Stabilization of Discrete Time Systems with a Fold or Period Doubling Control Bifurcation

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Abstract

In this paper we derive the controlled center dynamics and synthesize stabilizing controllers for nonlinear discrete time systems with fold and period-doubling control bifurcations.

Keywords Controlled Center Dynamics, Discrete-Time, Fold Control Bifurcation, Period-Doubling Control Bifurcation, Stabilization, Bird Foot Bifurcation for Maps.

1 Introduction

Center manifold theory plays an important role in the study of the stability of nonlinear systems when the equilibrium point is not hyperbolic. The center manifold is an invariant manifold of the differential (difference) equation which is tangent at the equilibrium point to the eigenspace of the neutrally stable eigenvalues. In practice, one does not compute the center manifold and its dynamics exactly, since this requires the resolution of a quasilinear partial differential (nonlinear functional) equation which is not easily solvable. In most cases of interest, an approximation of degree two or three of the solution is sufficient. Then, we determine the reduced dynamics on the center manifold, study its stability and then conclude about the stability of the original system [5]. This theory combined with the normal form approach of Poincaré was used extensively to study parameterized dynamical systems exhibiting bifurcations [20].

For nonlinear systems with control bifurcations (see [16]) a similar approach was used for the analysis and stabilization of systems with one or two uncontrollable modes in continuous and discrete-time [13, 16, 7, 10, 8, 17, 9]. This approach was generalized to systems with any number of uncontrollable modes by introducing the *Controlled Center Dynamics* in continuous time [11], and in discrete time [12]. The Controlled Center Dynamics is a reduced order control system whose stabilizability properties determine the stabilizability properties of the full order system. The approach based on the controlled center dynamics can also be viewed as a reduction technique for some classes of controlled differential (difference) equations. After reducing the order of these equations, the synthesis of a stabilizing controller is performed based on the reduced order control system.

In this paper, we continue the study in [12] by deriving the controlled center dynamics and stabilizing discrete time systems with a fold control bifurcation, i.e. systems with an uncontrollable mode whose modulus is slightly greater than one, and systems with a period doubling control bifurcation. We shall, also, introduce the discrete-time version of the bird foot bifurcation introduced in [15].

The paper is organized as follows: In section §2, we review the results on the controlled center dynamics, in sections §3 we apply this technique to stabilize systems with a fold and a period doubling control bifurcations. We shall treat the bird foot bifurcation for maps in the appendix.

2 Review of The Controlled Center Dynamics

Consider the following nonlinear system

$$\zeta^+ = f(\zeta, v) \tag{2.1}$$

the variable $\zeta \in \mathbb{R}^n$ is the state, $v \in \mathbb{R}$ is the input variable. The vectorfield $f(\zeta)$ is assumed to be C^k for some sufficiently large k.

Assume $f(0,0) = 0$, and suppose that the linearization of the system at the origin is (A, B),

$$A = \frac{\partial f}{\partial \zeta}(0,0), \quad B = \frac{\partial f}{\partial v}(0,0),$$

with

$$\text{rank}([B \; AB \; A^2B \; \cdots \; A^{n-1}B]) = n - r, \tag{2.2}$$

and $r > 0$. Let us denote by $\Sigma_{\mathcal{D}}$ the system (2.1) under the above assumptions.

The system $\Sigma_{\mathcal{D}}$ is not linearly controllable at the origin, and a change of some control properties may occur around this equilibrium point, this is called a control bifurcation if it is linearly controllable at other equilibria [16].

From linear control theory, we know that there exist a linear change of coordinates and a linear feedback transforming the system $\Sigma_{\mathcal{D}}$ to

$$\begin{aligned} x_1^+ &= A_1x_1 + \bar{f}_1(x_1, x_2, u), \\ x_2^+ &= A_2x_2 + B_2u + \bar{f}_2(x_1, x_2, u), \end{aligned} \tag{2.3}$$

with $x_1 \in \mathbb{R}^r$, $x_2 \in \mathbb{R}^{n-r}$, $u \in \mathbb{R}$, $A_1 \in \mathbb{R}^{r\times r}$ is in the Jordan form and its eigenvalues are on the imaginary axis, $A_2 \in \mathbb{R}^{(n-r)\times(n-r)}$, $B_2 \in \mathbb{R}^{(n-r)\times 1}$ are in the Brunovskỳ form, i.e.

