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A DECOMPOSITION THEORY FOR DIFFERENTIABLE SYSTEMS*

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Abstract. A theory analogous to the Krohn–Rhodes theory of finite automata is developed for systems described by a finite dimensional ordinary differential equation. It is shown that every such system with a finite dimensional Lie algebra can be decomposed into the cascade of systems with simple or one dimensional algebras. Moreover, in some sense these systems admit no further decomposition. No knowledge of Krohn–Rhodes theory is assumed of the reader.

Introduction. The Krohn–Rhodes theory [1] of finite automata is a very elegant way of describing how a machine can be decomposed as the cascade of two or more simpler machines. Moreover it gives a complete classification of the fundamental building blocks of such cascades. We refer the reader to [2] and [3] for extensive treatment of this and related topics. This paper develops as far as possible a similar theory for differentiable systems, i.e., control systems defined by a nonlinear ordinary differential equation on a finite dimensional manifold.

The first step in this program is to view each constant input as not affecting a particular state but all possible states, that is, to consider the state transition map defined by the input. The family of these maps forms a semigroup which acts on the state space in the obvious fashion and this semigroup has a natural completion to a group. One can try to lift the dynamics from the state space to the group. For a finite state machine, this can always be done resulting in another finite state machine, but for a differentiable system the group of state transition maps need not be finite dimensional. This is a major difference between the two theories.

For a differentiable system it is natural to consider the infinitesimal version of this group, the Lie algebra of vector fields corresponding to constant inputs. This algebra, which has no analogue in Krohn–Rhodes theory, determines the local dynamics of the system. We shall show that if the algebra can be split into an ideal and finite dimensional subalgebra then the system can be split into a cascade. It follows that every system with finite dimensional Lie algebra can be decomposed into a cascade of systems with simple or one dimensional algebras. It is also shown that such systems admit no further decomposition into systems with less complicated Lie algebras. They do however admit decompositions where the controls are split between the elements of the cascade.

1. Nonlinear control systems. In the last few years it has become apparent through the work of Sussman, Brockett, Hermes, Elliott, Lobry, and others that the appropriate state space for nonlinear systems is not $\mathbb{R}^n$. For this paper we adopt a terminology and notation similar to that introduced by Sussmann in his important papers on the existence and uniqueness of minimal realizations of nonlinear systems [4], [5]. We restrict our discussion to real analytic systems for

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This allows us to define a semigroup homomorphism from $\mathcal{U}_{pc}$ into Diff (M) in the obvious fashion. The range of this homomorphism is a subsemigroup S of Diff (M) and we refer to this as the *semigroup of the system* Σ. The smallest subgroup G of Diff (M) containing S is called the *group of the system*.

Given a point $x_0 \in M$, we can consider the orbits of x_0 under the semigroup S and group G respectively

$$S(x_0) = \{\phi(x_0) : \phi \in S\},$$
$$G(x_0) = \{\phi(x_0) : \phi \in G\}.$$

$S(x_0)$ is often referred to as the set of points accessible from x_0 under Σ by controls in $\mathcal{U}_{pc}$. Because we have assumed $f(x, u)$ to be analytic in x it can be shown using the Hermann–Nagano theorem [17], [7] that $G(x_0)$ is an analytic submanifold of M. In some sense $G(x_0)$ is the natural submanifold of M on which to consider the problem for it contains all trajectories of the system emanating from x_0. Chow's theorem [8] tells us that every point in $G(x_0)$ can be reached from x_0 along trajectories of the system going both forward and backward in time. Moreover it is of minimal dimension among submanifolds of M containing $S(x_0)$ because $S(x_0)$ has a nonempty interior in the topology of $G(x_0)$ [9], [10]. (Note the topology of $G(x_0)$ is not necessarily its relative topology inherited from M.) If $S(x_0) = M (G(x_0) = M)$ then the system is said to be *controllable* (*weakly controllable*), see [18]. If $G(x_0) \neq M$ then we redefine the Lie algebra L of Σ to be the smallest subalgebra of $V(Gx_0)$ which contains all the vector fields $f(\cdot, u)$, $u \in \Omega$.

Let $u(\cdot) \in \mathcal{U}_m$; then given any $x_0 \in M$ there exists a compact neighborhood K of x_0, an open interval I containing 0 and a map $\phi_u : I \times K \to M$ satisfying

$$\frac{d}{dt}\phi_u(t, x) = f(\phi_u(t, x), u(t)),$$
$$\phi_u(0, x) = x. \tag{2.1}$$

Since $u(\cdot)$ is only a bounded measurable function the curve $t \mapsto \phi_u(t, x)$ is generally only absolutely continuous. For each $t \in I$ the map $\phi_u(t, \cdot) : K \to M$ is 1–1 and analytic. If it can be defined on all of M then it is an element of Diff (M). Since we have assumed that Σ is complete, if $u(\cdot) \in \mathcal{U}_{pc}$ then ϕ_u can be defined on $\mathbb{R} \times M$. However if $u(\cdot) \in \mathcal{U}_m - \mathcal{U}_{pc}$ this need not be true. (See Sussmann [4, p. 14] for a counterexample). The effect of $u(\cdot) \in \mathcal{U}_m$ can always be approximated by piecewise constant controls.

