Chapter 1

On the convergence of normal forms for analytic control systems

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1.1 Background

A fruitful technique for the local analysis of a dynamical system consists of using a local change of coordinates to transform the system to a simpler form, which is called a normal form. The qualitative behavior of the original system is equivalent to that of its normal form which may be easier to analyze. A bifurcation of a parameterized dynamics occurs when a change in the parameter leads to a change in its qualitative properties. Therefore normal forms are useful in analyzing when and how a bifurcation will occur. In his dissertation, Poincaré studied the problem of linearizing a dynamical system around an equilibrium point, linear dynamics being the simplest normal form. Poincaré's idea is to simplify the linear part of a system first, using a linear change of coordinates. Then, the quadratic terms in the system are simplified, using a quadratic change of coordinates, then the cubic terms, and so on. For systems that are not linearizable, the Poincaré-Dulac Theorem provides the normal form.

Given a C^∞ dynamical system in its Taylor expansion around $x = 0$,

$$\dot{x} = f(x) = Fx + f^{[2]}(x) + f^{[3]}(x) + \cdots \tag{1.1}$$

where $x \in \Re^n$, F is a diagonal matrix with eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_n)$, and $f^{[d]}(x)$ is a vector field of homogeneous polynomial of degree d. The dots $+\cdots$ represent the rest of the formal power series expansion of f. Let $\mathbf{e}_k$ be the k-th unit vector in $\Re^n$. Let $m = (m_1, \ldots, m_n)$ be a vector of nonnegative integers. In the following, we define $|x|$ and x^m by $|m| = \sum |m_i|$ and $x^m = x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$. A nonlinear term $x^m \mathbf{e}_k$ is said to be resonant if $m \cdot \lambda = \lambda_k$ for some nonzero vector of nonnegative integers m and some $1 \leq k \leq n$.

Definition 1.1 The eigenvalues of F are in the Poincaré Domain if their convex hull does not contain zero, otherwise they are in the Siegel Domain.

Definition 1.2 The eigenvalues of F are of type (C, ν) for some $C > 0, \nu > 0$ if

$$|m \cdot \lambda - \lambda_k| \geq \frac{C}{|m|^\nu}$$

For eigenvalues in the Poincaré Domain, there are at most a finite number of resonances. For eigenvalues in the Siegel Domain, there are no resonances and as $|m| \to \infty$ the rate at which resonances are approached is controlled.

A formal change of coordinates is a formal power series

$$z = Tx + \theta^{[2]}(x) + \theta^{[3]}(x) + \cdots \tag{1.2}$$

where T is invertible. If $T = I$, then it is called a near identity change of coordinates. If the power series converges locally, then it defines a real analytic change of coordinates.

Theorem 1.1*(Poincaré-Dulac) If the system (1.1) is C^∞ then there exists a formal change of coordinates (1.2) transforming it to*

$$\dot{z} = Az + w(z)$$

where A is in Jordan form and $w(z)$ consists solely of resonant terms. (If some of the eigenvalues of F are complex then the change of coordinates will also be complex). In this normal form $w(z)$ need not be unique.

If the system (1.1) is real analytic and its eigenvalues lie in the Poincaré Domain (1.2), then $w(z)$ is a polynomial vector field and the change of coordinates(1.2) is real analytic.

Theorem 1.2 (Siegel) *If the system (1.1) is real analytic and its eigenvalues are of type (C, ν) for some $C > 0, \nu > 0$, then $w(z) = 0$ and the change of coordinates(1.2) is real analytic.*

As is pointed out in [1], even in cases where the formal series are divergent, the method of normal forms turns out to be a powerful device in the study of nonlinear dynamical systems. A few low degree terms in the normal form often give significant information on the local behavior of the dynamics.

1.2 The open problem

In [3], [4], [5].[10] and [8], Poincaré's idea is applied to nonlinear control systems. A normal form is derived for nonlinear control systems under change of state coordinates and invertible state feedback. Consider a C^∞ control system

$$\dot{x} = f(x,u) = Fx + Gu + f^{[2]}(x,u) + f^{[3]}(x,u) + \cdots \tag{1.3}$$

where $x \in \Re^n$ is the state variable, $u \in \Re$ is a control input. We only discuss scalar input systems but the problem can be generalized to vector input systems. Such a system is called *linearly controllable* at the origin if the linearization (F, G) is controllable.