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

and $\bar{f}_k(x_1, x_2, u)$, for $k = 1, 2$, designates a vector field which is a homogeneous polynomial of degree $d \geq 2$.

Now, consider the feedback given by

$$u(x_1, x_2) = \kappa(x_1) + K_2x_2, \tag{2.4}$$

with κ a smooth function and $K_2 = \begin{bmatrix} k_{2,1} & \cdots & k_{2,n-r} \end{bmatrix}$.
Because (A_2, B_2) is controllable, the eigenvalues in the closed-loop system associated with the equation of x_2 can be placed at arbitrary given points in the complex plane by selecting values for K_2. If one of these controllable eigenvalues is placed outside the unit disk, the closed-loop system is unstable around the origin. Therefore, we assume that K_2 has the following property.

Property $\mathcal{P}$: The modulus of the eigenvalues of the matrix $\bar{A}_2 = A_2 + B_2K_2$ is less or equal than one.

Let us denote by $\mathcal{F}$ the feedback (2.4) with the property $\mathcal{P}$.

The closed loop system (2.3)-(2.4) possesses r eigenvalues on the unit circle, and $n - r$ eigenvalues strictly inside the unit disk. Thus, a center manifold exists. It is represented locally around the origin as

$$W^c = \{(x_1, x_2) \in \mathbb{R}^r \times \mathbb{R}^{n-r} | x_2 = \pi(x_1), |x_1| < \delta, \pi(0) = 0\}.$$

This means that π and κ satisfy the nonlinear functional equation

$$\begin{aligned} &\bar{A}_2\pi(x_1) + B_2\kappa(x_1) + \bar{f}_2(x_1, \pi(x_1), \kappa(x_1) + K_2\pi(x_1)) \\ &= \pi(A_1x_1 + \bar{f}_1(x_1, \pi(x_1), \kappa(x_1) + K_2\pi(x_1))) \end{aligned} \tag{2.5}$$

The center manifold theorem ensures that this equation has a local solution for any smooth $\kappa(x_1)$. The reduced dynamics of the closed loop system (2.3)-(2.4) on the center manifold is given by

$$x_1^+ = f_1(x_1; \kappa) \tag{2.6}$$

where

$$f_1(x_1; \kappa) = A_1x_1 + \bar{f}_1(x_1, \pi(x_1), \kappa(x_1) + K_2\pi(x_1))$$

According to the center manifold theorem, we know that if the dynamics (2.6) is locally asymptotically stable then the closed loop system (2.3)-(2.4) is locally asymptotically stable.

The part of the feedback $\mathcal{F}$ given by $\kappa(x_1)$ determines the controlled center manifold $x_2 = \pi(x_1)$ which in turn determines the dynamics (2.6). Hence the problem of stabilization of the system (2.3) reduces the problem to stabilizing the system (2.6) after solving the equation (2.5), i.e. finding $\kappa(x_1)$ such that the origin of the dynamics (2.6) is asymptotically stable. Thus we can view $\kappa(x_1)$ as a pseudo control.
But the equation (2.5) need not be solved exactly, frequently it suffices to compute the low degree terms of the Taylor series expansion of π and κ around $x_1 = 0$. Because κ starts with linear terms

$$\kappa(x_1) = K_1x_1 + \kappa^{[2]}(x_1) + \ldots \tag{2.7}$$

π starts with linear terms

$$\pi(x_1) = \pi^{[1]}x_1 + \pi^{[2]}(x_1) + \ldots \tag{2.8}$$

The equation (2.5) implies that

$$\bar{A}_2\pi^{[1]} + B_2K_1 = \pi^{[1]}A_1, \tag{2.9}$$

and

$$\begin{array}{l}\bar{A}_2\pi^{[2]}(x_1) + B_2\kappa^{[2]}(x_1)\\ +\bar{f}_2^{[2]}(x_1,\pi^{[1]}x_1,K_1x_1+K_2\pi^{[1]}x_1) =\\ \pi^{[2]}(A_1x_1) + \pi^{[1]}\bar{f}_1^{[2]}(x_1,\pi^{[1]}x_1,K_1x_1+K_2\pi^{[1]}x_1),\end{array} \tag{2.10}$$

