Nonlinear Observer Design for Smooth Systems[*]

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Abstract: Recently Kazantzis-Kravaris and Kreisselmeier-Engel have suggested two apparently different approaches to constructing observers for nonlinear systems. We show that these approaches are closely related and lead to observers with linear error dynamics in transformed variables. In particular we give sufficient conditions for the existence of smooth solutions to the Kazantzis-Kravaris PDE.

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1 Introduction

We consider the problem of constructing an observer for a smooth system without controls

$$\begin{array}{rcccl} \dot{x} &=& f(x) &=& Fx+\bar{f}(x) \\ y &=& h(x) &=& Hx+\bar{h}(x) \\ x(0) &=& x^0 \end{array} \tag{1.1}$$

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where $f : \mathcal{X} \to I\!\!R^n$ and $h : \mathcal{X} \to I\!\!R^p$ are C^r functions with $r \geq 1$ and $\bar{f}(x) = o(x)$, $\bar{h}(x) = o(x)$. The set $\mathcal{X} \subset I\!\!R^n$ is assumed to be an invariant subset of the dynamics and a neighborhood of the equilibrium $x = 0$. We let $\mathcal{Y} = h(\mathcal{X}) \subset I\!\!R^p$. Typically $p \leq n$.

An observer is a second dynamical system

$$\begin{array}{rcccl} \dot{\hat{z}} &=& a(\hat{z}, y) &=& A\hat{z} + By + \bar{a}(\hat{z}, y) \\ \hat{x} &=& c(\hat{z}, y) &=& C\hat{z} + Dy + \bar{c}(\hat{z}, y) \\ \hat{z}(0) &=& \hat{z}^0 && \end{array} \tag{1.2}$$

where $a : \mathcal{Z} \times \mathcal{Y} \to I\!\!R^k$ and $c : \mathcal{Z} \times \mathcal{Y} \to I\!\!R^n$ are C^r functions, $\mathcal{Z} \subset I\!\!R^k$ and $\bar{a}(\hat{z}, y) = o(\hat{z}, y)$, $\bar{c}(\hat{z}, y) = o(\hat{z}, y)$.

The goal is to choose the observer in such a way that the estimation error $\tilde{x}(t) = x(t) - \hat{x}(t) \to 0$ as $t \to \infty$. The dimension k of the observer can be different from the dimension n of the system. For nonlinear systems one expects that $k \geq n$. There is a vast literature on this topic, we refer the reader to the recent survey paper [3] and conference proceedings [8].

Kazantzis and Kravaris [1] have introduced a method for constructing an observer which has linear error dynamics in transformed coordinates. We briefly review their method. Suppose the system (2.1) is real analytic. One selects an $n \times p$ matrix B and an invertible $n \times n$ matrix T such that the matrix

$$A = (TF - BH)T^{-1}$$

is Hurwitz and such that the eigenvalues of A are distinct from those of F. Then one seeks an real analytic solution of the Kazantzis-Kravaris partial differential equation (KK PDE)

$$\frac{\partial \theta}{\partial x}(x) f(x) \quad = \quad A\theta(x) + \beta(h(x)). \tag{1.3}$$

where $\beta : \mathcal{Y} \to I\!\!R^n$ is real analytic and

$$\frac{\partial \beta}{\partial y}(y) \quad = \quad B$$

If θ satisfies this PDE then

$$\frac{\partial \theta}{\partial x}(0) \quad = \quad T$$

and so θ is a local diffeomorphism. If we define a change of coordinates

$$z = \theta(x)$$

then

$$\begin{array}{rcl} \dot{z} &=& Az + \beta(y) \\ y &=& h(\theta^{-1}(z)) \end{array} \quad (1.4)$$

One can construct a local observer for the transformed system (1.4),

$$\begin{array}{rcl} \dot{\hat{z}} &=& A\hat{z} + \beta(y) \\ \hat{x} &=& \theta^{-1}(\hat{z}) \\ \hat{z}(0) &=& 0 \end{array} \quad (1.5)$$

which has linear error dynamics in the transformed coordinates

$$\dot{\tilde{z}} = A\tilde{z} \quad (1.6)$$

where $\tilde{z}(t) = z(t) - \hat{z}(t)$. Since A is Hurwitz, the error goes to zero as $t \to \infty$ provided that $x(t)$ stays sufficiently small.

The observer can also be implemented in the original coordinates,

$$\dot{\hat{x}} = f(\hat{x}) + \left(\frac{\partial\theta}{\partial x}(\hat{x})\right)^{-1} (\beta(y) - \beta(h(\hat{x}))\,. \quad (1.7)$$

This is a standard form for an observer, a copy of this dynamics driven by a gain times the estimation error of some function of y. In this case, the gain varies with $\hat{x}$.