APPROXIMATION LEMMA [4]. *Let $u(\cdot) \in \mathcal{U}_m$ and $\phi_u : I \times K \to M$ be its flow. Suppose $\{u^i(\cdot)\} \subset \mathcal{U}_{pc}$ is a sequence of piecewise constant controls such that $u^i(t) \to u(t)$ for almost all $t \in I$. If $\phi_i(\cdot, \cdot)$ is the flow of $u^i(\cdot)$ and J is a compact subinterval of I then $\phi_i \to \phi_u$ uniformly on $J \times K$.*

The above lemma indicates why S and G can be defined without regard to the class of admissible inputs, $\mathcal{U}$ as long as $\mathcal{U}_{pc} \subset \mathcal{U} \subset \mathcal{U}_m$.

The semigroup S plays a similar role in the theory of differentiable systems as the semigroup of a machine in the theory of finite automata. They both describe the action of the semigroup of inputs on all states of the system/machine. There are some important distinctions however; one is the problem of finite escape time.

Another is that the state transition map $\phi_u(t, \cdot)$ is always in Diff (M) for differentiable systems and hence S can naturally be extended to a subgroup G of Diff (M). On the other hand the state transition maps of a finite automaton are not necessarily invertible. The group of the machine is generated by the invertible ones and may not contain the semigroup of the machine.

The Lie algebra L has no analogue in the theory of finite automata; it is an infinitesimal version of G. By this we mean L completely determines $\phi_u(t, \cdot)$ for small t.

In both theories it is desirable to lift the dynamics from the state space to the group/semigroup of the system/machine. That is, we view inputs as not affecting a particular state but rather affecting all possible states. In the case of a finite automaton this results in the semigroup of the machine becoming the new state space. This state space is again finite and hence the new machine is again a finite automaton.

In the case of a differentiable system, G is not always a finite dimensional manifold and therefore the dynamics lifted to G is not described by a finite dimensional differential equation. We define Σ to be *finite dimensional* if L is finite dimensional. In this case G can be given the structure of a finite dimensional real analytic manifold compatible with the group operation (Palais [6]). This makes G into a Lie group and L can be viewed as the Lie algebra of right invariant vector fields on G. This allows us to lift the dynamics from the state space M where they are given by

$$\dot{x} = f(x, u), \qquad x(0) = x_0$$

to a new state space G where they are given locally by a matrix differential equation.

We shall elaborate on this in the next section but first we illustrate these points with some familiar examples, the first of which is a linear system.

Σ: Let the state space $M = \mathbb{R}^n$, the control set $\Omega = \mathbb{R}^k$, the initial point be x_0 and the dynamics be given by

$$\dot{x} = Ax + \sum_{i=1}^{k} u_i b_i \tag{2.2}$$

where A is $n \times n$ real matrix and b_i are n-vectors.

The Lie bracket is given by

$$[Ax, b_i] = Ab_i, \qquad [b_i, b_j] = 0,$$
$$ad^l(Ax)b_i = A^l b_i, \qquad [b_i, ad^l(Ax)b_j] = 0$$

and so the Lie algebra is finite dimensional with a particularly simple form which characterizes a locally linear system, [11]. The zero control defines a matrix differential equation

$$\dot{\Phi}(t) = A\Phi(t)$$

with $\Phi(0) = I$ the identity matrix. The solution is $\Phi(t) = \exp(tA)$. For any control

$u(\cdot) \in \mathcal{U}_m$ the flow ϕ_u is given by the variation of constants formula

$$\phi_u(t, x) = \Phi(t)x + \int_0^t \Phi(t-s) \sum_i u_i(s) b_i \, ds,$$

which for fixed t defines an invertible affine map $\phi_u(t, \cdot): \mathbb{R}^n \to \mathbb{R}^n$. S and G are the smallest subsemigroup and subgroup of the group of invertible affine motions of $\mathbb{R}^n$ which contain all such $\phi_u(t, \cdot)$, $t \geqq 0$. Lifting the dynamics from the state space $\mathbb{R}^n$ to G means replacing the vector differential equation (2.2) by the matrix differential equation

$$\dot{X} = AX + \left(\sum_i u_i b_i \cdots \sum_i u_i b_i\right).$$

Now consider a bilinear system

Σ: Let the state space be $M = \mathbb{R}^n$, the control set be $\Omega = \mathbb{R}^k$, the initial point be x_0 and the dynamics be given by

$$\dot{x} = \left(A + \sum_i u_i B_i\right)x \tag{2.3}$$

when A and B_i are $n \times n$ real matrices.

If C and D are $n \times n$ real matrices the Lie bracket of the vector fields Cx and Dx is given by the commutator $[C, D] = CD - DC$,

$$[Cx, Dx] = [C, D]x.$$

Therefore, L is isomorphic to the smallest subalgebra of $gl(n, \mathbb{R})$ (the Lie algebra of all $n \times n$ real matrices) containing A and B_i. Each control $u(\cdot) \in \mathcal{U}$ defines a matrix differential equation

$$\dot{\Phi}_u(t) = \left(A + \sum_i u_i(t) B_i\right)\Phi_u(t) \tag{2.4}$$

where $\Phi_u(t) \in GL(n, \mathbb{R})$, the Lie group of invertible matrices in $gl(n, \mathbb{R})$ and

$$\phi_u(t, x) = \Phi_u(t)x.$$

S and G are the smallest subsemigroup and subgroup of $GL(n, \mathbb{R})$ containing Φ_u for each $u \in \Omega$. Lifting the dynamics from M to G again means replacing the vector differential equation (2.3) by the matrix differential equation (2.4). The finiteness of the Lie algebra locally characterizes bilinear systems [12].

It is well known that every linear control system can be turned into a bilinear system by the addition of an extra coordinate which is identically one, so henceforth when we refer to bilinear systems we include linear ones.