and so on.
For any $\kappa^{[k]}(x_1)$, these linear equations are solvable for $\pi^{[k]}(x_1)$ because $|\sigma(\bar{A}_2)| < 1 = |\sigma(A_1)|$.
The degree k equations are

$$\begin{array}{l}\bar{A}_2\pi^{[k]}(x_1) + B_2\kappa^{[k]}(x_1) + \tilde{f}_2^{[k]}(x_1)\\ = \pi^{[1]}\tilde{f}_1^{[k]}(x_1) + \zeta^{[k]}(x_1) + \pi^{[k]}(A_1x_1)\end{array} \tag{2.11}$$

where $\tilde{f}_i(x_1) = \bar{f}_i(x_1,\pi(x_1),\kappa(x_1) + K_2\pi(x_1))$, and $\zeta(x_1) = \sum_{i=2}^{k-1}\pi^{[i]}(A_1x_1 + \tilde{f}_1(x_1))$.

Notice that $\tilde{f}_i^{[j]}(x_1)$ only depends on $\pi^{[1]}(x_1)$, $\ldots$, $\pi^{[j-1]}(x_1)$ and $\kappa^{[1]}(x_1)$, $\ldots$, $\kappa^{[j-1]}(x_1)$.

For $1 \le i \le n-r-1$, the i^{th} row of these equations is

$$\begin{array}{rcl}\pi_{i+1}^{[k]}(x_1) &=& \pi_i^{[k]}(A_1x_1) + \zeta_i^{[k]}(x_1) - \tilde{f}_{2,i}^{[k]}(x_1)\\ &+& \sum_{j=1}^{r}\pi_{i,j}^{[1]}(x_1)\tilde{f}_{1,j}^{[k]}(x_1).\end{array} \tag{2.12}$$

The $(n-r)^{th}$ row is

$$\begin{array}{rcl}\kappa^{[k]}(x_1) &=& \pi_{n-r}^{[k]}(A_1x_1) + \zeta_{n-r}^{[k]}(x_1) - \tilde{f}_{2,n-r}^{[k]}(x_1)\\ &+& \sum_{j=1}^{r}\pi_{n-r,j}^{[1]}(x_1)\tilde{f}_{1,j}^{[k]}(x_1) - \sum_{i=1}^{n-r}k_{2,i}\pi_i^{[k]}(x_1)\end{array} \tag{2.13}$$

Notice that $\pi_1^{[k]}(x_1)$ determines $\pi_2^{[k]}(x_1)$, $\ldots, \pi_r^{[k]}(x_1)$, $\kappa^{[k]}(x_1)$. Therefore we may change our point of view. Instead of viewing $\kappa^{[k]}(x_1)$ as determining $\pi_1^{[k]}(x_1), \ldots, \pi_r^{[k]}(x_1)$, we can view $\pi_1^{[k]}(x_1)$ as determining $\pi_2^{[k]}(x_1), \ldots, \pi_r^{[k]}(x_1), \kappa^{[k]}(x_1)$.
In other words, instead of viewing the feedback as determining the center manifold, we can view the first coordinate function of the center manifold as determining the other coordinate functions and the feedback.
Alternatively we can view π_1 as a pseudo control and write the dynamics as

$$x_1^+ = A_1x_1 + \bar{f}_1(x_1;\pi_1). \tag{2.14}$$

We shall call this dynamics the *Controlled Center Dynamics*.
Now let us write explicitly the solution of equations (2.9) and (2.10).