In contrast with Poincar'e's theory, a homogeneous transformation for (1.3) consists of both change of coordinates and invertible state feedback,

$$z = x + \theta^{[d]}(x), \quad v = u + \kappa^{[d]}(x,u) \tag{1.4}$$

where $\theta^{[d]}(x)$ represents a vector field whose components are homogeneous polynomials of degree d. Similarly, $\kappa^{[d]}(x)$ is a polynomial of degree d. A formal transformation is defined by

$$z = Tx + \textstyle\sum_{d=2}^{\infty} \theta^{[d]}(x), \quad v = Ku + \sum_{d=2}^{\infty} \kappa^{[d]}(x,u) \tag{1.5}$$

where T and K are invertible. If T and K are identity matrices then this is called a near identity transformation.

The following theorem for the normal form of control systems is a slight generalization of that proved in [3], see also [8] and [10].

Theorem 2.1 *Suppose (F, G) in (1.3) is a controllable pair. Under a suitable transformation (1.5), (1.3) can be transformed into the following normal form*

$$\begin{array}{rcl} \dot{z}_i & = & z_{i+1} + \sum_{j=i+2}^{n+1} p_{i,j}(\bar{z}_j) z_j^2 \quad 1 \leq i \leq n-1 \\ \dot{z}_n & = & v \end{array} \tag{1.6}$$

where $z_{n+1} = v$, $\bar{z}_j = (z_1, z_2, \cdots, z_j)$, and $p_{i,j}(\bar{z}_j)$ is a formal series of $\bar{z}_j$.

Once again, the convergence of the formal series $p_{i,j}$ in (1.6) is not guaranteed hence nothing is known about the convergence of the normal form.

Open Problem (The Convergence of Normal Form) *Suppose the controlled vector field $f(x,u)$ in (1.3) is real analytic and F, G is a controllable pair. Find verifiable necessary and sufficient conditions for the existence of a real analytic transformation (1.5) that transforms the system to the normal form (1.6).*

Normal forms of control systems have proven to be a powerful tool in the analysis of local bifurcations and local qualitative performance of control systems. A convergent normal form will make it possible to study a control system over the entire region in which the normal form converges. Global or semi-global results on control systems and feedback design can be proved by studying analytic normal forms.

1.3 Related results

The convergence of the Poincaré normal form was an active research topic in dynamical systems. According to Poincaré's Theorem and Siegel's Theorem, the location of eigenvalues determines the convergence. If the eigenvalues are located in the Poincaré Domain with no resonances, or if the eigenvalues are located in the Siegel Domain and are of type (C, ν), then the analytic vector field that defines the system is biholomorphically equivalent to a linear vector field. Clearly the normal form converges because it has only a linear part. The Poincaré-Dulac Theorem deals with a more complicated case. It states that if the eigenvalues of an analytic vector field belong to the Poincaré domain, then the field is biholomorphically equivalent to a polynomial vector field. Therefore, the Poincaré normal form has only finite many terms, and hence is convergent.

As for control systems, it is proved in [5] that if an analytic control system is linearizable by a formal transformation, than it is linearizable by an analytic transformation. It is also proved in [5] that a class of three dimensional analytic control systems, which are not necessarily linearizable, can be transformed to their normal forms by analytic transformations. No other results on the convergence of control system normal forms are known to us.

The convergence problem for control systems has a fundamental difference from th convergence results of Poincaré-Dulac. For the latter the location of the eigenvalues are crucial and the eigenvalues are invariant under change of coordinates. However, the eigenvalues of a control system can be changed by linear state feedback. It is unknown what intrinsic factor in control systems determines the convergence of their normal form or if the normal form is always convergent.

The convergence of normal forms is an important problem to be addressed. Applications of normal forms for control systems are proved to be successful. In [6], the normal forms are used to classify the bifurcation of equilibrium sets and controllability for uncontrollable systems. In [7], the control of bifurcations using state feedback is introduced based on normal forms. For discrete-time systems, normal form and the stabilization of Naimark-Sacker bifurcation are addressed in [2]. In [10], a complete characterization for the symmetry of nonlinear systems is found for linearly controllable systems.

In addition to linearly controllable systems, the normal form theory has been generalized to larger family of control systems. Normal forms for systems with uncontrollable linearization are derived in several papers ([6], [7], [8], and [10]). Normal forms of discrete-time systems can be found in [9], and [2]. The convergence of these normal forms is also an open problem.

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