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# Local Approximation of Control Systems

ARTHUR J. KRENER

*Department of Mathematics, University of California, Davis, California 95616*

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Given two control systems where the control enters linearly, a sufficient condition is derived that one system locally approximates the other, i.e., there exists a map between the state spaces which carries the trajectory of the first system for any control into the trajectory of the second system for the same control with an error that grows like a power of $t$.

## 1. INTRODUCTION

We consider the two control systems

$$\dot{x} = \dot{x}(x, u) = \sum_{i=0}^{h} u_i a_i(x), \tag{1.1}$$

$$x(0) = x^0, \qquad u(t) \in \Omega,$$

and

$$\dot{y} = \dot{y}(y, u) = \sum_{i=0}^{h} u_i b_i(y), \tag{1.2}$$

$$y(0) = y^0, \qquad u(t) \in \Omega,$$

where $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_n)$, $a_0(x), \ldots, a_h(x)$, $b_0(y), \ldots, b_h(y)$ are $C^\infty$ vector fields, $u(t) = (u_0(t), \ldots, u_h(t))$ is a measurable control and

$$\Omega = \{u : |u_i| \leqslant 1, i = 0, \ldots, h\}.$$

We intend to give a sufficient condition for the existence of a $C^\infty$ map, $\lambda: x \mapsto y$, an integer, $\mu$, and real numbers, $M$ and $T \geqslant 0$, such that for any solutions, $x(t)$ and $y(t)$, of (1.1) (1.2) corresponding to the same control, we have

$$|\lambda(x(t)) - y(t)| \leqslant Mt^{\mu+1} \tag{1.3}$$

for $t \in [0, T]$. Notice that this implies a similar result for the subsystem of (1.1) and (1.2) obtained by constraining $u_0(t)$ to be identically 1.

This paper is an extension of our earlier work [1], which gave necessary and sufficient conditions for $\lambda(x(t)) = y(t)$ for small $t$. That result has been extended by Sussmann [2]. We conjecture that our sufficient conditions is also necessary.

## 2. Preliminaries

If $a_i(x)$, $a_j(x)$ are $m$-dimensional vector fields, we define the *Lie bracket*, $[a_i, a_j](x)$ another $m$-dimensional vector field by

$$[a_i, a_j](x) = \frac{\partial a_j}{\partial x}(x)\, a_i(x) - \frac{\partial a_i}{\partial x}(x)\, a_j(x) \tag{2.1}$$

where $(\partial a_j/\partial x)(x)$ is the matrix of partial derivatives.

Let $(t, x) \to \alpha_i(t)x$ denote the *flow* or *family of integral curves* of $a_i(x)$, that is,

$$\frac{d}{dt}\alpha_i(t)x = a_i(\alpha_i(t)x), \qquad \alpha_i(0)x = x. \tag{2.2}$$

A more standard notation is $\alpha_i(t, x)$ since $\alpha_i(t)$ is not a linear operator on $x$ but we will be concatenating these flows and so $\alpha_i(t)x$ will be more convenient.

For fixed $t$, the map $x \to \alpha_i(-t)x$ is a diffeomorphism from a neighborhood of $\alpha_i(t)\, x^0$ onto a neighborhood of $x^0$ and has a tangent map denoted by $\alpha_i(-t)_*$. The derivative of the vector valued curve $t \mapsto \alpha_i(-t)_*\, a_j(\alpha_i(t)\, x^0)$ at $t = 0$ is $[a_i, a_j](x^0)$, [3, p. 17]. Therefore the Taylor series of this curve is given by the *Campbell–Baker–Hausdorff formula*,

$$\alpha_i(-t)_* a_j(\alpha_i(t)\, x^0) = \sum_{k=0}^{\nu} \frac{t^k}{k!}\, ad^k(a_i)\, a_j(x^0) + \mathcal{O}(t^{\nu+1}) \tag{2.3}$$

where

$$ad^0(a_i)\, a_j = a_j \qquad \text{and} \qquad ad^k(a_i)\, a_j = [a_i,\ ad^{k-1}(a_i)\, a_j]. \tag{2.4}$$

The *order of a bracket* of $a_0, \ldots, a_h$ is defined as follows: $a_i$ is of order 1, $[a_i, a_j]$ is of order 2 and in general $[a_{i_1}[\ldots[a_{i_{k-1}}, a_{i_k}]\ldots]]$ is of order $k$. A linear map $l\colon \mathbb{R}^m \to \mathbb{R}^n$ *preserves brackets* of (1.1) and (1.2) to order $\mu$ if

$$l([a_{i_1}[\cdots[a_{i_{k-1}}, a_{i_k}]\cdots]](x^0)) = [b_{i_1}[\cdots[b_{i_{k-1}}, b_{i_k}]\cdots]](y^0) \tag{2.5}$$

for $1 \leqslant k \leqslant \mu$, $0 \leqslant i_j \leqslant h$.