2.1 Linear Center Manifold

Suppose the entries in K_2 are $K_{2,1}, K_{2,2}, \cdots, K_{2,n-r}$. Then the characteristic polynomial, $p(\lambda)$, of the matrix $A_2 + B_2K_2$ is defined by

$$\begin{array}{rcl}p(\lambda) &=& \det\left(\lambda I_{(n-r)\times(n-r)} - A_2 - B_2K_2\right)\\ &=& \lambda^{n-r} - K_{2,n-r}\lambda^{n-r-1} - \cdots, K_{2,2}\lambda - K_{2,1}\end{array} \tag{2.15}$$

The linear part of the feedback (2.4) is given by

$$u(x_1,x_2) = K_1x_1 + K_2x_2 + O(x_1,x_2)^2. \tag{2.16}$$

Theorem 2.1 *([12]) Given the feedback $\mathcal{F}$, the center manifold (2.8) is given by*

$$x_2 = \pi^{[1]}x_1 + O(x_1^2)$$

with the components of $\pi^{[1]}$ uniquely determined by

$$\begin{array}{rcl}\pi_1^{[1]} &=& K_1p(A_1)^{-1}\\ \pi_i^{[1]} &=& \pi_1^{[1]}A_1^{i-1}, \quad \textit{for } i = 2,\cdots,n-r\end{array} \tag{2.17}$$

where $\pi_i^{[1]}$ is the ith row vector in $\pi^{[1]}$.

The matrix $p(A_1)$ is always invertible as discussed in [12].

Now, consider the following change of coordinates

$$\tilde{x}_{2,i} = x_{2,i} - \pi_1^{[1]}A_1^{i-1}x_1, \quad i = 1,\cdots,n-r \tag{2.18}$$

then,

$$\begin{array}{rcl}\tilde{x}_i^+ &=& \tilde{x}_{i+1}, \quad i = 1,\cdots,n-r\\ \tilde{x}_{n-r}^+ &=& \sum_{i=1}^{n-r}k_{2,i}\tilde{x}_i\end{array}$$

Hence the coefficient K_1 has been removed from the x_2-part of the dynamics (2.3)-(2.16) by a change of coordinates. With $K_1 = 0$, we deduce from (2.17) that $\pi^{[1]} = 0$. So, the linear terms of the center manifold have been removed.

Proposition 2.1 *Given any feedback (2.16) satisfying Property $\mathcal{P}$, and the change of coordinates (2.18), then the center manifold is given by*

$$\tilde{x}_2 = O(x_1^2) \tag{2.19}$$

2.2 Quadratic Center Manifold

In the next, we derive the quadratic center manifold. Under a linear change of coordinates (2.18), the system is transformed into

$$x_1^+ = A_1x_1 + f_1^{[2]}(x_1, z_2 + \pi^{[1]}x_1, \kappa^{[2]}(x_1)) + O(\cdot)^3$$

$$
\begin{aligned}
\tilde{x}_2^+ &= A_2(\tilde{x}_2 + \pi^{[1]}x_1) - \pi^{[1]}A_1x_1 \\
&+B_2(K_1x_1 + K_2\tilde{x}_2 + K_2\pi^{[1]}x_1 + \kappa^{[2]}(x_1)) \\
&+f_2^{[2]}(x_1, \tilde{x}_2 + \pi^{[1]}x_1, u(x_1, \tilde{x}_2 + \pi^{[1]}x_1)) \\
&-\pi^{[1]}f_1^{[2]}(x_1, \tilde{x}_2 + \pi^{[1]}x_1, u(x_1, \tilde{x}_2 + \pi^{[1]}x_1)) \\
&+O(x_1, \tilde{x}_2, u)^3
\end{aligned}
$$

in which u is the feedback defined by (2.4). Define a quadratic vector field $\bar{f}_2^{[2]}(x_1, \tilde{x}_2)$ by

$$
\begin{aligned}
&\bar{f}_2^{[2]}(x_1, \tilde{x}_2) = \\
&f_2^{[2]}(x_1, \tilde{x}_2 + \pi^{[1]}x_1, K_1x_1 + K_2\tilde{x}_2 + K_2\pi^{[1]}x_1) \\
&-\pi^{[1]}f_1^{[2]}(x_1, \tilde{x}_2 + \pi^{[1]}x_1, K_1x_1 + K_2\tilde{x}_2 + K_2\pi^{[1]}x_1)
\end{aligned} \tag{2.20}
$$

