Low Codimension Control Singularities

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Abstract

Key Words:

1 Introduction

We consider the class $C^k(\mathcal{X} \times \mathcal{U}, I\!\!R^n)$ of control systems of the form

$$\dot{x} = f(x, u) \tag{1.1}$$

where $x \in \mathcal{X}$, an open subset of $I\!\!R^n$, $u \in \mathcal{U}$, an open subset of $I\!\!R^m$ and f is C^k where $0 \leq k \leq \infty$. The equilibrium set $\mathcal{Z}$ of the control system is the set of all $(x^0, u^0) \in \mathcal{X} \times \mathcal{U}$ such that $f(x^0, u^0) = 0$. We are interested in studying control bifurcations, i.e., equilibria that are more difficult to stabilize than most of their neighboring equilibria. In order to do so we need to study the singular equilibria of all possible control systems. The set of all equilibria of all smooth systems is infinite dimensional. It is most convenient to pose this study in a k-jet space which is finite dimensional.

A linear system of the form

$$\dot{x} = Fx + Gu \tag{1.2}$$

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is *controllable* if the smallest F invariant subspace containing the columns of G is $I\!R^n$. A controllable linear system can be steered from any state to any other state in any $t > 0$.

The *linear part* of the nonlinear system (1.1) around the equilibrium (x^0, u^0) is the system

$$\dot{x} \;=\; F(x - x^0) + G(u - u^0) \tag{1.3}$$

where

$$F = \frac{\partial f}{\partial x}(x^0, u^0), \qquad G = \frac{\partial f}{\partial u}(x^0, u^0).$$

If this linear system is controllable then the nonlinear system can steered from any nearby state to any other nearby state in any $t > 0$.

The generic equilibrium of (1.1) has a controllable linear part so if an equilibrium is not linearly controllable then it is more difficult to control than most of its neighboring equilibria and hence a control singularity. But as we shall see there are linearly controllable equilibria that are more difficult to control than most of their neighboring equilibria. Hence they are also control singularities. The goal of this paper is to study and classify the low codimension control singularities of nonlinear control systems.

The set of all equilibria of all control systems like (1.1) is infinite dimensional but the nature of control singularity frequently depends on the low degree terms of the Taylor series of f at the equilibrium. Therefore instead of studing the infinite dimensional object we study the Taylor series through degree k of all possible systems at all possible equilibria. The latter is called the system k-jet space which we introduce in the next section. In Section 3 we define the extended contollability indices and in Section 4 we study the action of the linear feedback group.

2 Equilibria in the k-Jet Space of Systems

The *system k-jet space* $\mathcal{S}^k(\mathcal{X} \times \mathcal{U}, I\!R^n)$ is the space of all tuples of the form

$$\Big((x, u), f(x, u), f^{(1)}(x, u), \ldots, f^{(k)}(x, u)\Big) \tag{2.1}$$

where $f \in C^k(\mathcal{X} \times \mathcal{U}, I\!R^n)$ and

$$f^{(j)}(x, u) \;=\; \frac{\partial^j f}{\partial (x, u)^j}(x, u).$$

Frequently when there is no chance of ambiguity we shall use the shortened notation $\mathcal{S}^k$. The terminology is a bit misleading, this is not a collection of systems but is a vector bundle with base $\mathcal{X} \times \mathcal{U}$ and fiber a real linear space of dimension $N(n, m, k)$ where

$$N(n, m, k) = n \begin{pmatrix} n + m + k \\ k \end{pmatrix}.$$

Moreover the notation (2.1) is very convenient but can be misleading. Each $f^{(j)}(x, u) = (f_1^{(j)}, \ldots, f_n^{(j)})'$ and each $f_i^{(j)}$ is actually a symmetric tensor of degree j in $n + m$ indices. There is a natural projection of $\mathcal{S}^k$ onto $\mathcal{S}^l$ when $k \geq l \geq 0$.

A system (1.1) *realizes* a point in $\mathcal{S}^k$ if it has those derivatives at $\mathcal{X} \times \mathcal{U}$. Any point in $\mathcal{S}^k$ can be realized by a polynomial system of degree k but there are many other realizations.

It is more convenient to work with the systems jet space $\mathcal{S}^k(\mathcal{X} \times \mathcal{U}, I\!R^n)$ which is finite dimensional than the space of systems $C^k(\mathcal{X} \times \mathcal{U}, I\!R^n)$ which is infinite dimensional, particularly when studying a system locally around a particular (x, u) such as an equilibrium. The equilibrium set $\mathcal{Z}(f)$ of a system (1.1) is the set of all pairs $(x, u) \in \mathcal{X} \times \mathcal{U}$ such that $f(x, u) = 0$.

The equilibrium set $\mathcal{E}^k(\mathcal{X} \times \mathcal{U}, I\!R^n) \subset \mathcal{S}^k(\mathcal{X} \times \mathcal{U}, I\!R^n)$ is the space of all tuples of the form

$$\left((x, u), 0, f^{(1)}(x, u), \ldots, f^{(k)}(x, u)\right). \tag{2.2}$$

Again when there is no chance of ambiguity we shall use the shortened notation $\mathcal{E}^k$. The equilibrium set $\mathcal{E}^k$ is also a vector bundle with base $\mathcal{X} \times \mathcal{U}$ and fiber a real linear space of dimension $N(n, m, k) - n$. Clearly $\mathcal{E}^k$ is a subbundle of $\mathcal{S}^k$ and it is carried onto $\mathcal{E}^l$, $0 \leq l \leq k$ by the natural projection. Notice that $\mathcal{Z}(f) \subset \mathcal{X} \times \mathcal{U}$ and depends on the system (1.1) but $\mathcal{E}^k \subset \mathcal{S}^k$ and we don't need a system to define it.

A k-jet in $\mathcal{E}^k$ is a *control singularity* it is more difficult to control than most of its neighboring tuples in $\mathcal{E}^k$. Our goal is to study the classes of control singularities that are of low codimension in $\mathcal{E}^k$. In particular we will classify all of codimension one or two.

There is another jet bundle that is of interest. The feedback k-jet bundle $\mathcal{K}^k$ is the set of all tuples of the form

$$\left(x, \kappa(x), \kappa^{(1)}(x), \ldots, \kappa^{(k)}(x)\right) \tag{2.3}$$

where $\kappa : x \mapsto u$ is C^k mapping from $\mathcal{X}$ to $\mathcal{U}$ and

$$\kappa^{(j)}(x) \quad = \quad \frac{\partial^j \kappa}{\partial x^j}(x).$$

The maps $\kappa(x)$ are feedbacks and $\mathcal{K}^k$ is a fiber bundle with base $\mathcal{X}$ and fiber $\mathcal{U} \times I\!R^{M(n,k)}$ where

$$M(n,k) = m \begin{pmatrix} n+k \\ k \end{pmatrix} - m.$$

Given an equilibrium (x^0, u^0) of the system (1.1), a typical goal is to find a smooth feedback such that the closed loop system

$$\dot{x} \quad = \quad f(x, \kappa(x)) \tag{2.4}$$
$$u \quad = \quad \kappa(x) \tag{2.5}$$

is locally asymptotically stable to (x^0, u^0). Frequently the stability of the closed loop system can be decided by its k-jet at x^0 for small k. And the k-jet of the closed loop system can be computed from the k-jet of the system at (x^0, u^0) and the k-jet of the feedback at x^0 assuming that $\kappa(x^0) = u^0$.