3. Simulation. Given an initialized system $I = (M, \Omega, f, \mathcal{U}, N, g, x_0)$ let $\mathcal{X}_{x_1}$ denote the space of all absolutely continuous functions $x(\cdot): [0, T] \to M$ for any $T \geqq 0$ such that $x(0) = x_1$. Similarly define $\mathcal{Y}_{x_1}$ as the space of all absolutely continuous functions $y(\cdot): [0, T] \to N$ for any $T \geqq 0$ such that $y(0) = g(x_1)$. Given any $x_1 \in M$ the system Σ defines a pair of maps $\mathcal{F}_{x_1}: \mathcal{U} \to \mathcal{X}_{x_1}$ and $\mathcal{G}_{x_1}: \mathcal{U} \to \mathcal{Y}_{x_1}$ in the obvious fashion:

$$\mathcal{F}_{x_1}(u(t)) = \phi_u(t, x_1) \quad \text{and} \quad \mathcal{G}_{x_1}(u(t)) = g(\phi_u(t, x_1)).$$

Two points $x_1, x_2 \in M$ are said to be *indistinguishable* if $\mathscr{G}_{x_1}(u(t)) = \mathscr{G}_{x_2}(u(t))$ for all $u(\cdot) \in \mathscr{U}_{pc}$. The system Σ is *observable* if x_1 and x_2 indistinguishable implies that $x_1 = x_2$. The system Σ is *minimal* if it is weakly controllable and observable.

Given a pair of initialized systems

$$\Sigma^i = (M^i, \Omega^i, f^i, \mathscr{U}^i, N^i, g^i, x_0^i) \quad \text{for } i = 1, 2,$$

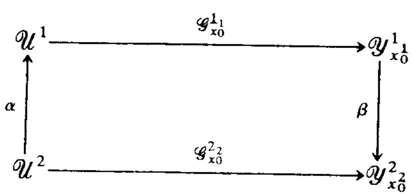
the maps $\mathscr{F}^i_{x_0^i}$ and $\mathscr{G}^i_{x_0^i}$ describe the state space and input/output behavior of the systems. A question of some importance is when the behavior of one system simulates the behavior of the other in either of the above senses. There are several alternative ways of approaching this problem. In the classical theory of minimal realizations of linear systems one assumes that $\Omega^1 = \Omega^2$, $N^1 = N^2$ and studies when two input-output equivalent systems differ by a homomorphism or isomorphism of the state space, $\mathbb{R}^n$. A similar theory has been developed by H. Sussmann for nonlinear systems which we shall discuss in a moment.

In the theory of finite automata the input and output spaces are allowed to differ by encoding and decoding functions. One wishes to know when an automaton can be made to simulate the input-output behavior of another automaton by a suitable choice of encoder and decoder. For algebraic reasons it is considerably easier to discuss when the state behavior of an automaton can be simulated by another automaton. We describe a similar theory for nonlinear systems.

Given a pair of initialized systems Σ^i and functions $\alpha : \Omega^2 \to \Omega^1$, $\beta : N^1 \to N^2$ with α continuous and β analytic we obtain induced maps (also denoted by α and β) $\alpha : \mathscr{U}^2 \to \mathscr{U}^1$ and $\beta : \mathscr{Y}^1_{x_0^1} \to \mathscr{Y}^2_{x_0^2}$ in the obvious fashion:

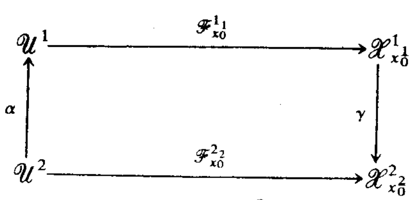
$$\alpha(u(\cdot))(t) = \alpha(u(t)) \quad \text{and} \quad \beta(y(\cdot))(t) = \beta(y(t)).$$

Σ^1 *simulates* Σ^2 with encoder α and decoder β if the following diagram commutes:



Σ^1 is *equivalent* to Σ^2 if Σ^1 simulates Σ^2 with $\alpha : \Omega^2 = \Omega^1$ and $\beta : N^1 = N^2$ the identity maps.

Suppose Σ^1 is weakly controllable, i.e., $G(x_0^1) = M^1$. Given a pair $\alpha : \Omega^2 \to \Omega^1$ and $\gamma : M^1 \to M^2$ with α continuous and γ analytic we say that Σ^1 is homomorphic to Σ^2 if the following diagram commutes:



Σ^1 is isomorphic to Σ^2 if Σ^1 is homomorphic to Σ^2 with $\alpha : \Omega^1 = \Omega^2$ the identity and $\gamma : M^1 \to M^2$ a diffeomorphism. If Σ^1 is not weakly controllable it is sufficient to find $\gamma : G(x_0^1) \to M^2$ such that the appropriate diagram commutes.

There is also a local form of the above definitions. For example suppose Σ^1 is weakly controllable, define $\mathcal{U}_T^i = \{u(\cdot) \in \mathcal{U}^i : u(\cdot)$ defined on $[0, t]$ when $t \leqq T\}$. Σ^1 is locally homomorphic to Σ^2 if there exists $T > 0$, a neighborhood V of x_0^1 in M^1 and a map $\gamma : V \to M^2$ such that the appropriate diagram commutes.