Then from (2.18) and (2.20), the equation (2.3) is equivalent to

$$
\begin{aligned}
x_1^+ &= A_1x_1 + f_1^{[2]}(x_1, \tilde{x}_2 + \pi^{[1]}x_1, u(x_1, \tilde{x}_2 + \pi^{[1]}x_1)) \\
&+O(x_1, \tilde{x}_2)^3 \\
\tilde{x}_2^+ &= A_2\tilde{x}_2 + B_2(K_2z_2 + \alpha^{[2]}(x_1, \tilde{x}_2 + \pi^{[1]}x_1)) \\
&+\bar{f}_2^{[2]}(x_1, \tilde{x}_2) + O(x_1, \tilde{x}_2)^3
\end{aligned} \tag{2.21}
$$

In the $(x_1, \tilde{x}_2)$ coordinates, the center manifold has the form (2.19). It satisfies the center manifold equation

$$
\bar{A}_2\pi^{[2]}(x_1) + B_2\kappa^{[2]}(x_1) + \bar{f}_2^{[2]}(x_1, 0) = \pi^{[2]}(A_1x_1)
$$

Let us adopt the following matrix notations,

$$
\begin{aligned}
\pi_i^{[2]}(x_1) &= x_1^TQ_ix_1 \\
\bar{f}_{2,i}^{[2]}(x_1, 0) &= x_1^TR_ix_1 \\
\kappa(x_1) &= x_1^TLx_1
\end{aligned} \tag{2.22}
$$

where Q_i, R and L are symmetric $r \times r$ matrices. Let $\mathcal{S}$ be the operator defined by

$$
\mathcal{S}_{A_1}(Q) = A_1^TQA_1 \tag{2.23}
$$

for all symmetric $r \times r$ matrices Q.

Theorem 2.2 *([12]) If*

$$
x_2 = \pi^{[1]}(x_1) + \pi^{[2]}(x_1) + O(x_1)^3
$$

is the center manifold of (2.3), then $\pi^{[2]}(x_1)$ is uniquely determined by the following equations:

$$
\pi_i^{[2]}(x_1) = x_1^TQ_ix_1, \quad \textit{for } i = 1, 2, \cdots, n-r
$$

where

$$
Q_1 = p(\mathcal{S}_{A_1})^{-1}\left(L + R_{n-r} + \sum_{i=2}^{n-r}\sum_{j=0}^{i-2} K_{2,i}\mathcal{S}_{A_1}^j(R_{i-j-1})\right)
$$

and

$$
Q_i = \mathcal{S}_{A_1}^{i-1}(Q_1) - \sum_{j=0}^{i-2}\mathcal{S}_{A_1}^j(R_{i-j-1})
$$

in which $\mathcal{S}_{A_1}$ is the operator defined by (2.23); R_i is from the quadratic dynamics and it is defined by (2.22) and (2.20); L is from the quadratic feedback and it is defined by (2.22).

We can also show that the operator $p(\mathcal{S}_{A_1})$ is always invertible [12].

There are some special cases in which the center manifold is simpler. For instance, if (2.21) is in quadratic normal form (see [17]), then $\bar{f}_2^{[2]}$ is independent of x_1. In this case, $\bar{f}_2^{[2]}(x_1, 0) = 0$. Therefore, $R_i = 0$. Under the feedback

$$
u = K_2x_2 + x_1^TQ_{fb}x_1
$$

the center manifold of (2.21) is

$$
x_2 = \pi^{[2]}(x_1)
$$

where $\pi_i^{[2]}(x_1) = x_1^TQ_ix_1$, $Q_1 = p(\mathcal{S}_{A_1})^{-1}(Q_{fb})$, and $Q_i = \mathcal{S}_{A_1}^{i-1}(Q_1)$.