Therefore we say an equilibrium k-jet with base point (x^0, u^0) is *stabilizable* if there exists a feedback k-jet with base point x^0 so that every realization of the former makes every realization of the latter locally asymptotically stable to (x^0, u^0).

3 Extended Controllability Indices

Given an equilibrium (x^0, u^0) of the system (1.1) we can define its tuple of *controllability indices* (also called Kronecker indices) as follows. Define

$$F = \frac{\partial f}{\partial x}(x^0, u^0), \qquad G = \frac{\partial f}{\partial u}(x^0, u^0)$$

then the controllability matrix of this pair is

$$\left[\begin{array}{cccc} G & FG & \dots & F^{n-1}G \end{array} \right].$$

This is an $n \times nm$ matrix of rank r, $0 \leq r \leq n$. The span of the columns of this matrix is an F invariant subspace of dimension r denoted by $\mathcal{V}$. If $r = n$

then the pair F, G is said to be *controllable* and the system (1.1) is said to be *linearly controllable* at (x^0, u^0).

Starting from the left, we delete columns of the controllability matrix which are linearly dependent on the columns to the left. At the end of this process, we obtain an $n \times r$ matrix of rank r which after reordering of the columns is of the form

$$\left[\begin{array}{ccccccccc} G_{.1} & FG_{.1} & \dots & F^{r_1-1}G_{.1} & \dots & G_{.m} & FG_{.m} & \dots & F^{r_m-1}G_{.m} \end{array}\right]$$

where $G_{.j}$ denotes the j^{th} column of G and $r = r_1 + \cdots + r_m$. After reordering the columns of G we can assume that $r \geq r_1 \geq r_2 \geq \ \cdots \geq r_m \geq 0$. The *controllability indices* of the equilibrium (x, u) is the m tuple $(r_1, \dots, r_m)$.

We define the *extended controllability indices* of the equilibrium (x^0, u^0) as the $m+1$ tuple $(r_0, r_1, \dots, r_m)$ where $r_0 = n - r$. The interpertation of this is as follows, r_0 is the number of state dimensions that can't be controlled by linear effects and r_j is the number of dimensions that can be controlled by the linear effects of u_j with the least number of overall integrations. Even if the equilibrium is linearly contollable, it is difficult to control if it has any large controllablity indices.

A $m + 1$ tuple of nonnegative integers $(s_0, s_1, \dots, s_m)$ is an admissible tuple of controllability indices for a system with n dimensional state and m dimensional control if $n \geq s_1 \geq s_2 \geq \cdots \geq s_m \geq 0$ and $s_0 + s_1 + \cdots + s_l = n$. The set of all such admissible indices is partially ordered as follows. The tuple $(r_0, \dots, r_m)$ is smaller than the tuple $(s_0, \dots, s_m)$ if they are not identical and

$$\sum_{i=0}^{k} r_i \leq \sum_{i=0}^{k} s_i$$

for $k = 0, \dots, r_m$.

The easiest equilibrium of a system with n states and m inputs to control is the one where $(r_0, r_1, \dots, r_m)$ is as small as possible. For then $r_0 = 0$ so there are no uncontrollable states and the control of the n states is as evenly distributed as possible over the m controls. The larger the extended controllabilities indices are, the more difficult it is to control the system.

In a similar fashion we can define the extended controllability indices of a point of $\mathcal{E}^k$. We define $\mathcal{C}^k(r_0, r_1, \dots, r_m)$ as the subset of $\mathcal{E}^k$ with extended controllability indices $(r_0, r_1, \dots, r_m)$. The tuple of extended controllability indices of a generic system with n dimensional state and m dimensional input is the smallest admissible tuple, i.e., $r_0 = 0$ and $r_j,\ 1 \leq j \leq m$ is as close to $\frac{n}{m}$ as possible. Let l be the largest integer not exceeding $\frac{n}{m}$, then the generic $m+1$ tuple of extended controllability indices is $(0, l+1, \ldots, l+1, l, \ldots, l)$ where the numbers of l and $l+1$ indices are chosen to make the sum of the indices equal to n. Let $\mathcal{C}^{k*} = \mathcal{C}^k(0, l+1, \ldots, l+1, l, \ldots, l)$ then this is a generic subset (open and dense) of the equilibrium set $\mathcal{E}^k$. Therefore we call this tuple the *generic tuple of extended controllability indices* of system with n states and m inputs.

The singular subset of $\mathcal{E}^k$ is in the compliment $\mathcal{C}^{kc}$ of $\mathcal{C}^{k*}$. An element of this set is more difficult to control than most of its neighbors because it always has neighbors with smaller extended controllability indices. A 1-jet in $\mathcal{C}^k(r_0, r_1, \ldots, r_m)$ can be perturbed to any 1-jet with smaller extended controllability indices. We are interested in studying particular types of singularities, those of low codimension in $\mathcal{E}^k$.

4 Linear Feedback Group and Linear Normal Form

The linear feedback group acts on systems and so it acts on system k-jets. An element of the *linear feedback group* is of the form

$$\begin{bmatrix} T & 0 \\ K & L \end{bmatrix} \tag{4.1}$$

where T is an $n \times n$ invertible matrix, K is an $m \times n$ matrix and L is an $m \times m$ invertible matrix. Together they define a linear change of state coordinates and a linear feedback on (1.1)

$$\begin{array}{rcl} x - x^0 & = & Tz \\ u - u^0 & = & Kz + Lv \end{array} \tag{4.2}$$

which takes the equilibrium at (x^0, u^0) of the system (1.1) to the equilibrium at $(0, 0)$ of

$$\dot{z} \;=\; \bar{f}(z, v) \;=\; T^{-1} f(x^0 + Tz, u^0 + Kz + Lv). \tag{4.3}$$

This induces a mapping from $\mathcal{E}^k$ to $\mathcal{E}^k$. The k jet

$$\left((x^0, u^0), 0, \frac{\partial f}{\partial (x,u)}(x^0, u^0), \ldots, \frac{\partial^k f}{\partial (x,u)^k}(x^0, u^0) \right) \tag{4.4}$$

goes to

$$\left((0,0), 0, \frac{\partial \bar{f}}{\partial(z,v)}(0,0), \dots, \frac{\partial^k \bar{f}}{\partial(z,v)^k}(0,0)\right) \tag{4.5}$$