A $C^\infty$ map $\lambda : x \mapsto y$ *preserves solutions* of (1.1) and (1.2) to order $\mu$ if there exists a $T > 0$ and $M$ such that for any solutions, $x(t)$ and $y(t)$, of (1.1) and (1.2) using the same control, we have

$$|\lambda(x(t)) - y(t)| \leqslant Mt^{\mu+1} \tag{2.6}$$

for $t \in [0, T]$.

The *rank* of (1.1) at $x^0$ is the dimension of the span of $a_1, \ldots, a_h$ and their brackets evaluated at $x^0$. Henceforth we shall assume that the rank of (1.1) at $x^0$ is $m$, if this is not true then possibly the system (1.1) can be restricted to a submanifold of $x$ space where this rank condition will hold (see [1]).

THEOREM. *If there exists a linear map* $l: \mathbb{R}^m \to \mathbb{R}^n$ *which preserves brackets of* (1.1) *and* (1.2) *to order* $\mu$ *then there exists a* $C^\infty$ *map* $\lambda: \mathbb{R}^m \to \mathbb{R}^n$ *which preserves solutions of* (1.1) *and* (1.2) *to order* $\mu$.

## 3. BOUNDS ON $x(t)$ AND $\dot{x}(x, u)$

Before proving this theorem it is necessary to obtain uniform bounds on $x(t)$ and $\dot{x}(x, u)$. These are most conveniently expressed after a change of coordinates. To define the new coordinates we choose from $a_0, \ldots, a_h$ a maximal set of vector fields which are linearly independent at $x^0$, relabeling them $c_1, \ldots, c_j$. From the brackets of order 2 of $a_0, \ldots, a_h$, we choose a maximal set of vector fields, relabeled $c_{j-1}, \ldots, c_k$, such that $c_1, \ldots, c_k$ are linearly independent at $x^0$. Continuing on in this fashion eventually, because of the rank assumption, we obtain a set of vector fields $c_1, \ldots, c_m$ which are linearly independent at $x^0$ and hence span $\mathbb{R}^m$. Let $\theta(i)$ denote the order of $c_i$; from the way these vector fields were chosen any bracket of $a_0, \ldots, a_h$ of order $\varphi$ is a linear combination at $x^0$ of $\{c_i(x^0): \theta(i) \leqslant \varphi\}$.

Let $(t, x) \mapsto \alpha_i(t)x$ be the flow of $c_i$ and $s = (s_1, \ldots, s_m)$. Define a map $s \mapsto x(s)$ by

$$x(s) = \alpha_m(s_m) \cdots \alpha_1(s_1)\, x^0. \tag{3.1}$$

Since $\partial x/\partial s_i(0) = c_i(x^0)$, this map has an inverse $x \mapsto s(x)$ defined in some compact neighborhood, $B$, of $x^0$. In this neighborhood, $s_1, \ldots, s_m$ are coordinates. Under this change of coordinates (1.1) becomes

$$\dot{s} = \dot{s}(s, u) = \frac{\partial s}{\partial x}(x(s)) \sum_{i=0}^{h} u_i a_i(x(s)). \tag{3.2}$$

Let $|s| = \max\{|s_1|, \ldots, |s_m|\}$ and choose $M$, $N$ such that if $|s| \leqslant M$ then $x(s) \in B$ and $|\dot{s}(s, u)| \leqslant N$ for all $u \in \Omega$. Let $T = M/N$. Any solution, $s(t)$, of (3.2) satisfies

$$|s(t)| \leqslant Nt, \qquad \text{for} \quad t \in [0, T]. \tag{3.3}$$

Because of the special character of the $s$ coordinates a stronger conclusion can be reached, namely that for any solution of (3.2),

$$|\dot{s}_i(t)| \leqslant Mt^{\theta(i)-1} \qquad \text{and} \qquad |s_i(t)| \leqslant Mt^{\theta(i)} \tag{3.4}$$

for some new constant $M$ and for $t \in [0, T]$. This is shown by an inductive argument.