3 Stabilization of Systems with a Fold or Period Doubling Control Bifurcation

In this section we use the precedent results to stabilize systems with a fold or period doubling control bifurcation i.e. those where the system has a single uncontrollable mode, $\lambda \in \mathbb{R}$, such that, $|\lambda| > 1$ or $\lambda = -1$, respectively.

When there is only one uncontrollable mode $\lambda \notin \{0, 1\}$ in (2.3), we know, from [6, 17], that there exists a cubic change of coordinates and feedback bringing the system to its cubic normal form

$$
\begin{aligned}
z_1^+ &= \lambda z_1 + \gamma z_1z_{21} + \sum_{i=1}^{r+1}\delta_iz_{2i}^2 + \bar{\gamma}z_1^2z_{21} + \sum_{i=1}^{r+1}\bar{\delta}_iz_1z_{2i}^2 \\
&+ \sum_{i=1}^{r+1}\sum_{j=i}^{r+1}\bar{\epsilon}_{ij}z_{21}z_{2j}z_{2i} + O(z_1, z_2, v)^4, \\
z_2^+ &= A_2z_2 + B_2v + O(z_1, z_2, v)^2,
\end{aligned} \tag{3.24}
$$

with $z_{2,r+1} = v$. We know also that this system exhibits a control bifurcation provided the transversality condition $\widetilde{\delta} = \sum_{i=1}^{r+1}(1 + \lambda^{i-1})\delta_i \neq 0$ is satisfied [17]. Let $\widehat{\delta} = \sum_{i=1}^{r+1}\delta_i$.

Suppose that we use the piecewize linear feedback

$$v = K_1 z_1 + K_2 z_2, \tag{3.25}$$

with $K_1 = \begin{cases} \bar{k}_1, & z \geq 0 \\ \tilde{k}_1, & z < 0 \end{cases}$.

Theorem 3.1 *Consider the system (3.24). If $\gamma\widetilde{\delta\delta} \neq 0$, then the feedback (3.25) practically stabilizes the system (3.24) around the origin when $\lambda > 1$ or $\lambda < -1$. The feedback asymptotically stabilizes the system around the origin when $\lambda = -1$.*

Proof: Let us write λ as $\lambda = (1+\epsilon)\text{sign}(\lambda)$, with ϵ is a slightly positive number. If we consider ϵ as an extra state whose equation is $\epsilon^+ = \epsilon$, the term ϵz_1 will be considered of order two. Then, the linear part of the closed loop system (3.24)-(3.25) has the form

$$\begin{array}{rcl} \epsilon^+ &=& \epsilon, \\ z_1^+ &=& \text{sign}(\lambda) z_1 + O(z_1, z_2, \epsilon)^2, \\ z_2^+ &=& \bar{A}_2 z_2 + O(z_1, z_2)^2. \end{array} \tag{3.26}$$

Hence, for the closed loop system (3.24)-(3.25), a center manifold exists. It is defined by $z_2 = \pi(\epsilon, z_1)$. Since there is no linear term in ϵ in the z_1-subdynamics of the system (3.26), the linear part of the center manifold can be written as

$$z_2 = \pi^{[1]} z_1.$$

¿From (2.17), the components of $\pi^{[1]}$ are given by

$$\begin{array}{rcl} \pi_i^{[1]} &=& \pi_1^{[1]}, \quad i = 2, \ldots, r, \\ K_1 &=& p(\text{sign}(\lambda))\pi_1^{[1]}, \end{array} \tag{3.27}$$

since $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & \text{sign}(\lambda) \end{bmatrix}$ for the dynamics in the (ϵ, z_1, z_2) space. Thus, the controlled center dynamics is

$$z_1^+ = \begin{cases} \lambda z_1 + \Phi(\bar{\pi}_1^{[1]}) z_1^2 + O(z_1)^3, & z_1 \geq 0, \\ \lambda z_1 + \Phi(\widetilde{\pi}_1^{[1]}) z_1^2 + O(z_1)^3, & z_1 < 0. \end{cases}$$

with $\Phi(X) = X(\gamma + \hat{\delta} X)$, $\bar{\pi}_1^{[1]} = \frac{\bar{k}_1}{p(\text{sign}(\lambda))}$, and $\widetilde{\pi}_1^{[1]} = \frac{\tilde{k}_1}{p(\text{sign}(\lambda))}$.