If the linear part of the k-jet is $[F \;\; G]$ then it is changed to

$$\left[\begin{array}{cc} \bar{F} & \bar{G} \end{array}\right] = T^{-1}\left[\begin{array}{cc} F & G \end{array}\right]\left[\begin{array}{cc} T & 0 \\ K & L \end{array}\right].$$

Given any system (1.1) there is an element of the linear feedback group which takes the linear part of the system into linear normal form. A system

$$\dot{z} = Az + Bv + O(z,v)^2 \tag{4.6}$$

is in *linear normal form* at the equilibrium $(z,v) = (0,0)$ if

$$A = \left[\begin{array}{cc} A_0 & 0 \\ 0 & A_1 \end{array}\right], \qquad B = \left[\begin{array}{c} 0 \\ B_1 \end{array}\right] \tag{4.7}$$

where the $r_0 \times r_0$ matrix A_0 is in real Jordan form and the pair consisting of the $r \times r$ matrix A_1 and the $r \times m$ matrix B_1 is in Brunovsky form.

The former means that A_0 is a block diagonal matrix with diagonal blocks of the form

$$\left[\begin{array}{ccccc} \Lambda_1 & I & 0 & \dots & 0 \\ 0 & \Lambda_2 & I & \dots & 0 \\ & & \ddots & \ddots & \\ 0 & 0 & 0 & \ddots & I \\ 0 & 0 & 0 & \dots & \Lambda_s \end{array}\right] \tag{4.8}$$

where Λ_i is a scalar

$$\Lambda_i = a_i$$

or a 2×2 matrix of the form

$$\Lambda_i = \left[\begin{array}{cc} a_i & -\omega_i \\ \omega_i & a_i \end{array}\right]$$

with $\omega_i \neq 0$

The latter means that

$$
\begin{aligned}
A_1 &= \mathrm{Diag}\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}^{r_j \times r_j} \\
B_1 &= \mathrm{Diag}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}^{r_j \times 1}
\end{aligned}
$$

for those $r_j > 0$. If $r_{\tau+1} = \ldots = r_m = 0$ then $m - \tau$ identically zero columns are added on the left of B_1.

An equilibrium 1-jet in normal form is

$$
\left((0,0), 0, \begin{bmatrix} A_0 & 0 & 0 \\ 0 & A_1 & B_1 \end{bmatrix} \right) \tag{4.9}
$$

where A_0, A_1, B_1 are as above.

As an example consider the scalar input system

$$
\begin{aligned}
\dot{z}_0 &= a z_0 + O(z,v)^2 && (4.10) \\
\dot{z}_1 &= A_1 z_1 + B_1 v + O(z,v)^2 && (4.11)
\end{aligned}
$$

where the $z_0 \in I\!R,\ z_1 \in I\!R^{n-1}$ and

$$
\begin{aligned}
A_0 &= a \neq 0 \\
A_1 &= \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}^{r_1 \times r_1} \\
B_1 &= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}^{r_1 \times 1} .
\end{aligned} \tag{4.12}
$$

Its 1-jet at the origin is

$$\left((0,0),0,\begin{bmatrix} a & 0 & 0 \\ 0 & A_1 & B_1 \end{bmatrix}\right) \tag{4.13}$$

This equilibrium 1-jet is the simplest example of a control singularity and is called a fold. The subset $\mathcal{F} \subset \mathcal{E}^k$ of *fold singularities* is the set of all 1-jets of the form

$$\left((x^0,u^0),0,\begin{bmatrix} F_{00} & F_{01} & G_0 \\ F_{10} & F_{11} & G_1 \end{bmatrix}\right) \tag{4.14}$$

whose normal form is (4.13) for some $a \neq 0$. We shall show that $\mathcal{F}$ is of codimension one in $\mathcal{E}^k$.

There is also a nonlinear feedback group that acts on (1.1). It includes the linear feedback group. It consists of C^k changes of state coordinates and state feedback of the form

$$\begin{array}{rcl} x & = & \theta(z) \\ u & = & \kappa(z,v) \end{array} \tag{4.15}$$

where $(z,v) \mapsto (x,u)$ is a local diffeomorphism. This induces a corresponding action on points of $\mathcal{E}^k$. The nonlinear feedback group is stratified. There are near identity transformations of degree d which are of the form

$$\begin{array}{rcl} x & = & z + \theta^{[d]}(z) \\ u & = & v + \kappa^{[d]}(z,v) \end{array} \tag{4.16}$$

where the superscript $^{[d]}$ indicates a polynomial vector field homogeneous of degree d. These do not form a subgroup as the composition of two such transformations typically has terms of degree d through d^2. A near identity transformation of degree d does not change the $d-1$ jet but it does modify the d-jet and higher jets. This allows one to bring the higher degree terms to normal form [2]

5 The Codimension of Orbits of the Linear Feedback Group

Control singularities are invariant under the linear and nonlinear feedback groups. If the original system (1.1) has a control singularity at an equilibrium

(x^0, u^0) then the transformed system has the same type of control singularity at the transformed equilibrium.

A class of *linear control singularities* is most conveniently defined by conditions on the 1 jet of the system in normal form For such singularities, we are only interested in the action of the linear feedback group. The fold $\mathcal{F}$ above is a linear control singularity.

To compute the codimension of a class of linear control singularities we proceed as follows. As we said before the singular class is most conveniently defined by certain conditions on the linear normal form at the equilibrium $(0, 0)$. The normal form may depend on one or more parameters. For the fold singularity there is one parameter a. All other elements of the singular class are obtained by a linear feedback transformation acting on a singularity in normal form.

Hence we must study the action of the linear feedback group

$$
\begin{aligned}
z &= Tx && (5.1)\\
v &= Kx + Lu && (5.2)
\end{aligned}
$$

on systems in linear normal form (4.6). We partition (4.1) compatibly with (4.7)

$$
\begin{bmatrix} T & 0 \\ K & L \end{bmatrix} = \begin{bmatrix} T_{00} & T_{01} & 0 \\ T_{10} & T_{11} & 0 \\ K_0 & K_1 & L \end{bmatrix} \qquad (5.3)
$$

The result is a new system

$$
\dot{x} = Fx + Gu + O(x, u)^2
$$

where

$$
\begin{aligned}
\begin{bmatrix} F & G \end{bmatrix} &= T^{-1} \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} T & 0 \\ K & L \end{bmatrix}\\
\begin{bmatrix} F_{00} & F_{01} & G_0 \\ F_{10} & F_{11} & G_1 \end{bmatrix} &= \begin{bmatrix} S_{00} & S_{01} \\ S_{10} & S_{11} \end{bmatrix} \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_1 & B_1 \end{bmatrix} \begin{bmatrix} T_{00} & T_{01} & 0 \\ T_{10} & T_{11} & 0 \\ K_0 & K_1 & L \end{bmatrix}
\end{aligned}
$$

with $S = T^{-1}$.