The core of the argument is to demonstrate that if for $t \in [0, T]$ and

$$|s_i| \leqslant \begin{cases} Mt^{\theta(i)}, & \theta(i) \leqslant \nu - 1, \\ Mt^{\nu-1}, & \theta(i) \geqslant \nu, \end{cases} \tag{3.5}$$

for some $M$ and $\nu \geqslant 1$ then for some $N$

$$|\dot{s}_i(s, u)| \leqslant \begin{cases} Nt^{\theta(i)-1}, & \theta(i) \leqslant \nu, \\ Nt^{\nu-1}, & \theta(i) \geqslant \nu + 1. \end{cases} \tag{3.6}$$

Once this is shown then any solution, $s(t)$, of (3.2) satisfies (3.3) and hence (3.5) with $\nu = 2$. This implies (3.6) which, when integrated, yields (3.5) with $\nu = 3$, and so on until (3.4) is demonstrated. Of course the constants $M$ and $N$ are constantly changing during the argument but the interval $[0, T]$ remains fixed.

To show that (3.5) implies (3.6) we proceed by induction on $\nu$, starting with $\nu = 1$. If $|s| \leqslant M$, then by compactness there exists $N$ such that $|\dot{s}(s, u)| \leqslant N$. Assume (3.5) implies (3.6) up to $\nu - 1$ so that for every $M$ there exists an $N$ such that if for $t \in [0, T]$ and

$$|s_i| \leqslant \begin{cases} Mt^{\theta(i)}, & \theta(i) \leqslant \nu - 2, \\ Mt^{\nu-2}, & \theta(i) \geqslant \nu - 1, \end{cases} \tag{3.7}$$

then

$$|\dot{s}_i(s, u)| \leqslant \begin{cases} Nt^{\theta(i)-1}, & \theta(i) \leqslant \nu - 1 \\ Nt^{\nu-2}, & \theta(i) \geqslant \nu. \end{cases} \tag{3.8}$$

For some $M$ assume (3.5). The unit vector in $s_i$ direction at $x(s)$ is given in the $x$ coordinate system by

$$\frac{\partial x}{\partial s_i}(s) = \alpha_m(s_m)_* \cdots \alpha_{i+1}(s_{i+1})_* c_i(\alpha_i(s_i) \cdots \alpha_1(s_1)\, x^0) \tag{3.9}$$

and $\dot{s}(s, u)$ in the $x$ coordinate system is given by (1.1). The coefficients of (1.1) in terms of (3.9) for $i = 1,..., m$ are precisely $\dot{s}_1(s, u),..., \dot{s}_m(s, u)$.

Let $g_0(s, u)$ and $g_i(s)$ denote the pull backs of (1.1) and (3.9) to $x^0$ by means of the linear isomorphism $\alpha_1(-s_1)_* \cdots \alpha_m(-s_m)_*$. Then

$$g_0(s, u) - \sum_{\theta(i)<\nu} \dot{s}_i(s, u)\, g_i(s) = \sum_{\theta(i)\geqslant\nu} \dot{s}_i(s, u)\, g_i(s). \tag{3.10}$$

Since $c_1(x^0),..., c_m(x^0)$ form a basis, there exists functions $\sigma_1(s, u),..., \sigma_m(s, u)$, linear in $u$ such that

$$g_0(s, u) - \sum_{\theta(i)<\nu} \dot{s}_i(s, u)\, g_i(s) = \sum_{i=1}^{m} \sigma_i(s, u)\, c_i(x^0). \tag{3.11}$$

Together they yield

$$\sum_{\theta(i)\geqslant\nu} \dot{s}_i(s, u)\, g_i(s) = \sum_{i=1}^{m} \sigma_i(s, u)\, c_i(x^0), \tag{3.12}$$

and subtracting $\sum_{\theta(i)\geqslant\nu} \dot{s}_i(s, u)\, c_i(x^0)$ from both sides we have

$$\sum_{\theta(i)\geqslant\nu} \dot{s}_i(s, u)(g_i(s) - c_i(x^0)) = \sum_{\theta(i)<\nu} \sigma_i(s, u)\, c_i(x^0) + \sum_{\theta(i)\geqslant\nu} (\sigma_i(s, u) - \dot{s}_i(s, u))\, c_i(x^0). \tag{3.13}$$