Kazantzis and Kravaris [1] gave sufficient conditions for the solvability of (1.3). To state them we need some definitions. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ denote the spectrum of F. We say that a complex number μ_j is *resonant of degree $d > 0$ with the spectrum of F* if there is a tuple $m = (m_1, \ldots, m_n)$ of nonnegative integers

$$\sum_{i=1}^{n} m_i\lambda_i = \mu_j, \qquad \sum_{i=1}^{n} m_i = d.$$

The spectrum of F is in the *Siegel domain* if 0 is in the convex hull of $\lambda_1, \ldots, \lambda_n$ in $\mathbb{C}$. The spectrum of F is in the *Poincaré domain* if 0 is not in the convex hull of $\lambda_1, \ldots, \lambda_n$. Since F is real, the spectrum of F is in the *Poincaré domain* iff it is in the open left half plane of $\mathbb{C}$ or it is in the open right half plane of $\mathbb{C}$.

Let $\mu = (\mu_1, \dots, \mu_n)$ denote the spectrum of A. Kazantzis and Kravaris [1] showed that if no μ_j is resonant of any degree d with the spectrum of F and if the spectrum of F is in the Poincaré domain then for given real analytic $\beta(y)$ the KK PDE has a unique real analytic solution defined in some neighborhood of $x = 0$.

Suppose $C > 0,\ \nu > 0$. A complex number μ_j is *type* (C, ν) with respect to the spectrum of F if for any tuple $m = (m_1, \dots, m_n)$ of nonnegative integers with $\sum m_i = d > 0$, one has

$$\left| \sum_{i=1}^{n} m_i \lambda_i - \mu_j \right| \geq \frac{C}{d^\nu}$$

Recently we claimed that if the spectrum of A is of type (C, ν) with respect to the spectrum of F then for given real analytic $\beta(y)$ the KK PDE has a unique real analytic solution defined in some neighborhood of $x = 0$ but we have found a error in our proof [5]. One needs the additional assumption that the spectrum of F is of type (C, ν) with respect to the spectrum of F [7]

Another approach to observer design has recently been presented by Kreisselmeier and Engel [2]. The purpose of our paper is to show that their approach is closely related to that of Kazantzis and Kravaris and the two approaches taken together yield the existence of smooth solutions to the KK PDE under suitable conditions. Given the system (2.1), Kreisselmeier and Engel construct an observer as follows.

Choose a $k \times k$ Hurwitz matrix A and a $k \times p$ matrix B where $k \geq n$. Given x^0, let $x(s, x^0),\ y(s, x^0)$ denote the corresponding state and output trajectories of (2.1). Define

$$z^0 = \theta(x^0) \ = \ \int_{-\infty}^{0} e^{-As} B y(s, x^0)\, ds. \tag{1.8}$$

One has to impose suitable conditions so that the integral exists.

If one can find a mapping $x^0 = \psi(z^0)$ which is a left inverse of θ,

$$\psi(\theta(x^0) = x^0$$

then one can construct the Kreisselmeier and Engel observer

$$\begin{aligned} \dot{\hat{z}} &= A\hat{z} + By \\ \hat{x} &= \psi(\hat{z}). \end{aligned} \tag{1.9}$$

Kreisselmeier and Engel showed that for a suitable choice of A, B, the mapping θ is injective so that ψ exists. We shall show that the error in the transformed variables $\tilde{z}(t) = z(t) - \hat{z}(t)$ has linear error dynamics (1.6).

Let us review the differences in the approaches of Kazantzis-Kravaris and Kreisselmeier-Engel. The Kazantzis-Kravaris construction applies to real analytic systems and defines an observer of the same dimension as the system. The observer is constructed by a real analytic change of coordinates found by solving the KK PDE. The KK PDE is locally solvable if the spectrum of the linear part of the dynamics is in the Poincaré domain and if the spectrum of the linear part of the observer is not resonant with that of the dynamics. In our recent work [5], [5], it known that it is also locally solvable if the spectrum of both F and A are of type (C, ν) with respect to the spectrum of F. This local solution leads to a local observer. Conditions for the global solvability of the KK PDE are not known. The Taylor series of the solution of the KK PDE can be found to any degree by solving a sequence of linear equations for the coefficients and therefore the Taylor series of the inverse can be found up to the same degree. This is essential if the observer is to be implemented in the transformed coordinates. The inverse change of coordinates need not be found if the observer is implemented in the original coordinates, but the Jacobian of the change of coordinates must be inverted. If the system (1.1) is only C^r and if the spectrum of A is not resonant up to degree $d \leq r$ with the spectrum of F, an approximate solution to the KK PDE, polynomial of degree d, can be found and used to construct a local observer with error dynamics linear to degree d in the transformed coordinates.

The Kreisselmeier-Engel construction applies to Lipschitz continuous systems (2.1) and defines an observer whose dimension is at least as large as that of the system. The observer is constructed by a change of variables found through an integral equation but the change of variables need not be a change of coordinates. It is not guaranteed to be smooth even if the system is. It can be hard to compute explicitly. Its existence depends on growth conditions for the output of the system in negative time. A left inverse of the change of variables must be found. The observer is implemented in the transformed variables where it has linear error dynamics. The Kreisselmeier-Engel observer requires a choice of A and B to define θ by (1.8). When the system (1.1) is C^1 then the convergence of this integral implies that the spectra of A and F are disjoint. We shall show that if (1.3) does not hold for some invertible T then θ is not differentiable.