Tannenbaum [3] has descibed the action of the group of linear changes of state coordinates, i. e. $K_1 = 0, L = I$, acting on linearly controllable systems, $r_0 = 0$. But we are interested in the full feedback group acting on possibly linearly uncontrollable systems.

When computing the codimension of an orbit of this action, it is simpler to compute the codimension the infinitesmal action which is a linear calculation. Consider a curve $T = T(\mu),\ K = K(\mu),\ L = L(\mu)$ in the linear feedback group parameterized by $\mu \in I\!R$ where $T(0) = I,\ K(0) = 0,\ L(0) = I$. Let $'$ denote differentiation with respect to μ at $\mu = 0$ then

$$\begin{bmatrix} F_{00} & F_{01} & G_0 \\ F_{10} & F_{11} & G_1 \end{bmatrix}' = \left(T^{-1} \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} T & 0 \\ K & L \end{bmatrix} \right)'$$

$$= \begin{bmatrix} A_0 T'_{00} - T'_{00} A_0 & A_0 T'_{01} - T'_{01} A_1 & -T'_{01} B_1 \\ A_1 T'_{10} - T'_{10} A_0 + B_1 K'_0 & A_1 T'_{11} - T'_{11} A_1 + B_1 K'_1 & -T'_{11} B_1 + B_1 L' \end{bmatrix}$$

This action splits into four mappings

$$T'_{00} \mapsto F'_{11} = A_0 T'_{00} - T'_{00} A_0, \tag{5.4}$$

$$T'_{01} \mapsto \begin{bmatrix} F_{01} & G_0 \end{bmatrix}' = \begin{bmatrix} A_0 T'_{01} - T'_{01} A_1, & -T'_{01} B_1 \end{bmatrix}, \tag{5.5}$$

$$\begin{bmatrix} T'_{10} \\ K'_0 \end{bmatrix} \mapsto F'_{21} = A_1 T'_{10} - T'_{10} A_0 + B_1 K'_0, \tag{5.6}$$

$$\begin{bmatrix} T'_{11} & 0 \\ K'_1 & L' \end{bmatrix} \mapsto \begin{bmatrix} F_{11} & G_1 \end{bmatrix}' = \begin{bmatrix} A_1 T'_{11} - T'_{11} A_1 + B_1 K'_1, & -T'_{11} B_1 + B_1 L' \end{bmatrix}. \tag{5.7}$$

Each mapping is from a real vector space to another. For each we wish to compute the codimension of its range and find a maximal set of linearly independent vectors which are transverse to the range.

The first linear mapping (5.4) is the action of infinitesmal linear changes of the uncontrollable coordinates. It goes from $I\!R^{r_0^2}$ to $I\!R^{r_0^2}$. This is the same mapping that occurs when studying dynamical systems without controls. It is never an isomorphism and the codimension of its range depends on A_0. An analysis of this map can be found in Wiggins [4] on page 315. We state the results for the cases where A_0 is 1×1 or 2×2.

If A_0 is 1×1 then the map (5.4) is identically zero so the range is of codimension one. A 1×1 matrix transverse to the range is 1.

If A_0 is 2×2 then there are several possibilities. We enumerate those of codimension two.

If A_0 has distinct, nonzero real eigenvalues then the range of (5.4) is of codimension two. Two matrices transverse to the range are

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

If A_0 has two complex eigenvalues whose real and imaginary parts are both nonzero then the range of (5.4) is of codimension two. Two matrices transverse to the range are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

If A_0 is a 2×2 Jordan block

$$A_0 = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$$

where $a \neq 0$ then the codimesion is also two. Two matrices transverse to the range are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Next consider the linear mapping (5.5). This is a mapping from $I\!R^{r_0 r}$ to $I\!R^{r_0(r+m)}$ which is clearly not onto. But it is one to one which we now demonstrate for the case $m = 1$. The general case follows similarly. Let $X = T'_{01}$ and $X_{\cdot,j}$ denote the j^{th} column of the matrix X. Suppose

$$\begin{aligned} A_0 X - X A_1 &= 0 \\ X B_1 &= 0 \end{aligned}$$

then these equations become

$$\begin{aligned} A_0 X_{\cdot,j} - X_{\cdot,j-1} &= 0 \\ X_{\cdot,r} &= 0 \end{aligned}$$

so $X = 0$. Therefore the range of the mapping (5.5) has codimension r_0. One choice of r_0 independent $r_0 \times (r+1)$ matrices transverse to the range is

$$\begin{bmatrix} F_{01} & G_0 \end{bmatrix} = \begin{bmatrix} \mathbf{e_i} & 0 & \ldots & 0 \end{bmatrix}, \quad i = 1, \ldots, r_0 \tag{5.8}$$

where $\mathbf{e_i}$ is the i^{th} vector in $I\!R^{r_0}$.

Next consider the linear mapping (5.6). This is a mapping from $I\!R^{rr_0+mr_0}$ to $I\!R^{rr_0}$ which we now show is onto. Let $X = T_{10}'$ and $X_{i,\cdot}$ denote the i^{th} row of the matrix X. If $m = 1$ then for $1 \le i \le r_1 - 1$

$$\begin{aligned} (A_1 X - X A_0 + B_1 K_0')_{i,\cdot} &= X_{i+1,\cdot} - X_{i,\cdot} A_0 \\ (A_1 X - X A_0 + B_1 K_0')_{n,\cdot} &= -X_{n,\cdot} A_0 + K_1' \end{aligned}$$

Hence we choose $X_{1,\cdot}$ arbitrarily, then for for $2 \le i \le r_1$ we choose $X_{i,\cdot}$ to arbitrarily fix the $i - 1^{th}$ row and K_1' to arbitrarily fix the r^{th} row. The case when $m > 1$ follows similarly.

Finally we look at the linear mapping (5.7). This is a mapping from $I\!R^{r^2+mr+m^2}$ to $I\!R^{r^2+rm}$ This is the infinitesmal action of the linear feedback group acting on the controllable part of the system. The controllability indices are a complete set of invariance for the feedback group action. Any linear controllable system can be transformed into the Brunovsky normal form with the same controllability indices. Hence the codimension of the range of the mapping (5.7) is same as the codimension of the set of all (F_{11}, G_1) with the same tuple of controllabity indices $(r_1, \ldots, r_m)$. If the tuple $(0, r_1, \ldots, r_m)$ is the smallest admissible tuple of extended controllability indices for systems with r dimensional state and m dimensional control then the map (5.7) is onto. Otherwise the codimension is the number of admissible tuples of controllability indices of a system with r states and m inputs that are smaller than $(r_1, \ldots, r_m)$.