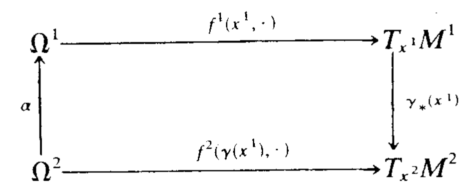
It is difficult to give conditions for Σ^1 to simulate Σ^2 because of the complications caused by the presence of the encoder and decoder. However if one is interested in equivalence these difficulties are somewhat mitigated and H. Sussmann has proved the following generalization of the existence and uniqueness theorem for minimal realization of linear systems.

SUSSMAN'S THEOREM [5]. *Every initialized analytic system is equivalent to a minimal system. Any two equivalent minimal systems are isomorphic.* For related results see [18].

We now focus in one the question of when Σ^1 is homomorphic to Σ^2, Suppose Σ^1 has the accessibility property and is homomorphic to Σ^2. Then it is easy to show that the Jacobian γ_*

$$\gamma_*(x^1) = \frac{\partial \gamma}{\partial x^1}(\cdot)$$

must define a homomorphism $\gamma_* : L^1 \to L^2$ of the Lie algebras L^i of Σ^i. Furthermore, the maps α and $\gamma_*(x^1)$ must satisfy the following commutative diagram



where $T_{x^i}M^i$ is the tangent space to M^i at x^i.

Suppose $u_0, u_1, \cdots, u_l \in \Omega^2$ such that

$$f^1(x^1, \alpha(u_0)) = \sum_{i=1}^{l} \lambda_i f^1(x^1, \alpha(u_i));$$

then the linearity of $\gamma_*(x^1)$ implies that

$$f^2(\gamma(x^1), u_0) = \sum_{i=1}^{l} \lambda_i f^2(\gamma(x^1), u_i).$$

In particular if $\alpha(u_0) = \alpha(u_1)$ then $f^2(x^2, u_0) = f^2(x^2, u_1)$ for every $x^2 \in G^2(x_0^2)$. It follows that the map $f^1(\cdot, \alpha(u)) \mapsto f^2(\cdot, u)$ has a well-defined linear extension from span $\{f^1(\cdot, \alpha(u)) : u \in \Omega^2\}$ to span $\{f^2(\cdot, u) : u \in \Omega^2\}$.

Let $\alpha(L^2)$ denote the subalgebra of L^1 generated by $f^1(\cdot, \alpha(u))$ for each $u \in \Omega^2$. Let L_0^i denote the *isotropy subalgebra* of L^i at x_0^i, i.e.,

$$L_0^i = \{h(\cdot) \in L^i : h(x_0^i) = 0\}$$

and let $\alpha(L^2)_0 = L_0^1 \cap \alpha(L^2)$.

If L^1 and L^2 are arbitrary Lie algebras we say that L^2 *divides* L^1 if there exists a subalgebra $L \subset L^1$ and a Lie algebra homomorphism of L onto L^2. If L^i is the Lie algebra of system Σ^i we say L^2 G-divides L^1 if there exists a continuous map

$\alpha:\Omega^2\to\Omega^1$ such that the map $f^1(\cdot,\alpha(u))\mapsto f^2(\cdot,u)$ generates a Lie algebra homomorphism of $\alpha(L^2)$ onto L^2. If in addition this homomorphism carries $\alpha(L^2)_0$ into L_0^2 then we say L^2 *Σ-divides* L^1.

THEOREM 1 (Krener [11]). *Σ^1 is locally homomorphic to Σ^2 if L^2 Σ-divides L^1. Moreover if L^2 Σ-divides L^1 and $G^1(x_0^1)$ is simply connected then Σ^1 is homomorphic to Σ^2.*

Suppose L^2 Σ-divides L^1 but $G^1(x_0^1)$ is not simply connected; then we can ask if there is any way to lift Σ^1 to a system that is simply connected. The answer is yes; suppose for convenience $G^1(x_0^1)=M^1$. Then M^1 has a unique simply connected covering manifold M with covering map $\pi:M\to M^1$. Given any $x\in M$ we can find a sufficiently small neighborhood V of x such that the map $\pi:V\to M^1$ is a diffeomorphism, and so the Jacobian π_* is invertible. This allows us to lift the dynamics from M^1 to M by defining

$$f(x,u)=\pi_*^{-1}(x)f^1(\pi(x),u)$$

for $x\in M$ and $u\in\Omega=\Omega^1$. We choose any initial point $x_0\in\pi^{-1}(x_0^1)$ and by Chow's theorem we know that $G(x_0)=M$. The result is a system $\Sigma=(M,\Omega,f,x_0)$ which is called the *simply connected cover* of Σ^1 and is homomorphic to Σ^1 under $\alpha:\Omega^1\to\Omega$ the identity and $\pi:M\to M^1$ the covering map. If Σ^1 is locally homomorphic to Σ^2 then Σ is homomorphic to Σ^2.

For example consider the following system: $\Sigma^1:M^1=S^1$ the unit circle with angular coordinate θ, $\Omega=\mathbb{R}$, $\theta^0=0$ and $\dot\theta=u$. The simply connected covering space of S^1 is $\mathbb{R}$ and so the simply connected cover of Σ^1 is $\Sigma:M=\mathbb{R}$, $\Omega=\mathbb{R}$, $x^0=0$ and $\dot x=u$.