Since $c_i(x^0)$ is precisely the constant term in the Taylor series of $g_i(s)$ given by repeated application of the Campbell-Baker-Hausdorff formula, then (3.5) implies that for some $N$ and $t \in [0, T]$

$$| g_i(s) - c_i(x^0)| \leqslant Nt. \tag{3.14}$$

This, together with (3.8), implies that for some $N$

$$\left| \sum_{\theta(i)\geqslant\nu} \dot{s}_i(s, u)(g_i(s) - c_i(x^0))\right| \leqslant Nt^{\nu-1} \tag{3.15}$$

for all $u \in \Omega$. Since $c_1(x^0),\ldots, c_m(x^0)$ are linearly independent, applying (3.15) to (3.13) yields for some new $N$,

$$| \sigma_i(s, u) - \dot{s}_i(s, u)| \leqslant Nt^{\nu-1} \tag{3.16}$$

for all $u \in \Omega$ and $\theta(i) \geqslant \nu$. Therefore (3.6) follows from (3.8) and (3.16) if

$$| \sigma_i(s, u)| \leqslant Nt^{\nu-1}, \qquad \theta(i) \geqslant \nu. \tag{3.17}$$

Expanding $g_0(s, u)$ and $g_i(s, u)$ in a power series by repeated application of the Campbell-Baker-Hausdorff formula, we have

$$g_0(s, u) = \sum_{j=0}^{h} u_j \left(\sum_{k_1=0}^{\mu} \frac{s_1^{k_1}}{k_1} ad^{k_1}(c_1)\right) \cdots \left(\sum_{k_m=0}^{\mu} \frac{s_m^{k_m}}{k_m!} ad^{k_m}(c_m)\right) a_j(x^0) + \mathcal{O}(| s |^{\mu+1}) \tag{3.18}$$

and

$$g_i(s) = \left(\sum_{k_1=0}^{\mu} \frac{s_1^{k_1}}{k_1!} ad^{k_1}(c_1)\right) \cdots \left(\sum_{k_i=0}^{\mu} \frac{s_i^{k_i}}{k_i!} ad^{k_i}(c_i)\right) c_i(x^0) + \mathcal{O}(| s |^{\mu+1}) \tag{3.19}$$

Since $a_j$ is of order 1, the coefficient of a bracket of order $\varphi \leqslant \mu$ in (3.18) is the sum of monomials of the form $K_{r_1} \cdots {}_{r_p} s_{r_1} \cdots s_{r_p}$ where $K_{r_1} \cdots {}_{r_p}$ is a constant and $\theta(r_1) + \cdots + \theta(r_p) = \varphi - 1$. Because $s$ satisfies (3.5), we can conclude that there exists an $N$ such that the coefficient is bounded by $Nt^{\varphi-1}$, if $1 \leqslant \varphi \leqslant \nu$, and by $Nt^{\nu-1}$, if $\nu + 1 \leqslant \varphi \leqslant \mu$. There are only a finite number of brackets of order $\leqslant \mu$ and $u$ enters linearly so an $N$ can be found which works for all brackets in (3.18) and all $u \in \Omega$.

Similarly, since $c_i$ is of order $\theta(i)$, there exists a new $N$ such that the coefficient of a bracket of oder $\varphi$ in (3.19) is bounded by $Nt^{\varphi-\theta(i)}$, if $\theta(i) \leqslant \varphi \leqslant \nu + \theta(i) - 1$ and by $Nt^{\nu-1}$, if $\nu + \theta(i) \leqslant \varphi \leqslant \mu$. By the induction hypothesis, (3.8), and the above remarks we see that the coefficient of a bracket of order $\varphi$ in the series expansion of the left side of (3.11) is bounded by $Nt^{\varphi-1}$, if $1 \leqslant \varphi \leqslant \nu$ and by $Nt^{\nu-1}$, if $\nu + 1 \leqslant \varphi \leqslant \mu$.

Since each bracket of order $\varphi$ is a linear combination at $x^0$ of $\{c_i(x^0)\colon \theta(i) \leqslant \varphi\}$ we conclude that

$$| \sigma_i(s, u) | \leqslant \begin{cases} Nt^{\theta(i)-1}, & 1 \leqslant \theta(i) \leqslant \nu, \\ Nt^{\nu-1}, & \theta(i) \geqslant \nu, \end{cases} \tag{3.18}$$

as was desired.