We consider the cases where the codimension is one or two. Let $(r_1, \ldots, r_m)$ be the smallest admissible tuple for a system with r states and m inputs. The next smallest admissible tuple is

$$(r_1 + 1, r_2, \ldots, r_{m-1}, r_m - 1)$$

so for A_1, B_1 with this tuple of controllability indices, the range of the map (5.7) is of codimension one. An $r \times (r + m)$ matrix transverse to range is zero except for its $(1, r_1 + \ldots + r_{m-1} + 2)$ entry which is one.

If $m = 2$ the next smallest admissible tuple is

$$(r_1 + 2, r_2 - 2)$$

so for A_1, B_1 with this tuple of controllability indices, the range of the map (5.7) is of codimension two. An $r \times (r + 2)$ matrix transverse to range is zero except for its $(1, r_1 + 2)$ entry which is one. Another $r \times (r + m)$ matrix transverse to range is zero except for its $(2, r_1 + 2)$ entry which is one.

If $m = 3$ the next smallest admissible tuple is

$$(r_1 + 1, r_2 + 1, r_3 - 2)$$

so for A_1, B_1 with this tuple of controllability indices, the range of the map (5.7) is of codimension two. An $r \times (r + m)$ matrix transverse to range is zero except for its $(1, r_1 + r_2 + 3)$ entry which is one. Another $r \times (r + m)$ matrix transverse to range is zero except for its $(r_1 + 2, r_1 + r_2 + 3)$ entry which is one.

If $m \geq 4$ the next smallest admissible tuple is

$$(r_1 + 1, r_2 + 1, r_3, \ldots, r_{m-2}, r_{m-1} - 1, r_m - 1)$$

so for A_1, B_1 with this tuple of controllability indices the range of the map (5.7) is of codimension two. An $r \times (r + m)$ matrix transverse to range is zero except for its $(1, r_1 + \ldots + r_{m-2} + 3)$ entry which is one. Another $r \times (r + m)$ matrix transverse to range is zero except for its $(r_1 + 2, r_1 + \ldots + r_{m-1} + 1)$ entry which is one.

6 Versal Deformations

Given a class of control singularities $\mathcal{G} \in \mathcal{E}^k$, one would like to study the types of equilibrium k jets that can be obtained by small perturbations. A family of control singularities is always invariant under the action of the feedback group.

Let $\mathcal{G} \subset \mathcal{E}^k$ which is invariant under the action of the feedback group. A C^k *versal deformation* of $\mathcal{G}$ is a C^k parametrized subset of $\mathcal{E}^k$ of the form

$$\begin{array}{rcl} \phi : \mathcal{P} & \rightarrow & \mathcal{E}^k \\ \phi : \mu & \mapsto & \phi(\mu) = ((x, u), 0, \phi^{(1)}(\mu), \ldots, \phi^{(k)}(\mu)) \end{array} \tag{6.1}$$

defined for μ in some neighborhood $\mathcal{P}$ of $0 \in I\!R^p$, which intersects $\mathcal{G}$ at $\mu = 0$ and which is transversal to $\mathcal{G}$. The versal deformation is said to be *miniversal* if the dimension p is minimal among all versal deformations. The minimal p is the *codimension* of $\mathcal{G}$.

A C^k *versal feedback* for a versal deformation (6.1) is a mapping

$$\begin{array}{rcl} \psi : \mathcal{P} & \rightarrow & \mathcal{K}^k \\ \psi : \mu & \mapsto & \psi(\mu) = ((x, u), 0, \psi^{(1)}(\mu), \ldots, \psi^{(k)}(\mu)). \end{array} \tag{6.2}$$

A versal feedback is *stabilizing* if at each $\mu \in \mathcal{P}$, $\psi(\mu)$ stabilizes $\phi(\mu)$.

7 Low Codimension Linear Control Singularities of Scalar Input Sytems

7.1 Fold Control Singularities

The simplest example of a control singularity is a fold and the class of all fold singularities is of codimension one as we now show. Recall that an equilibrium 1-jet (4.14) is a fold singularity if its normal form is (4.14) for some $a \neq 0$. The reason for the terminology will be discussed later. We complete the above analysis to understand the infinitesmal action of the linear feedback group on A, B (4.12). The range of the linear mapping (5.5) is of codimension one so from (5.8) we obtain a nonzero $n \times (n+1)$ matrix transverse to the orbit of A, B under the linear feedback group. The linear mapping (5.4) is identically zero so there is another linearly independent $n \times (n+1)$ matrix transverse to the orbit of A, B under the linear feedback group.

But perturbations in one of these directions can be accomplished by varying a and so it is not transverse to $\mathcal{F}$. Therefore $\mathcal{F}$ is codimension one set of control singularities and a miniversal deformation of it is

$$\mu \quad \mapsto \quad \left[\begin{array}{cc} F(\mu) & G(\mu) \end{array} \right]$$

$$F(\mu) = \left[\begin{array}{c|cccccc} a & \mu & 0 & 0 & \dots & 0 \\ \hline 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ & & & & \ddots & \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{array} \right], \qquad G(\mu) = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ \\ 0 \\ 1 \end{array} \right]$$

Now we see why the fold terminology. The controllabilty matrix of the versal deformation in reverse order is

$$\left[\begin{array}{cccc} F(\mu)^{n-1}G(\mu) & \dots & F(\mu)G(\mu) & G(\mu) \end{array} \right] = \left[\begin{array}{c|c} \mu & 0 \\ \hline 0 & I \end{array} \right].$$

Notice that all these 1-jets are controllable except for $\mu = 0$ and the controllability reverses orientation (folds over) at $\mu = 0$.

There are two subclasses of fold singularities, those where $a < 0$ and those with $a > 0$. There is an important distinction between these subclasses. The former are linearly stabilizable, i.e., there exists a linear feedback $u = Kx$ so that all the poles of the linear part of the closed loop dynamics $A + BK$ are in the left half plane. In fact the versal deformation of a fold singularity with $a < 0$ is versally stabilize by a versal linear feedback of the form

$$((x, u), 0, [K_0, K_1])$$

where $K_0 = 0$, $A_1 + B_1 K_1$ is Hurwitz.

But if $a > 0$ then the closed loop dynamics will always have at least one unstable eigenvalue a.

7.2 Transcontrollable Singularities

A transcontrollable singularity is a degenerate fold where $a = 0$. This means that the stabilizability of a system realizing this 1-jet is decided by its higher order terms. The class of transcontrollable singularities denoted by $\mathcal{TC}$ is of codimension two and a versal deformation of it is

$$(\mu_1, \mu_2) \quad \mapsto \quad \begin{bmatrix} F(\mu_1, \mu_2) & G(\mu_1, \mu_2) \end{bmatrix}$$

$$F(\mu_1, \mu_2) = \left[\begin{array}{c|cccccc} \mu_1 & \mu_2 & 0 & 0 & \dots & 0 \\ \hline 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ & & & & \ddots & \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{array}\right], \qquad G(\mu_1, \mu_2) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \\ 0 \\ 1 \end{bmatrix}$$

Notice that a transcontrollable singularity can be perturbed into a fold by changing μ_1 from zero and can be perturbed into a linearly controllable 1-jet by changing μ_2 from zero.

