Consider the nonlinear control system

$$\dot{x} = f_o(x) + uf_1(x) \quad ,$$

$$y = g_o(x) + ug_1(x) \quad ,$$

$$x(0) = 0 \quad \text{and} \quad |u| \leq 1 \quad ,$$

where $x = (x_1,\ldots,x_n)$, $y = (y_1,\ldots,y_k)$ and u is a scalar. For any integer $q \geq 0$ there exists a bilinear control system

$$\dot{x} = Ax + uBx$$
$$y = Cx + uDx$$
$$x(0) = (1,0,\ldots,0) \quad \text{and} \quad |u| \leq 1$$

where $x = (x_0,\ldots,x_m)$ $(m \geq n)$, $y = (y_1,\ldots,y_k)$ and u a scalar such that for some constants M, $T > 0$ and for any admissible control $u(t)$ the outputs $y(t)$ and $\overline{y}(t)$ of the nonlinear and bilinear system satisfy

$$|y(t) - \overline{y}(t)| \leq Mt^q \quad , \quad \forall\, t \in [0,T] \quad .$$

Moreover if $x(t)$ is the state of the nonlinear system and $\overline{x}(t)$ consists of x_1 to x_n of the state of the bilinear system then

$$|x(t) - \overline{x}(t)| \leq Mt^{q+1} \quad , \quad \forall\, t \in [0,T] \quad .$$

The proof is based on the following.

Lemma: Consider the two control systems

$$\text{(i)} \quad \dot{z} = h_1(z,u) \quad \text{and} \quad \text{(ii)} \quad \dot{z} = h_2(z,u)$$

where $\quad |u| \leq 1, \quad z = (z_o,\ldots,z_m) \quad$ and $\quad z(0) = (1,0,\ldots,0)$.

Suppose there exists constants K_1 and $T_1 > 0$ and an integer $q \geq 0$ such that for any solution $z(t)$ of (i) where $0 \leq t \leq T_1$,

$$|h_1(z(t),u(t)) - h_2(z(t),u(t))| \leq K_1 t^q \quad .$$

Furthermore suppose there exists constants K_2, $\varepsilon > 0$ such that for any

u, z, $\bar{z}$ where $|u| \leqslant 1$, $|z| \leqslant \varepsilon$, $|\bar{z}| \leqslant \varepsilon$

$$|h_2(z,u)-h_2(\bar{z},u)| \leqslant K_2|z-\bar{z}| .$$

Then there exist M, $T > 0$ such that if $z(t)$ and $\bar{z}(t)$ are the solution of (i) and (ii) for the same $u(t)$ then

$$|z(t)-\bar{z}(t)| \leqslant Mt^{q+1} , \quad \forall \quad 0 \leqslant t \leqslant T$$

Proof of Lemma: Choose $0 < T \leqslant T_1$ such that any solution $z(t)$ and $\bar{z}(t)$ of (i) and (ii) satisfy $|x(t)| \leqslant \varepsilon$, $|\bar{x}(t)| \leqslant \varepsilon$ for $0 \leqslant t \leqslant T$.

$$|z(t)-\bar{z}(t)| \leqslant \int_0^t |h_1(z(s),u(s))-h_2(\bar{z}(s),u(s))|ds$$

$$\leqslant \int_0^t |h_1(z(s),u(s))-h_2(z(s),u(s))|ds$$

$$+ \int_0^t |h_2(z(s),u(s))-h_2(\bar{z}(s),u(s))|ds$$

$$|z(t)-\bar{z}(t)| \leqslant K_1 t^{q+1} + K_2 \int_0^t |z(s)-\bar{z}(s)|ds$$

Choose $M_1 \geqslant K_2$ such that $M_1^{q+2} \geqslant (q+1)!K_1$. Let

$$\phi(t) = \int_0^t |z(s)-\bar{z}(s)|ds$$

and

$$\psi(t) = \exp(Mt)-\sum_{j=0}^{q+1} \frac{(M_1 t)^j}{j!} = \sum_{j=q+2}^{\infty} \frac{(M_1 t)^j}{j!}$$