4. *Proof.* Having verified (3.4) we proceed with the proof of the theorem. Each $c_j$ is of the form $[a_{i_1}[\cdots[a_{i_{k-1}}, a_{i_k}]\cdots]]$, so let

$$d_j = [b_{i_1}[\cdots [b_{i_{k-1}}, b_{i_1}] \cdots]].$$

If $(t, y) \mapsto \beta_j(t)y$ is the flow of $d_j$, define $y(s) = \beta_m(s_m) \cdots \beta_1(s_1) y^0$ and $\lambda(x) = y(s(x))$ for $s \in B$.

The tangent map, $\lambda_* = \partial y/\partial x$, maps tangent vectors at $x(s)$ to tangent vectors at $y(s)$ and in particular

$$\lambda_* \frac{\partial x}{\partial s_i}(s) = \frac{\partial y}{\partial s_i}(s). \tag{4.1}$$

A tangent vector at $x(s)$ can be pulled back to a tangent vector at $x^0$ by $\alpha_1(-s_1)_* \cdots \alpha_m(-s_m)_*$ then mapped into a tangent vector at $y^0$ by the linear map $l$ and finally pulled out to a tangent vector at $y(s)$ by $\beta_m(s_m)_* \cdots \beta_1(s_1)_*$. Let $\pi$ denote this map

$$\pi = \beta_m(s_m)_* \cdots \beta_1(s_1)_* \, l\alpha_1(-s_1)_* \cdots \alpha_m(-s_m)_* . \tag{4.2}$$

The two maps $\lambda_*$ and $\pi$ are almost the same in the following sense, if for some $M$

$$| s_i | \leqslant Mt^{\theta(i)} \qquad \text{for} \quad i = 1,\ldots, m \tag{4.3}$$

then for some $N$

$$\left| \lambda_* \frac{\partial x}{\partial s_i}(s) - \pi \frac{\partial x}{\partial s_i}(s) \right| \leqslant Nt^{\mu-\theta(i)+1}. \tag{4.4}$$

This can be easily shown by comparing the Taylor series of

$$l\alpha_1(-s_1)_* \cdots \alpha_m(-s_m)_* \,\partial x/\partial s_i$$

with the Taylor series of $\beta_1(-s_1)_* \cdots \beta_m(-s_m)_* \,\partial y/\partial s_i$ using the assumption that $l$ preserves brackets to order $\mu$ and noting that the coefficient of a bracket of order $\varphi$ in both these series is bounded by $Nt^{\varphi-\theta(i)}$. Then use the fact that $s(B)$ is compact to obtain a bound on the norms of the linear maps

$$\beta_m(s_m)_* \cdots \beta_1(s_1)_* \qquad \text{for} \quad s \in s(B).$$

If $x(t)$ is the solution of (1.1) for $u(t)$ and $s(t) = s(x(t))$ then

$$\dot{x}(t) = \sum_{i=1}^{m} \dot{s}_i(t) \frac{\partial x}{\partial s_i} (s(x(t))), \tag{4.5}$$

where $s(t)$ and $\dot{s}(t)$ satisfy (3.4). Therefore

$$|\lambda_* \dot{x}(t) - \pi \dot{x}(t)| = \left| \sum_{i=1}^{m} \dot{s}_i(t) \left(\lambda_* \frac{\partial x}{\partial s_i} (s(t)) - \pi \frac{\partial x}{\partial s_i} (s(t))\right)\right| \leqslant Mt^{\mu} \tag{4.6}$$

for some $M$ and $t \in [0, T]$.

Next we show that

$$\left| \pi \sum_{i=0}^{h} u_i(t)\, a_i(x(t)) - \sum_{i=0}^{h} u_i(t)\, b_i(\lambda(x(t)))\right| \leqslant Mt^{\mu}, \tag{4.7}$$

for some $M$ by comparing the Taylor series of

$$l\alpha_1(-s_1)_* \cdots \alpha_m(-s_m)_* \sum u_i(t)\, a_i(x(t))$$

and $\beta_1(-s_1)_* \cdots \beta_m(-s_m)_* \sum u_i(t)\, b_i(y(x(t)))$ once again using the assumption that $l$ preserves brackets to order $\mu$, noting that the coefficient of a bracket of order $\varphi$, in both these series, is bounded by $Nt^{\varphi-1}$ and bounding the norms of $\beta_m(s_m)_* \cdots \beta_1(s_1)_*$. From (4.6) and (4.7) we have

$$\left| \lambda_* \sum_{i=0}^{h} u_i(t)\, a_i(x(t)) - \sum_{i=0}^{h} u_i(t)\, b_i(\lambda(x(t)))\right| \leqslant Mt^{\mu}. \tag{4.8}$$