7.3 Two Real Roots Control Singularities

There are two other classes of control singularities of codimension two for scalar input systems. They are the class of two real roots control singularities and the class of two complex roots control singularities.

We denote the class of two real roots control singularities by $\mathcal{TRR}$. These are equilibrium 1- jets that have the linear normal form (4.13) where $n \geq 2,\ m = 1,\ r_0 = 2,\ r = r_1 = n-2$,

$$A_0 \;=\; \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix},$$

A_1, B_1 are in Brunovsky form and $a_1 \neq a_2, a_1 a_2 \neq 0$.

The linear mapping (5.4) is of rank two so there are two linearly independent $n \times (n+1)$ matrices transverse to the orbit of A, B under the linear feedback group. The range of the linear mapping (5.5) is of codimension two so from (5.8) we obtain two additional linearly independent $n \times (n+1)$ matrices transverse to the orbit of A, B under the linear feedback group.

But perturbations in the first two of these directions can be accomplished by varying $a_1,\ a_2$ and so they are not transverse to $\mathcal{TRR}$. Therefore $\mathcal{TRR}$ is of codimension two and a miniversal deformation of it is

$$(\mu_1, \mu_2) \mapsto \begin{bmatrix} F(\mu_1,\mu_2) & G(\mu_1,\mu_2) \end{bmatrix}$$

$$F(\mu_1,\mu_2) = \left[\begin{array}{cc|cccccc} a_1 & 0 & \mu_1 & 0 & \dots & 0 & 0 \\ 0 & a_2 & \mu_2 & 0 & \dots & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ & & & & & \ddots & \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{array}\right], \qquad G(\mu_1,\mu_2) = \left[\begin{array}{c} 0 \\ 0 \\ \hline 0 \\ 0 \\ \\ 0 \\ 1 \end{array}\right].$$

When $a_1 = a_2$ and/or $a_1 a_2 = 0$ then we obtain classes of singularities that are of higher codimension. Some of these are discussed below. When $a_1 < 0,\ a_2 < 0$, the singularity can be stabilized by a linear feedback and the versal deformation can versally stabilized by

$$((x,u), 0, [K_0, K_1])$$

where $K_0 = 0,\ A_1 + B_1 K_1$ is Hurwitz. But if an $a_i > 0$ then the singularity is not stabilizable.

7.4 Two Complex Roots Control Singularity

The class $\mathcal{TCR}$ of two complex roots control singularities is of codimension two. These are equilibrium 1- jets that have the linear normal form (4.13) where $n \geq 2,\ m = 1,\ r_0 = 2,\ r = r_1 = n-2,$

$$A_0 \;=\; \begin{bmatrix} a & -\omega \\ \omega & a \end{bmatrix},$$

A_1, B_1 are in Brunovsky form and $a \neq 0, \omega \neq 0$.

The linear mapping (5.4) is of rank two so there are two linearly independent $n \times (n+1)$ matrices transverse to the orbit of A, B under the linear feedback group. The range of the linear mapping (5.5) is of codimension two so from (5.8) we obtain two additional linearly independent $n \times (n+1)$ matrices transverse to the orbit of A, B under the linear feedback group. But perturbations in the first two of these directions can be accomplished by varying $a,\ \omega$ and so they are not transverse to $\mathcal{TCR}$. Therefore $\mathcal{TCR}$ is of codimension two and a miniversal deformation of it is

$$(\mu_1, \mu_2) \mapsto \begin{bmatrix} F(\mu_1,\mu_2) & G(\mu_1,\mu_2) \end{bmatrix}$$

$$F(\mu_1,\mu_2) = \left[\begin{array}{cc|ccccc} a & -\omega & \mu_1 & 0 & \dots & 0 & 0 \\ \omega & a & \mu_2 & 0 & \dots & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ & & & & & \ddots & \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{array}\right], \qquad G(\mu_1,\mu_2) = \left[\begin{array}{c} 0 \\ 0 \\ \hline 0 \\ 0 \\ \\ 0 \\ 1 \end{array}\right].$$

When $a = 0$ and/or $\omega = 0$ then we obtain classes of singularities that are of higher codimension. When $a < 0$ then the singularity can be stabilized by a linear feedback and the versal deformation can versally stabilized by

$$((x,u), 0, [K_0, K_1])$$

where $K_0 = 0,\ A_1 + B_1 K_1$ is Hurwitz. But if an $a > 0$ then the singularity is not stabilizable.

7.5 Double Root Control Singularity

The set $\mathcal{DR}$ of double root control singularities is of codimension three. These are equilibrium 1- jets that have the linear normal form (4.13) where $n \geq 2,\ m = 1,\ r_0 = 2,\ r = r_1 = n - 2,$

$$A_0 \;=\; \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix},$$

A_1, B_1 are in Brunovsky form and $a \neq 0$.

The linear mapping (5.4) is of rank two so there are two linearly independent $n \times (n+1)$ matrices transverse to the orbit of A, B under the linear feedback group. The range of the linear mapping (5.5) is of codimension two so from (5.8) we obtain two additional linearly independent $n \times (n+1)$ matrices transverse to the orbit of A, B under the linear feedback group. But perturbations in one of these directions can be accomplished by varying a and so it is not transverse to $\mathcal{DR}$. Therefore this is codimension three control singularity and a miniversal deformation of it is

$$(\mu_1, \mu_2, \mu_3) \mapsto \begin{bmatrix} F(\mu_1, \mu_2, \mu_3) & G(\mu_1, \mu_2, \mu_3) \end{bmatrix}$$

$$F(\mu_1, \mu_2, \mu_3) = \left[\begin{array}{cc|cccccc} a & 1 & \mu_2 & 0 & \dots & 0 & 0 \\ \mu_1 & a & \mu_3 & 0 & \dots & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ & & & & & \ddots & \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{array}\right], \qquad G(\mu_1, \mu_2, \mu_3) = \left[\begin{array}{c} 0 \\ 0 \\ \hline 0 \\ 0 \\ \\ 0 \\ 1 \end{array}\right]$$

When $a < 0$ then the singularity can be stabilized by a linear feedback and the versal deformation can versally stabilized by

$$((x, u), 0, [K_0, K_1])$$

where $K_0 = 0,\ A_1 + B_1 K_1$ is Hurwitz. But if an $a > 0$ then the singularity is not stabilizable.