Since $\gamma \neq 0$ and $\widehat{\delta} \neq 0$, there are two distinct solutions for the equation $\Phi(\pi_1^{[1]}) = 0$, hence $\Phi(\pi_1^{[1]})$ changes its sign. So we can choose $\overline{\pi}_1^{[1]}$ and $\widetilde{\pi}_1^{[1]}$ such that $\Phi(\overline{\pi}_1^{[1]}) = -\Phi(\widetilde{\pi}_1^{[1]}) = -\Phi_0$, with $\Phi_0 > 0$ if $\lambda > 1$, and $\Phi_0 < 0$ if $\lambda < 1$. In this case, the controlled center dynamics will have the form

$$z_1^+ = \lambda z_1 - \Phi_0 |z_1| z_1 + O(z_1)^3, \tag{3.28}$$

which is the normal form of the supercritical bird foot bifurcation for maps, as discussed in the appendix.

For λ such that $\lambda \notin \{0, 1\}$, we have three equilibrium points: the origin and $\bar{z}^* = \frac{\lambda-1}{\Phi_0}$, $\bar{z}^{**} = -\frac{\lambda-1}{\Phi_0} = -\bar{z}^*$. The origin is unstable for $\lambda > 1$ or $\lambda < -1$, and the two other equilibrium points are stable. Thus, the solution converges to $\bar{z}^*$ or $\bar{z}^{**}$. Hence, by making $\bar{z}^*$ sufficiently close to the origin, i.e. by choosing Φ_0 sufficiently large, we shall have practical stability for the origin of the controlled center dynamics. We can show that this implies practical stability of the origin of the system (3.24).

When $\lambda = -1$, the controlled center dynamics (3.28) reduces to

$$z_1^+ = -z_1 - \Phi_0 |z_1| z_1 + O(z_1)^3.$$

If we use the Lyapunov function $V(z_1) = z_1^2$, then

$$\Delta V = V(z^+) - V(z) = 2\Phi_0 |z_1| z_1^2 + O(z_1^3).$$

Hence choosing $\Phi_0 < 0$, permits to ensure that the origin is asymptotically stable. ■

Now let us consider the quadratic feedback

$$v = K_1 z_1 + K_2 z_2 + \kappa^{[2]}(z_1) \tag{3.29}$$

instead of the feedback (3.25). The coefficient K_2 is such that $|\sigma(A + B_2 K_2)| < 1$.

Theorem 3.2 *Consider the system (3.24). If $\gamma\widetilde{\delta} \neq 0$, then the feedback (3.29) with $K_1 = 0$ practically stabilizes the system (3.24) around the origin when $\lambda > 1$ or $\lambda < -1$. It asymptotically stabilizes the system around the origin when $\lambda = -1$.*

Proof: Adopting the same approach as precedently we show the existence of a center manifold in the (ϵ, z_1) plane. The feedback (3.29) shapes the linear and quadratic parts of the center manifold

$$z_2 = \pi^{[1]} z_1 + \pi^{[2]}(z_1)$$

which in turn shape the quadratic and cubic parts of the controlled center dynamics given by

$$z_1^+ = \lambda z_1 + \Phi(\pi_1^{[1]}) z_1^2 + O(z_1^3).$$

Since the equation $\Phi(X) = 0$ admits zero as a solution, we can choose the solution $\pi_1^{[1]} = 0$, which gives $K_1 = 0$ from (3.27). Then, by choosing $\pi_1^{[2]}(z_1) = c z_1^2$ arbitrarily, we deduce that the controlled center dynamics is given by