Now let $y(t)$ be the solution of (1.2) for $u(t)$, then

$$\begin{aligned} |\lambda(x(t)) - y(t)| &\leqslant \int_0^t \left| \lambda_* \sum_{i=0}^{h} u_i(\tau)\, a_i(x(\tau)) - \sum_{i=0}^{h} u_i(\tau)\, b_i(y(\tau))\right| d\tau \\ &\leqslant \int_0^t \left| \lambda_* \sum_{i=0}^{h} u_i(\tau)\, a_i(x(\tau)) - \sum u_i(\tau)\, b_i(\lambda(x(\tau)))\right| d\tau \\ &\quad + \int_0^t \left| \sum u_i(\tau)(b_i(\lambda(x(\tau))) - b_i(y(\tau))\right| d\tau. \end{aligned} \tag{4.9}$$

The first integral is bounded by $Mt^{\mu+1}$ from (4.8); as for the second, since each $b_i(y)$ satisfies a Lipschitz condition on the compact set, $\lambda(B)$, we can find an $N$ such that for all $u \in \Omega$,

$$\left| \sum u_i(b_i(y^1) - b_i(y^2)) \right| \leqslant N \mid y^1 - y^2 \mid. \tag{4.10}$$

Hence (4.9) becomes

$$\mid \lambda(x(t)) - y(t) \mid \leqslant Mt^{\mu+1} + N \int_0^t \mid \lambda(x(\tau)) - y(\tau) \mid d\tau. \tag{4.11}$$

Choose $K$ such that $K \geqslant N$ and $K^{\mu+2} \geqslant (\mu + 1)!M$ and let

$$f(t) = \int_0^t \mid \lambda(x(\tau)) - y(\tau) \mid d\tau. \tag{4.12}$$

Then

$$f'(t) \leqslant \frac{K^{\mu+2}}{(\mu+1)!} t^{\mu+1} + Kf(t). \tag{4.13}$$

By a standard comparison theorem [4, p. 25], $f(t) \leqslant g(t)$ where $g(t)$ is the solution of

$$g'(t) = \frac{K^{\mu+2}}{(\mu+1)!} t^{\mu+1} + Kg(t) \tag{4.14}$$

satisfying $g(0) = f(0) = 0$. The solution of (4.14) is

$$g(t) = e^{Kt} - \sum_{j=0}^{\mu+1} \frac{(Kt)^j}{j!} = \sum_{j=\mu+2}^{\infty} \frac{(Kt)^j}{j!} \tag{4.15}$$

and so

$$\begin{aligned} \mid \lambda(x(t)) - y(t) \mid = f'(t) &\leqslant \frac{K^{\mu+2}}{(\mu+1)!} t^{\mu+1} + Kf(t) \\ &\leqslant \frac{K^{\mu+2}}{(\mu+1)!} t^{\mu+1} + Kg(t) \\ &\leqslant K \sum_{j=\mu+1}^{\infty} \frac{(Kt)^j}{j!} \leqslant Mt^{\mu+1} \end{aligned}$$

for some new $M$ and $t \in [0, T]$. Q.E.D.

## 5. Conclusion

This theorem points out that the Lie brackets of $a_0, \ldots, a_h$ evaluated at $x^0$ determine the local behavior of (1.1) in the same fashion as the partial deri-

vatives of a function determine its local behavior. More precisely, the linear relationships between these brackets determine the local behavior of (1.1) up to an affine transformation of $x$ space. Furthermore, (1.1) approximately covers (1.2) if the low order brackets of (1.2) evaluated at $y^0$ have all the linear relations that the low order brackets of (1.1) at $x^0$ have. This allows us to construct a system of lower dimension locally approximating (1.1) by introducing linear relations among the brackets of (1.1).

Inequality (3.4) is of independent interest for it gives bounds on the set of points locally accessible from $x^0$.

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