7.6 Multiple Distinct Roots Control Singularity

The set $\mathcal{MDR}$ of multiple distinct roots control singularities is a generaliztion of $\mathcal{TRR}$ and $\mathcal{TCR}$. It is charatcterized by $m = 1,\ 1 \le r_0 \le n,\ r = r_1 = n - r_0$ and A_0 in Jordan form with distinct real and/or complex roots. If a root is real it must be nonzero, if it is complex then it must have nonzero real and imaginary parts. Its codimension is r_0, the number of linearly uncontrollable modes.

The linear mapping (5.4) is of rank $r_0^2 - r_0$ so there are r_0 linearly independent $n \times (n+1)$ matrices transverse to the orbit of A, B under the linear feedback group. But perturbations in these directions can be achieved by varying the r_0 eigenvalues of A_0 so they don't affect the codimension.

The range of the linear mapping (5.5) is of codimension r_0 so from (5.8) we obtain r_0 linearly independent $n \times (n+1)$ matrices transverse to the orbit of A, B under the linear feedback group. Therefore $\mathcal{MDR}$ is a control singularity of codimension r_0 and a miniversal deformation is of the form

$$\mu \mapsto \begin{bmatrix} F(\mu) & G(\mu) \end{bmatrix} = \begin{bmatrix} A_0 & F_{01}(\mu) & 0 \\ 0 & A_1 & B_1 \end{bmatrix}$$

where

$$F_{01}(\mu) = \begin{bmatrix} \mu_1 & 0 & \dots & 0 \\ \mu_2 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \mu_{r_0} & 0 & \dots & 0 \end{bmatrix}.$$

When A_0 is Hurwitz then the singularity can be stabilized by a linear feedback and the versal deformation can versally stabilized by

$$((x, u), 0, [K_0, K_1])$$

where $K_0 = 0$, $A_1 + B_1 K_1$ is Hurwitz. But if an A_0 has an unstable eigenvalue then the singularity is not stabilizable.

7.7 Multiple Repeated Roots Control Singularity

The set $\mathcal{MRR}$ of multiple repeated roots control singularities is a generaliztion of $\mathcal{DR}$. It is charatcterized by $m = 1,\ 1 \le r_0 \le n,\ r = r_1 = n - r_0$

and A_0 consists of a single Jordan block (4.8) of size $r_0 \times r_0$ with the real eigenvalue $a \neq 0$ repeated r_0 times. Its codimension is $2r_0 - 1$.

The range of the map (5.4) has codimension r_0 which we now show. Let $T_{00} = S$ and $()_{ij}$ denote the i, j component of the enclosed matrix, then

$$(A_0S - SA_0)_{ij} \quad = \quad S_{i+1,j} - S_{i,j-1}$$

under the convention

$$S_{i,j} = 0, \qquad \text{if either } i < 1 \text{ or } i > r_0,$$

$$S_{i,j} = 0, \qquad \text{if either } i, j < 1 \text{ or } i, j > r_0.$$

We shall consider $A_0S - SA_0$ by diagonals. Let $l = j - i$ index the diagonals. First we consider the main and lower diagonals where $1 - r_0 \leq l \leq 0$. If $1 \leq s \leq r_0 + l - 1$ then

$$(A_0S - SA_0)_{s-l,s} = S_{s-l+1,s} - \sum_{j=1}^{s-1} (A_0S - SA_0)_{j-l,j}\,.$$

Going down the main and lower diagonals, $(A_0S - SA_0)_{s-l,s}$ can be set arbitrarily by choice of $S_{s-l+1,s}$ until we get to the last row. The entries in the last row are then determined by

$$(A_0S - SA_0)_{r_0,r_0+l} = \sum_{j=1}^{r_0+l-1} (A_0S - SA_0)_{j-l,j}\,.$$

Next we consider the upper diagonals where $1 \leq l \leq r_0 - 1$. If $1 \leq s \leq r_0 - l$ then

$$(A_0S - SA_0)_{s,l+s} = S_{s+1,l+s} - S_{1,l} - \sum_{i=1}^{s-1} (A_0S - SA_0)_{i,l+i}\,.$$

Going down the upper diagonals, $(A_0S - SA_0)_{s,l+s}$ can be set arbitrarily by choice $S_{s+1,l+s}$.

So the range of the map (5.4) has codimension r_0 and a set of $r_0 \times r_0$ marices transverse to the range

$$F_{00} \quad = \quad \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mathbf{e_i}' \end{bmatrix}, \qquad i = 1, \ldots, r_0$$

where $\mathbf{e_i}$ is the i^{th} unit column vector in $\mathbb{R}^{r_0}$. One of these transverse directions can be achieved by varying a.

The range of the map (5.5) has codimesion r_0. Therefore $\mathcal{MRR}$ is a control singularity of codimension $2r_0 - 1$ and a miniversal deformation is of the form

$$\mu \mapsto \left[\begin{array}{cc} F(\mu) & G(\mu) \end{array} \right] = \left[\begin{array}{ccc} F_{00}(\mu) & F_{01}(\mu) & 0 \\ 0 & A_1 & B_1 \end{array} \right]$$

where $\mu = (\mu_1, \ldots, \mu_{2r_0-1})$ and

$$F_{00}(\mu) = \left[\begin{array}{cccccc} a & 1 & 0 & \ldots & 0 & 0 \\ 0 & a & 1 & \ldots & 0 & 0 \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ 0 & 0 & 0 & \ldots & a & 1 \\ \mu_1 & \mu_2 & \mu_3 & \ldots & \mu_{r_0-1} & a \end{array} \right]$$

$$F_{01}(\mu) = \left[\begin{array}{cccc} \mu_{r_0} & 0 & \ldots & 0 \\ \mu_{r_0+1} & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ \mu_{2r_0-1} & 0 & \ldots & 0 \end{array} \right]$$

.

When $a < 0$ then the singularity can be stabilized by a linear feedback and the versal deformation can versally stabilized by

$$((x, u), 0, [K_0, K_1])$$

where $K_0 = 0$, $A_1 + B_1 K_1$ is Hurwitz. But if an $a > 0$ then the singularity is not stabilizable.

8 Low Codimension Linear Control Singularities of Multiple Input Sytems

In this section we present the classes of control singularities of codimensions one and two for systems with more than one input

8.1 Exchange of Control Singularity

The generic equilibrium 1-jet of a system with n states and $m \geq 2$ inputs has the smallest admissible extended controllability indices $(s_0, s_1, \ldots, s_m) = (0, l{+}1, \ldots, l)$ where l is the greatest integer not exceeding n/m the number of $l{+}1$ indices is such that $s_1 + \cdots + s_m = n$. The next smallest admissible tuple of extended controllability indices is $(r_0, r_1, \ldots, r_m) = (0, s_1 + 1, \ldots, s_m - 1)$. In other words the first input has one more state to control and the last input has one less. We are assuming that $s_m > 0$, if not then the last input with nonzero index has one less state to control. The class $\mathcal{C}(r_0, r_1, \ldots, r_m)$ is singular and of codimension one. We give a versal deformation when $m = 2$ and $n = 4$ so $(r_0, r_1, r_2) = (0, 3, 1)$.

$$F(\mu) = \left[\begin{array}{ccc|c} 0 & 1 & 0 & \mu \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right], \qquad G(\mu) = \left[\begin{array}{c|c} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ \hline 0 & 1 \end{array}\right]$$

If $\mu \neq 0$ the perturbation has the generic extended controllability indices $(s_0, s_1, s_2) = (0, 2, 2)$.