Suppose L^2 G-divides L^1 but does not Σ-divide L^1, then we can ask if there is any way to lift Σ^1 to a different system such that L^2 Σ-divides L. The answer is yes if the Lie algebra L^1 of Σ^1 is finite dimensional. The group G^1 of Σ^1 is then a finite dimensional Lie group (Palais [6]) and by Ado's theorem L^1 is isomorphic to some subalgebra L of $gl(m,\mathbb{R})$ for some m possibly different from $n^1=$ dimension M^1. (For bilinear systems L^1 is already a subalgebra of $gl(n^1,\mathbb{R})$.) For each $u\in\Omega^1$, let $F(u)$ be the matrix corresponding to $f^1(\cdot,u)$ under this isomorphism. Let H be the subgroup of $GL(m,\mathbb{R})$ corresponding to L; then some neighborhood of the identity in G^1 is isomorphic as a Lie group to some neighborhood of the identity in H. This isomorphism can be viewed as defining H-valued local coordinates on G^1 in which the dynamics of Σ^1 is described by

$$\dot X=F(u)X \tag{3.1}$$

where $X\in H\subset GL(m,\mathbb{R})$.

If H and G^1 are globally isomorphic as in the case of bilinear systems then the matrix differential equation describes the lifted dynamics throughout G^1. For bilinear systems the action of $G^1=H$ on the state space $M^1=\mathbb{R}^{n^1}$ is the natural linear action, however H and G^1 globally isomorphic does not necessarily imply that the action of $G^1\subset\mathrm{Diff}(M^1)$ on M^1 is linear in any coordinates.

If H and G^1 are only locally isomorphic then this isomorphism can be used to define H-valued coordinates in a neighborhood of every $\phi\in G^1$. In these coordinates the dynamics is locally given by a matrix differential equation similar to (3.1). We define a new system $\Sigma=(M,\Omega,f,\phi_0)$ where $M=G^1$, $\Omega=\Omega^1$, the dynamics f is given locally by (3.1) and ϕ_0 is the identity of G^1, i.e., $\phi_0 : M^1 \to M^1$ the identity map. Σ is called the *group cover* of Σ^1.

Σ is homomorphic to Σ^1 under $\alpha : \Omega^1 \to \Omega$ the identity and $\gamma : G^1 \to M^1$ given by $\gamma(\phi) = \phi(x_0)$ for $\phi \in G^1$. Notice that the Lie algebra L if Σ is isomorphic to the Lie algebra L^1 of Σ^1 but the isotropy subalgebras need not be. If L^2 G-divides L^1 then L^2 Σ-divides L for $L_0 = 0$.

Recall that G^1 is the not necessarily closed subgroup of Diff (M) generated by the flows of constant controls. Suppose $u(\cdot) \in \mathcal{U}$ and for some t, the map $\phi_u(t, \cdot)$ defined by (2.1) is a diffeomorphism of M^1. The Approximation Lemma implies that $\phi_u(t, \cdot) \in$ closure G^1; however by lifting the dynamics to G^1 we have shown that $\phi_u(t, \cdot) \in G^1$.

Finally suppose L^2 divides L^1 but does not G-divide L^1, then we can ask if there is any way to lift Σ^1 to a different system Σ such that L^2 G-divides L. The answer is again yes provided L^1 is finite dimensional. Let $\Sigma = (M, \Omega, f, x_0)$ where $M = M^1$, $\Omega = L^1$, $x_0 = x_0^1$ and $f(x, h(\cdot)) = h(x)$ for $x \in M^1$ and $h(\cdot) \in L^1$. Define $\alpha : \Omega^1 \to \Omega$ by $\alpha(u) = f^1(\cdot, u)$ and $\gamma : M \to M^1$ the identity; then clearly Σ is homomorphic to Σ^1. Σ is called the *fully controllable cover* of Σ^1 and if $\Sigma = \Sigma^1$ then Σ^1 is said to be fully controllable. Notice that in a fully controllable system the dynamics is linear in the controls.

4. Cascades. Suppose $\Sigma^i = (M^i, \Omega^i, f^i, x_0^i)$ are control systems for $i = 1, 2$. Henceforth we assume $\mathcal{U}^i = \mathcal{U}_m^i$ and therefore do not mention it explicitly. Let $v : M^1 \times \Omega^1 \to \Omega^2$ be an analytic map of x^1, continuous with respect to u^1. We define the *cascade* $\Sigma^1 \ominus_v \Sigma^2$ of these two systems with *linking map* v as the system

$$(M^1 \times M^2, \Omega^1, f^1 \ominus_v f^2, (x_0^1, x_0^2))$$

where

$$f^1 \ominus_v f^2(x^1, x^2, u) = (f^1(x^1, u^1), f^2(x^2, v(x^1, u^1))).$$

$\Sigma^1 \ominus_v \Sigma^2$ is a *parallel cascade* if v is a function of u_1 alone and a *series cascade* if v only depends on x^1.

Cascades are a way of combining two or more systems to obtain a more complicated system. We would like to study when this technique can be used to represent a given system as the homomorphic image of a cascade of "simpler" systems. Of course any system is the homomorphic image of a cascade consisting of itself followed by an arbitrary system but this can hardly be called a cascade of "simpler" systems.

We must make rigorous the notion of "simpler"; the obvious choice is that Σ^i is "simpler" than Σ if Σ is homomorphic to Σ^i. However this is not the appropriate definition for if Σ is actually a cascade, $\Sigma^1 \ominus_v \Sigma^2$, then it is easy to see that Σ is homomorphic to Σ^1 but it need not be homomorphic to Σ^2. Therefore we are forced to a weaker definition—Σ^i is "simpler" than Σ if Σ is not a homomorphic image of Σ^i. A similar definition is used by Krohn and Rhodes.