$$z_1^+ = \lambda z_1 + \gamma c z_1^3 + O(z_1)^4. \tag{3.30}$$

Since $|\lambda| > 1$, the origin is unstable. If we choose c such that $(1-\lambda)\gamma c > 0$, the two equilibrium points $\hat{z}^* = \sqrt{\frac{1-\lambda}{\gamma c}}$ and $\hat{z}^{**} = -\sqrt{\frac{1-\lambda}{\gamma c}}$ are stable. The controlled center dynamics (3.30) has the form of a system with a supercritical pitchfork bifurcation. Since the solution converges to one of the equilibrium points $\hat{z}^*$ or $\hat{z}^{**}$, the origin of the controlled center dynamics can be made practically stable by having the equilibrium points $\hat{z}^*$ and $\hat{z}^{**}$ sufficiently close to the origin. We can show that this implies practical stability of the origin of the system (3.24).

When $\lambda = -1$, the controlled center dynamics (3.30) reduces to

$$z_1^+ = -z_1 + \gamma c z_1^3 + O(z_1^4).$$

We see that choosing c such that $\gamma c > 0$ permits to ensure that the origin is asymptotically stable. ■

The piecewize linear feedback (3.25) is more robust than the quadratic feedback (3.29). Indeed, using the quadratic feedback (3.29) requires having the exact solutions of the equation $\Phi(\pi_1^{[1]}) = 0$. If there exists a small uncertainty on the invariants γ and δ_i (with $i = 1, \cdots, r+1$), the quadratic terms generated by the uncertainty in the controlled center dynamics (3.30) will be a source of instability of the system. Using the piecewize linear feedback (3.25) does not necessitate the exact solutions of the equation $\Phi(\pi_1^{[1]}) = 0$, as we just have to find $\overline{\pi}_1^{[1]}$ and $\widetilde{\pi}_1^{[1]}$ such that $\Phi(\overline{\pi}_1^{[1]})\Phi(\widetilde{\pi}_1^{[1]}) < 0$. Thus the piecewize linear feedback is more robust.

4 Appendix: The Birdfoot Bifurcation for Maps

In this section we introduce the discrete-time version of the "bird foot bifurcation" (see [15] for a treatment of the continuous-time case).
Consider a dynamical system

$$x^+ = \mu x - \Phi_0 x|x| + O(x^3), \tag{4.31}$$

with $x \in \mathbb{R}$, $\mu \in \mathbb{R}$ a parameter, and $\Phi_0 \in \mathbb{R} \setminus \{0\}$ a constant. The fixed points of the system are the solutions of the equation

$$((1-\mu) + \Phi_0|x|)x = 0.$$

Provided $(\mu - 1)\Phi_0 > 0$, the dynamical system has three fixed points: the origin, $x^* = \frac{\mu-1}{\Phi_0}$, and $x^{**} = -\frac{\mu-1}{\Phi_0} = -x^*$. If $\mu = 1$, the dynamical system has the origin as the only fixed point.

Let us consider the Lyapunov function $V(x) = x^2$, then $\Delta V = V(x^+) - V(x) = (\mu^2-1)x^2 - 2\Phi_0\mu|x|x^2 + O(x^4)$.

If $|\mu| < 1$, then $\Delta V < 0$ and the origin is an asymptotically stable equilibrium point. If $|\mu| > 1$, then $\Delta V > 0$ and the origin is an unstable equilibrium point.
When $\Phi_0 > 0$ (resp. $\Phi_0 < 0$), the equilibrium points x^* and x^{**} appear when $\mu > 1$ (resp. $\mu < 1$). They are unstable when the origin is asymptotically stable, and are asymptotically stable when the origin is unstable. As for the pitchfork bifurcation, we have an exchange of the stability properties, at $\mu = 1$, between the origin and the two equilibrium points x^* and x^{**}.
If $\mu = 1$, the origin is the only equilibrium point. It is asymptotically stable when $\Phi_0 > 0$, and unstable when $\Phi_0 < 0$. If $\mu = -1$, the origin is asymptotically stable when $\Phi_0 < 0$, and unstable when $\Phi_0 > 0$.

When $\Phi_0 > 0$, we shall call the bifurcation a *supercritical bird foot bifurcation*. When $\Phi_0 < 0$, we shall call the bifurcation *subcritical bird foot bifurcation*.

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