8.2 Double Exchange of Control Singularity

This takes three different forms depending on wheteher $m = 2$, $m = 3$ or $m \geq 4$. If $m = 2$ the next smallest tuple of admissible extended controllabilty indices after the exchange of control singularity is $(r_0, r_1, r_2) = (0, s_1 + 2, s_2 - 2)$. The class $\mathcal{C}(r_0, r_1, r_2)$ is singular and of codimension two. We give a versal deformation when $n = 6$ so $(r_0, r_1, r_2) = (0, 5, 1)$.

$$F(\mu_1, \mu_2) = \left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 0 & \mu_1 \\ 0 & 0 & 1 & 0 & 0 & \mu_2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \qquad G(\mu) = \left[\begin{array}{c|c} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ \hline 0 & 1 \end{array}\right]$$

If $\mu_2 \neq 0$ the perturbation has the generic extended controllability indices $(s_0, s_1, s_2) = (0, 3, 3)$. If $\mu_2 = 0$ but $\mu_1 \neq 0$ the perturbation has the extended controllability indices of an exchange of control singularity $(s_0, s_1+1, s_2-1) = (0, 4, 2)$.

If $m = 3$ the next smallest tuple of admissible extended controllabilty indices after the exchange of control singularity is $(r_0, r_1, r_2, r_3) = (0, s_1 + 2, s_2 + 1, s_3 - 2)$. The class $\mathcal{C}(r_0, r_1, r_2)$ is singular and of codimension two. We give a versal deformation when $n = 6$ so $(r_0, r_1, r_2, r_3) = (0, 3, 3, 0)$.

$$F(\mu_1, \mu_2) = \left[\begin{array}{ccc|ccc|} 0 & 1 & 0 & \mu_1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], G(\mu_1, \mu_2) = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & \mu_2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

If $\mu_1 \mu_2 \neq 0$ the perturbation has the generic extended controllability indices $(s_0, s_1, s_2, s_3) = (0, 2, 2, 2)$. If $\mu_1 = 0$ but $\mu_2 \neq 0$ the perturbation has the extended controllability indices of an exchange of control singularity $(s_0 + 1, s_1, s_2, s_3 - 1) = (0, 3, 2, 1)$.

If $m \geq 4$ the next smallest tuple of admissible extended controllabilty indices after the exchange of control singularity is $(r_0, r_1, r_2, r_3, \ldots, r_{m-2}, r_{m-1}, r_m) = (0, s_1+1, s_2+1, s_3, \ldots, s_{m-2}, s_{m-1}-1, r_m-1)$. The class $\mathcal{C}(r_0, r_1, r_2, r_3, \ldots, r_{m-2}, r_{m-1}, r_m)$ is singular and of codimension two. We give a versal deformation when $n = 8$ and $m = 4$ so $(r_0, r_1, r_2, r_3, r_4) = (0, 3, 3, 1, 1)$.

$$F(\mu_1, \mu_2) = \left[\begin{array}{ccc|ccc|c|c} 0 & 1 & 0 & 0 & 0 & 0 & \mu_1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & \mu_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$G(\mu_1,\mu_2) = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

If $\mu_1\mu_2 \neq 0$ the perturbation has the generic extended controllability indices $(s_0, s_1, s_2, s_3, s_4) = (0, 2, 2, 2, 2)$. If $\mu_1 \neq 0$ but $\mu_2 = 0$ the perturbation has the extended controllability indices of an exchange of control singularity $(0, s_1 + 1, s_2, s_3, s_4 - 1) = (0, 3, 2, 2, 1)$. If $\mu_2 \neq 0$ but $\mu_1 = 0$ the perturbation has the same extended controllability indices $(0, s_1 + 1, s_2, s_3, s_4 - 1) = (0, 3, 2, 2, 1)$.

8.3 Multiple Fold Control Singularity

This singularity occurs when there are n states, m inputs and one uncontrollable mode r_0. The controllability indices $(r_1, r_2, \ldots, r_m)$ are generic, $A_0 = a \neq 0$ and A_1, B_1 is in Brunovsky form for a system with m inputs. When $r_1 = r_2 = \ldots = r_m$ then this singularity is of codimension one. Here is a versal deformation when $m = 2$ and $(r_0, r_1, r_2) = (1, 2, 2)$.

$$F(\mu) = \left[\begin{array}{c|cc|cc} a & \mu & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right], \qquad G(\mu) = \left[\begin{array}{c|c} 0 & 0 \\ 1 & 0 \\ \hline 0 & 0 \\ 0 & 1 \end{array}\right].$$

If $\mu \neq 0$ then the extended controllability indices are $(0, 3, 2)$

When $r_1 = r_2 + 1$ then this singularity is of codimension two. Here is a versal deformation when $(r_0, r_1, r_2) = (1, 2, 1)$.

$$F(\mu) = \left[\begin{array}{c|cc|c} a & \mu_1 & 0 & \mu_2 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right], \qquad G(\mu) = \left[\begin{array}{c|c} 0 & 0 \\ 1 & 0 \\ \hline 0 & 1 \end{array}\right]$$

If $\mu_2 \neq 0$ then the extended controllability indices are $(0, 2, 2)$. If $\mu_2 = 0$ but $\mu_1 \neq 0$ then the extended controllability indices are $(0, 3, 1)$.

9 Control Bifurcations

In this section we discuss the relationship of control singularities to control bifurcations [2]. Recall that the equilibrium set $\mathcal{Z}$ of a system (1.1) is the set of all (x^0, u^0) such that

$$0 = f(x^0, u^0). \tag{9.1}$$

Since the equilibrium condition (9.1) consists of n smooth equations in $n+m$ unknowns, we expect $\mathcal{Z}$ to be m dimensional smooth surface in $\mathcal{X} \times \mathcal{U}$. But this surface may have singularities. Additionally there may be equilibria in $\mathcal{Z}$ that are more difficult to stabilize than some of their neighboring equilibria. Frequently both happen on the same subset of equilibria. These equilibria that are relatively more difficult to stabilize are called *control bifurctaions*. The k-jet of the equilibrium can determine the type of control bifurcation.

The equilibrium set $\mathcal{Z}$ determines a subset $\mathcal{ZE} \subset \mathcal{E}^k$ consisting of the k-jets of all $(x^0, u^0) \in \mathcal{Z}$. We shall see that control bifurcations can occur when $\mathcal{ZE}$ intersects one of the singular classes discussed above.

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