Then $\phi(t)$ and $\psi(t)$ satisfy

$$\phi'(t) \leqslant \frac{M_1^{q+2}}{(q+1)!} t^{q+1} + M_1\phi(t)$$

$$\psi'(t) = \frac{M_1^{q+2}}{(q+1)!} t^{q+1} + M_1\psi(t)$$

so by a standard comparison theorem [6, p. 25] $\phi(t) \leqslant \psi(t)$ therefore

$$|z(t)-z(t)| = \phi'(t) \leqslant \frac{M_1^{q+2}}{(q+1)!} t^{q+1} + M_1\psi(t)$$

$$\leqslant M_t^{q+1} \quad \text{for some } M.$$

Proof of Theorem: Let $x = (x_1,\dots,x_n)$ and $x^{[p]}$ be defined as before. If $\dot{x} = f(x,u)$ then for any p there exists a function $h_1^p(x,u)$ such that $\dot{x}^{[p]} = h_1^p(x,u)$. Let $(z_o,\dots,z_m) = (x^{[0]},\dots,x^{[q-1]})$ then

$$\dot{z} = h_1(z,u) \quad .$$

Let $h_2(z,u)$ be the finite dimensional bilinear system obtained in (2.9).

It is straightforward to verify that these systems satisfy the hypotheses of the lemma. The theorem follows immediately.

3. LINEARIZATION AND BILINEARIZATION AROUND A REFERENCE TRAJECTORY

Consider the nonlinear control system

$$\dot{x} = f_o(t,x) + uf_1(t,x)$$
$$y = g(t,x) \tag{3.1}$$

where x is an n vector, y a k vector and u a scalar. The case of vector controls is a straightforward generalization. Time appears explicitly in (3.1) because we intend to consider perturbations around a reference trajectory which we conveniently take to be $x(t) \equiv 0$ generated by the control $u(t) \equiv 0$ for $0 \leq t \leq T$. expanding the right side of (3.1) as before we obtain a time dependent linear system,

$$\dot{x} = A(t)x + uB(t)$$
$$y = C(t)x \tag{3.2}$$

where $A(t) = \partial f_o/\partial x\ (t,0)$, $B(t) = \partial f_1/\partial u(t,0)$ and $C(t) = \partial g/\partial x\ (t,0)$. It is well-known (for example Halkin [6]) that the error of approximating (3.1) by (3.2) is proportioned to the L' norm of the control. That is there exists an M such that if x(t) and y(t) are the solutions of (3.1) for some u(t) and $\overline{x}(t)$ and $\overline{y}(t)$ are the corresponding solutions of (3.2) u(t) then

$$|x(t)-\overline{x}(t)| \leq M\varepsilon^2$$

and

$$|y(t)-\overline{y}(t)| \leq M\varepsilon^2$$

where

$$\varepsilon = \int_0^T |u(t)|dt$$

is sufficiently small.

For second order approximations consider

$$\dot{x} = \frac{\partial f_o}{\partial x}(t,0)x + x^T \frac{\partial^2 f_o}{\partial x^2}(t,0)x + uf(t,0) + u\frac{\partial f}{\partial x}(t,0)x,$$
$$y = \frac{\partial g}{\partial x}(t,0) + x^T \frac{\partial^2 g}{\partial x^2}(t,0)x \quad . \tag{3.3}$$

Using Carleman's technique again we obtain a time varying bilinear system

$$x^{[0]} \equiv 1, \qquad x^{[p]} \equiv 0 \qquad \text{for } p \geq 3,$$
$$\dot{x}^{[1]} = A_1^1(t)x^{[1]} + A_2^1(t)x^{[2]} + uB_o^1(t)x^{[0]} + uB_1^1(t)x^{[1]} \ ,$$
$$\dot{x}^{[2]} = A_2^2(t)x^{[2]} + uB_1^2(t)x^{[1]} \ ,$$
$$y = C_1(t)x^{[1]} + C_2(t)x^{[2]}$$

Notice that as in (3.2) terms involving u can be truncated sooner than those with x only without reducing the accuracy of the approximation.