A system Σ has a *nontrivial cascade decomposition* if there exists Σ^1, Σ^2 and v such that $\Sigma^1 \ominus_v \Sigma^2$ is homomorphic to Σ but neither Σ^1 nor Σ^2 alone is homomorphic to Σ.

We leave it to the reader to verify the following.

LEMMA. *Suppose* Σ *is homomorphic to* $\Sigma^1(\Sigma^2)$; *then* $\Sigma \ominus_w \Sigma^2(\Sigma^1 \ominus_w \Sigma)$ *is homomorphic to* $\Sigma^1 \ominus_v \Sigma^2$ *for some linking map* w.

COROLLARY. *Suppose* $\Sigma^1 \ominus_v \Sigma^2$ *is homomorphic to* Σ *and* $\Sigma^3 \ominus_w \Sigma^4$ *is homomorphic to* Σ^1 (Σ^2). *Then* $\Sigma^3 \ominus_w \Sigma^4 \ominus_v \Sigma^2$ $(\Sigma^1 \ominus_v \Sigma^3 \ominus_w \Sigma^4)$ *is homomorphic to* Σ.

We describe a way for decomposing a system into cascades which is based on a technique used by E. Wichmann [13] and K. T. Chen [19] and originally due to S. Lie [20]. Let $\Sigma = (M, \Omega, f, x_0)$ and suppose the Lie algebra L of Ω is a semidirect sum,

$$L = L^1 + L^2.$$

(L is a *semidirect sum* of L^1 and L^2 if L^1 is a subalgebra of L, L^2 is an ideal of L and L is the direct sum of L^1 and L^2 as vector spaces. We exclude the trivial case where either is 0.) For each $u \in \Omega$, define $f^1(\cdot, u) \in L^1$ and $g(\cdot, u) \in L^2$ by requiring that

$$f(\cdot, u) = f^1(\cdot, u) + g(\cdot, u).$$

Consider the control system $\Sigma^1 = (M^1, \Omega^1, f^1, x_0^1)$ where $M^1 = M$, $\Omega^1 = \Omega$ and $x_0^1 = x_0$. Let G^1 be the group of Σ^1. Clearly the Lie algebra of Σ^1 is L^1.

Define a second system $\Sigma^2 = (M^2, \Omega^2, f^2, x_0^2)$ where $M^2 = M$, $\Omega^2 = G^1 \times \Omega$, $x_0^2 = x_0$ and

$$f^2(x, \phi, u) = \phi_*^{-1}(x) g(\phi(x), u),$$

for $x \in M$, $\phi \in G^1$ and $u \in \Omega$. There is a problem with this definition for in general the control set Ω^2 is not finite dimensional since G^1 is not. However if we assume that L^1 is finite dimensional then G^1 is embeddable in some R^l by the Whitney theorem and hence Σ^2 is a control system according to our definition.

Moreover if L^1 is finite dimensional then we can redefine Σ^1 so it equals its group cover. This allows us to form the cascade $\Sigma^1 \ominus_v \Sigma^2$ where the linking map $v: G^1 \times \Omega \to \Omega^2$ is the identity.

For a fixed control $u(\cdot) \in \mathcal{U}$, let $x(t)$ be the trajectory in Σ, $\phi(t, \cdot)$ be the trajectory in Σ^1 and $x^2(t)$ be the trajectory in Σ^2. We claim $x(t) = \phi(t, x^2(t))$ and we show this by noting that $x(0) = x_0 = \phi(0, x^2(0))$ and using (2.1) we see that both satisfy the same differential equation

$$\begin{aligned} \frac{d}{dt}\phi(t, x^2(t)) &= f^1(\phi(t, x^2(t)), u(t)) + \phi_*(t, x^2(t))\dot{x}^2(t) \\ &= f^1(\phi(t, x^2(t)), u(t)) + g(\phi(t, x^2(t)), u(t)) \\ &= f(\phi(t, x^2(t)), u(t)). \end{aligned}$$

Therefore $\Sigma^1 \ominus_v \Sigma^2$ is homomorphic to Σ under $\alpha: \Omega \to \Omega$, the identity, and $\gamma: G^1 \times M^2 \to M$ given by $\gamma(\phi, x) = \phi(x)$. Since the Lie algebras of Σ^1 and Σ^2 are L^1 *and* L^2 it follows that neither system is homomorphic to Σ and hence the cascade decomposition is nontrivial.

If $g(\cdot, u)$ is independent of u then $\Sigma_1 \ominus_v \Sigma^2$ is a series cascade. On the other hand suppose L is a direct sum of L^1 and L^2 (that is, both are ideals of L). From

$[L^1, L^2] \subset L^1 \cap L^2 = 0$ it follows that for every $\phi \in G^1$, $f^2(x, \phi, u) = \phi_*^{-1}(x)g(\phi(x), u) = g(x, u)$. Therefore f^2 is independent of ϕ and $\Sigma^1 \ominus_v \Sigma^2$ is a parallel cascade. We sum this up in the following:

THEOREM 2. *If the Lie algebra of a system is the semidirect sum of a finite dimensional subalgebra and an ideal then it has a nontrivial cascade decomposition. If it is the direct sum of two ideals, then it has a parallel cascade decomposition.*

A particular application of the above result is when the system is finite dimensional, as in the case of a bilinear system. By Levi's theorem L is a semidirect sum of semisimple subalgebra L^1 and a maximal solvable ideal L^2. Therefore Σ has a cascade decomposition $\Sigma^1 \ominus_v \Sigma^2$.