In general one obtains an infinite dimensional time varying bilinear system

$$\dot{x}^{[p]} = \sum_{i=p}^{\infty} A_i^p(t)x^{[i]} + uB_{i-1}^p(t)x^{[i-1]}$$

$$y = \sum_{i=1}^{\infty} C_i(t)x^{[i]} \tag{3.5}$$

which when truncated yields a finite dimensional time varying bilinear system

$$\dot{x}^{[p]} = \sum_{i=p}^{q-1} A_i^p(t)x^{[i]} + uB_{i-1}^p(t)x^{[i-1]}$$

$$y = \sum_{i=1}^{q-1} C_i(t)x^{[i]} \quad . \tag{3.6}$$

<u>Theorem</u> (Bilinearization around a reference trajectory): Consider the nonlinear control system

$$\dot{x} = f_o(t,x) + uf_1(t,x)$$

$$y = g(t,x)$$

where $x = (x_1,\ldots,x_n)$, $y = (y_1,\ldots,y_k)$, u a scalar, and the reference trajectory $x(t) \equiv 0$ is generated by $u(t) \equiv 0$ for $0 \leqslant t \leqslant T$. For any $q \geqslant 0$ there exists a bilinear control system

$$\dot{x} = A(t)x + uB(t)x$$

$$y = C(t)x$$

where $x = (x_o,\ldots,x_m)$ $(m \geqslant n)$, $y = (y_1,\ldots,y_k)$ and u a scalar such that for some M any control u(t) the outputs y(t) and $\overline{y}(t)$ of the nonlinear and bilinear system satisfy

$$|y(t)-\overline{y}(t)| \leqslant M\varepsilon^q \ , \qquad \text{for } 0 \leqslant t \leqslant T \ ,$$

provided

$$\varepsilon = \int_0^T |u(t)|dt$$

is sufficiently small. Moreover if x(t) is the state of the nonlinear system and x(t) is coordinater x_1 through x_m of the state of the bilinear system then

$$|x(t)-\overline{x}(t)| \leqslant M\varepsilon^q \ , \qquad \text{for } 0 \leqslant t \leqslant T \ .$$

Once again the proof follows from a lemma.

<u>Lemma</u>: Consider the two systems

(i) $\dot{z} = h_1(t,z,u)$ (ii) $\dot{z} = h_2(t,z,u)$

where $z = (z_0,\ldots,z_m)$, u is a scalar and the reference trajectory $z(t) \equiv (1,0,\ldots,0)$ is generated by $u(t) \equiv 0$, for $0 \leqslant t \leqslant T$, for both systems. Suppose there exists an integer $q \geqslant 0$ and a constant K_1 such that for any solution z(t) of (i)

$$|h_1(t,z(t),u(t))-h_2(t,z(t),u(t))| \leqslant K_1(|z(t)|^q+|z(t)|^{q-1}|u(t)|)$$

Suppose there exists $\delta > 0$ and a constant K_2 such that for any u, $|z| \leq \delta$, $|\bar{z}| \leq \delta$, $0 \leq t \leq T$

$$|h_2(t,z,u)-h_2(t,\bar{z},u)| \leq K_2|z-\bar{z}|$$

Finally suppose there exists a constant K_3 such that for any u $|z| \leq \delta$ and $0 \leq t \leq T$

$$|h_1(s,z,u)| \leq K_3|u| \quad \text{and} \quad |h_2(s,z,u)| \leq K_3|u| \quad .$$

If z(t) and $\bar{z}(t)$ are the solution of (i) and (ii) for some u(t) then

$$|z(t)-\bar{z}(t)| \leq M\varepsilon^q \; , \qquad 0 \leq t \leq T$$

if $\varepsilon = \int_0^T |u(t)|dt$ is sufficiently small.

Proof of Lemma: If z(t) and $\bar{z}(t)$ arethe solutions of (i) and (ii) for some u(t) then

$$|z(t)| \leq \int_0^t |h_1(s,z(s),u(s))|ds \leq K_3 \int_0^t |u(s)|ds \leq K_3\varepsilon$$

$$|z(t)| \leq \int_0^t |h_2(s,\bar{z}(s),u(s))|ds \leq K_3\int_0^t |u(s)|ds \leq K_3\varepsilon$$

Restricted to ε small enough so that $|z(t)| \leq \delta$ and $|\bar{z}(t)| \leq \delta$, for $0 \leq t \leq T$,

$$\begin{aligned}|z(t)-\bar{z}(t)| &\leq \int_0^t |h_1(s,z(s),u(s))-h_2(s,z(s),u(s))|ds \\ &+ \int_0^t |h_2(s,z(s),u(s))-h_2(s,\bar{z}(s),u(s))|ds \\ &\leq K_1 \int_0^t |z(s)|^q+|z(s)|^{q-1}|u(s)|ds+K_2 \int_0^t |z(s)-\bar{z}(s)|ds\end{aligned}$$

$$|z(t)-\bar{z}(t)| \leq K_1(T+1)\varepsilon^q+K_2\int_0^t |z(s)-\bar{z}(s)|ds \quad .$$

Let

$$\phi(t) = \int_0^t |z(s)-\bar{z}(s)|ds$$

and

$$\psi(t) = \frac{K_1(T+1)\varepsilon^q}{K_2} \exp(K_2 t) - \frac{K_1(T+1)\varepsilon^q}{K_2}$$

Then $\phi(t)$ and $\psi(t)$ satisfy

$$\phi'(t) \leq K_1(T+1)\varepsilon^q + K_2\phi(t)$$

$$\psi'(t) = K_1(T+1)\varepsilon^q + K_2\psi(t)$$

so once again

$$|z(t)-\bar{z}(t)| = \phi'(t) \leq \psi'(t) \leq M\varepsilon^q$$

for some M and all t, $0 \leq t \leq T$.

Proof of Theorem: Let $x = (x_1,\ldots,x_n)$ and $x^{[p]}$ be defined as before. If $\dot{x} = f_o(t,x)+uf_1(t,x)$ then for any p let $h_1^p(x,u)$ be the function such that $\dot{x}^{[p]} = h_1^p(x,u)$. Let $(z_o,\ldots,z_m) = (x^{[0]},\ldots,x^{[q-1]})$ then $\dot{z} = h_1(z,u)$.

Let $h_2(z,u)$ denote the right side of finite dimensional time varying bilinear system (3.6). It is straightfoward to verify that h_1 and h_2 satisfy the hypothesis of the lemma.

4. CONCLUSION

These two theorems indicate that in a local sense around a point or reference trajectory the class of bilinear systems is dense in the class of nonlinear systems with control entering linearly. One views this with somewhat mixed feelings, on the one hand it implies that nonlinear behavior can be effectively approximated by bilinear models. On the other hand it implies that complexity of bilinear systems is almost as great as that of nonlinear systems and that theoretical results for the whole class of bilinear systems will be almost as difficult as for the class of nonlinear systems.

The above methods of bilinearizing around a point and reference trajectory are not the only ways of doing this. One advantage of these methods is that the nonlinear approximation to the state of the nonlinear system is easily read from the state of bilinear systems. However the bilinear system will generally not be of smallest state dimension among those bilinear systems which approximate the nonlinear system with a desired order of accuracy. For an alternate approach to bilinearizing around a point we refer the reader to [7].

Besides more accurate quantitative information that higher order bilinearization afford they also increase our qualitative knowledge of the system. For example, by linearizing around a reference trajectory one is able to deduce only first order necessary conditions for optimality, i.e. the Pontryagin maximal principle. By using essentially the quadratic bilinearization Kelley, Kopp and Mayer [8] were able to develop new necessary conditions, not implied by the Maximal Principle. From this we see that higher order bilinearizations yield new qualitative information about the boundary of the set of accessible points of a control system.

The error analysis of the above theorems did not assume any stability properties for the nonlinear systems being approximated. In many applications these systems do have some form of stability. Further research is needed to discover how these improve the approximations.

The above bilinearizations were accomplished using monomials in the state variables to approximate the nonlinear behavior of the system. It is possible (see [1] and [2]) to use other families of functions and the question arises when would it be advantageous to do so.

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