Every finite dimensional semisimple Lie algebra L^1 is a direct sum

$$L^1 = L^{11} + \cdots + L^{1l}$$

of simple ideals L^{1i}. Therefore by repeated application of the above theorem Σ^1 can be decomposed into the parallel cascade of a family of systems Σ^{1i}, $i = 1, \cdots, l$, each with a simple Lie algebra, L^{1i}. Recall a Lie algebra is simple if it is not Abelian and contains no nontrivial ideals; therefore the Σ^{1i} admit no further decomposition using Theorem 2. However as we show by example in a moment systems whose Lie algebra is simple can admit nontrivial cascade decompositions.

We now turn to the system Σ^2 whose Lie algebra L^2 is solvable. This implies that $[L^2, L^2]$ is a proper ideal of L^2 and hence one can find a linear subspace L^{22} of codimension one in L^2 which contains $[L^2, L^2]$. Since L^{22} contains $[L^2, L^2]$ it is an ideal of L^2, and since it is of codimension one any vector field in $L^2 \backslash L^{22}$ generates a one dimensional subalgebra L^{21} such that $L^2 = L^{21} + L^{22}$ is a semidirect sum.

Using Theorem 2, Σ^2 can be decomposed into the cascade of a one dimensional system Σ^{21} and a system Σ^{22} with solvable Lie algebra of one lower dimension. By induction Σ^2 is cascade decomposition of a family of one dimensional systems.

Moreover there are, up to isomorphism, only two one dimensional systems, those on the circle and line described in § 3. Therefore we have shown the following.

THEOREM 3. *If the Lie algebra L of Σ is finite dimensional then Σ admits a decomposition into the parallel cascade of systems with simple Lie algebras followed by a cascade of one dimensional systems.*

This result is somewhat stronger than that of Brockett [14] since all the component systems of Theorem 3 are either simple or one dimensional. In Brockett's work the component systems are reductive. The one dimensional algebra and all simple algebras are reductive but $gl(n, \mathbb{R})$ is not simple but reductive.

Theorem 3 is highly reminescent of the Krohn–Rhodes theorem which states that every finite state machine can be broken up as a cascade of machines with simple groups and flip-flops. The system with simple Lie algebras are analogous to machines with simple groups but the analogy breaks down between one dimensional systems and flip-flop machines, since flip-flops correspond to the nongroup part of the machine.

Recall that in § 1 we suggested that time varying systems be made autonomous by the introduction of time as another state variable. Unfortunately this can make the Lie algebra of a time varying bilinear system infinite dimensional and therefore not amenable to the application of Theorem 3. Instead suppose we consider time as another control variable and view the time varying bilinear system as the cascade of a trivial system and the bilinear system with the time control.

$$\Sigma^1: \quad \dot{x}^1 = 1,$$

$$\Sigma^2: \quad \dot{x}^2 = A(u_0)x^2 + \sum_{i=1}^{k} u_i B_i(u_0)x^2$$

where the linking map is $u_0 = x^1$. Then we can see Theorem 3 to decompose Σ^2 since its Lie algebra is contained in $gl(n, \mathbb{R})$ and hence is finite dimensional.

Actually the above technique can be used to generate a cascade decomposition even if L is not a semidirect sum. (For example see [19].) Instead of splitting the Lie algebra of Σ between Σ^1 and Σ^2, we split the controls. Let K denote the subspace of $V(M)$ which is the span $\{f(\cdot, u): u \in \Omega\}$ and suppose K admits a direct sum decomposition (as a real vector space)

$$K = K^1 + K^2.$$

For each $u \in \Omega$, define $f^1(\cdot, u) \in K^1$ and $g(\cdot, u) \in K^2$ by requiring that

$$f(\cdot, u) = f^1(\cdot, u) + g(\cdot, u).$$

Define Σ^1 as before; if its Lie algebra L^1 generated by K^1 is finite dimensional then we can also define Σ^2 and the cascade $\Sigma^1 \ominus_v \Sigma^2$. The same argument as before shows that $\Sigma^1 \ominus_v \Sigma^2$ is homomorphic to Σ.

We ask whether this is a nontrivial cascade decomposition. The Lie algebras L^1 and L^2 could each be equal to L so we must check if Σ^i is homomorphic to Σ. If the decomposition of K is nontrivial then K^1 is a proper subset of K so L could not possibly G-divide L^1. This implies that Σ^1 is not homomorphic to Σ.

On the other hand the generators of L^2 are contained in the orbit of K^2 under the group G^1 acting by conjugation. The Campbell–Baker–Hausdorff formula shows that this is equal to the orbit of K^2 under the Lie algebra L^1 acting by bracketing. Therefore if this orbit does not contain K then we can conclude that L does not G-divide L^2 and hence Σ^2 is not homomorphic to Σ.

THEOREM 4. *Let K be the linear span of the vector fields corresponding to constant controls of a system Σ and suppose K is a direct sum of linear subspaces K^1 and K^2. If the Lie algebra L^1 generated by K^1 is finite dimensional and orbit of K^2 under L does not include K then Σ admits a nontrivial cascade decomposition.*

Now we use this theorem to exhibit a system whose Lie algebra is simple but admits a nontrivial cascade decomposiition. Σ: Let $M = SL(2, \mathbb{R})$ the group of real 2×2 matrices of determinant 1, $\Omega = \mathbb{R}^3$, the initial point be any identity matrix and the dynamics be given by

$$\dot{X} = \sum_{i=1}^{3} u_i B_i X$$

where

$$B_1 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}, \quad B_2 = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}, \quad B_3 = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}.$$

Proof. One way follows from Theorem 3, so suppose the Lie algebra L of Σ is one dimensional or simple. Let $\Sigma^1 \ominus_v \Sigma^2$ be a finite cascade decomposition of Σ where the Σ^i's are finite dimensionable systems. If L is one dimensional then it clearly divides any nontrivial Lie algebra and at least one L^i must be nontrivial. Therefore we restrict to the case where L is simple.

If Σ is simulated by $\Sigma^1 \ominus_v \Sigma^2$ then Σ can be simulated by the cascade of the fully controllable covers of Σ^1 and Σ^2 using the above lemma (in § 4). Since Σ^i and its fully controllable cover have the same Lie algebra L^i it is with no loss of generality that we assume that each Σ^i is fully controllable. Therefore the controls of Σ^i enter linearly and the dynamics of the cascade must look like

$$\dot{x}^1 = \Sigma u_i a_i(x^1),$$
$$\dot{x}^2 = \Sigma v_j(u, x^1) b_j(x^2).$$

For each $u \in \Omega^1$ and each j, $v_j(u, \cdot)$ is a scalar-valued function of x^1 and any vector field $h^1(\cdot) \in L^1$ acts on it by partial differentiation. Let P denote the orbit of $\{v_j(u, \cdot): \forall j \text{ and } u \in \Omega\}$ under the action of L^1. In general this is a real infinite dimensional vector space.

Consider the product $P \otimes L^2$ consisting of all finite linear combinations of elements of L^2 with coefficients from P. In an obvious fashion this can be identified with the subalgebra of $V(M^1 \times M^2)$ consisting of vector fields whose projection in the M^1 direction are identically zero. Similarly L^1 can be identified with the subalgebra of $V(M^1 \times M^2)$ consisting of vector fields whose projections in M^2 direction are identically zero. It follows that

$$L^1 \ominus_v L^2 \subset L^1 + P \otimes L^2$$

where the right side is the semidirect sum of the subalgebra L^1 and ideal $P \otimes L^2$.

Let Q denote the subspace of P consisting of all functions which actually appear in the expansion of a vector field of $L^1 \ominus_v L^2$. Each such vector field involves only a finite number of functions of P and since $L^1 \ominus_v L^2$ is finite dimensional it follows that Q is also finite dimensional. Moreover

$$L^1 \ominus_v L^2 \subset L^1 + Q \otimes L^2.$$

Since Q is a finite dimensional space of functions on M^1 and L^2 is a Lie algebra of vector fields on M^2, $Q \otimes L^2$, as a Lie algebra, is a direct sum of a finite number of copies of L^2.

Since the cascade is homomorphic to Σ there exists a Lie algebra homomorphism $\gamma_*: L^1 \ominus_v L^2 \to L$. Let π_* denote the projection $\pi_*: L^1 + Q \otimes L^2 \to L^1$; by restricting π_* we obtain the following diagram:

$$\begin{array}{ccc} L^1 \ominus_v L^2 & \xrightarrow{\ \gamma_*\ } & L \\ \big\downarrow{\scriptstyle \pi_*} & & \\ L^1 & & \end{array}$$

Let $I = \ker \pi_*$ restricted to $L^1 \ominus_v L^2$ and $J = \gamma_*(I)$; these are ideals in $L^1 \ominus_v L^2$

and L respectively. Since L is simple there are only two possibilities, $J=0$ or $J=L$.

Suppose $J=0$; then $\ker \pi_* \subset \ker \gamma_*$. The first homomorphism theorem implies that there exists a natural homomorphism of $\pi_*(L^1 \ominus_v L^2)$ onto L, i.e., L divides L^i.

Suppose $J=L$; then we can conclude that $\gamma_*: I \to L$ is onto. Moreover $I \subset Q \otimes L^2$ so we conclude that L divides $Q \otimes L^2$. Since this is a direct sum of a finite number of copies of L^2, using the first homomorphism theorem as before shows that L divides L^2. Q.E.D.

Remark. For the reader familiar with Krohn–Rhodes theory, $L^1 + Q \otimes L^2$ plays the role of the wreath product. Notice that the choice of Q depends on the linking map and there is not one choice that works for all possible linking maps.

6. Conclusion. We would like to suggest two areas where extension of the current line of research might prove fruitful. One nice thing about cascade decomposition of systems is that it exposes the dynamic relations between the state variables. For example from Theorem 3 we know that a system with a finite dimensional solvable Lie algebra admits a cascade decomposition of one dimensional systems. When described in local coordinates this amounts to a lower triangular form for the dynamics. Kelley [15] has suggested that this form would be useful in applying singular perturbation techniques to nonlinear problems.

A second area of research would be in introducing dynamic compensation or state variable feedback as has been done in the linear case by numerous authors. (We refer the reader to [16] for an excellent treatment and bibliography.)

The feedback law $v: M \times \Omega \to \Omega$ would in general be a nonlinear function so that the dynamics becomes

$$\dot{x} = f(x, v(x, u)).$$

To a certain extent we have already considered this by allowing the control law of the second system of the cascade to depend on the control and state of the first system, that is, we have allowed state variable feedforward.

Such a dynamic compensator v completely changes the Lie algebra of the system as we have defined it. (If L were redefined as the Lie algebra generated by $f(\cdot, u(\cdot))$ for all analytic $u: M \to \Omega$ it would not change.) it would be of interest to know when two systems are equivalent under dynamic compensation or when a given system is equivalent to a simpler type of system, i.e., cascade, a bilinear, or a linear system.

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