2 Main Results

In this paper, we make the following standing assumptions about the system

$$\begin{aligned} \dot{x} &= f(x) \\ y &= h(x) \\ x(0) &= x^0. \end{aligned} \tag{2.1}$$

1. $f : \mathcal{X} \to I\!R^n$ and $h : \mathcal{X} \to I\!R^p$ are Lipschitz continuous on an invariant set $\mathcal{X} \in I\!R^n$ with Lipschitz constants L_f and L_h, respectively,
2. A is $k \times mk$ Hurwitz matrix and there are $M > 0, a > L_f$ such that
$$|e^{At}| \leq Me^{-at}$$
3. $\beta : \mathcal{Y} \to I\!R^k$ and Lipschitz continuous with Lipschitz constant L_β,
4. when the integral exists, θ is defined by
$$z^0 = \theta(x^0) = \int_{-\infty}^{0} e^{-As}\beta(y(s, x^0))\ ds. \tag{2.2}$$

Theorem 2.1 *Under the assumptions 1-4, the map $\theta : \mathcal{X} \to I\!R^n$ exists and is Lipschitz continuous.*

Proof: Given x^0, let $x(s, x^0), y(s, x^0)$ denote the corresponding state and output trajectories of (2.1). Now

$$\begin{aligned} x(t, x^0) &= x^0 + \int_0^t f(x(s, x^0))\ ds \\ |x(t, x^0)| &\leq |x^0| + \int_0^t |f(x(s, x^0))|\ ds \\ |x(t, x^0)| &\leq |x^0| + \int_0^t L_f |x(s, x^0)|\ ds \end{aligned}$$

so by Gronwall's inequality

$$|x(t, x^0)| \leq |x^0| e^{L_f|t|}.$$

Hence

$$\begin{aligned} |y(t, x^0)| &\leq L_h |x^0| e^{L_f|t|} \\ |\beta(y(t, x^0))| &\leq L_\beta L_h |x^0| e^{L_f|t|}. \end{aligned}$$

Since $a - L_f > 0$, the integral exists

$$
\begin{aligned}
\left|\theta(x^0)\right| &\leq \int_{-\infty}^{0} \left|e^{-As}\beta(y(s,x^0))\right| \, ds \\
&\leq \int_{-\infty}^{0} ML_\beta L_h |x^0| e^{(a-L_f)t} \, ds \\
&\leq \frac{ML_\beta L_h |x^0|}{a - L_f}.
\end{aligned}
$$

Next we show that θ is Lipschitz continuous. Given two initial conditions $x^0,\ x^1 \in \mathcal{X}$, with corresponding state and output trajectories $x(t,x^i),\ y(t,x^i)$, the Lipschitz assumptions imply that

$$
\begin{aligned}
|x(t,x^1) - x(t,x^0)| &\leq |x^1 - x^0| e^{L_f |t|} \\
|y(t,x^1) - y(t,x^0)| &\leq L_h |x^1 - x^0| e^{L_f |t|} \\
|\beta(y(t,x^1)) - \beta(y(t,x^0))| &\leq L_\beta L_h |x^1 - x^0| e^{L_f |t|}
\end{aligned}
$$

so

$$
\begin{aligned}
|\theta(x^1) - \theta(x^0)| &\leq \int_{-\infty}^{0} \left| e^{-As} \left(\beta(y(t,x^1)) - \beta(y(t,x^0)) \right) \right| \, ds \\
&\leq \int_{-\infty}^{0} |e^{-As}| L_\beta L_h |x^1 - x^0| e^{L_f |s|} \, ds \\
&\leq |x^1 - x^0| \int_{-\infty}^{0} ML_\beta L_h e^{(a-L_f)s} \, ds.
\end{aligned}
$$

The map θ is Lipschitz continuous with Lipschitz constant

$$
L_\theta = \int_{-\infty}^{0} ML_\beta L_h e^{(a-L_f)s} \, ds = \frac{ML_\beta L_h}{a - L_f}.
$$

□

Theorem 2.2 *Under the assumptions 1-4, let $x(t), y(t)$ be state and output trajectories of the system where $x(0) \in \mathcal{X}$. Let $z(t) = \theta(x(t))$ where θ is defined by (1.8). Then*

$$
\frac{d}{dt} z(t) = Az(t) + \beta(y(t)).
$$

Proof: Because the system is autonomous

$$z(t) = \int_{-\infty}^{0} e^{-As}\beta(y(s+t))\ ds$$

Let $r = s + t$ then

$$z(t) = e^{At}\int_{-\infty}^{t} e^{-AR}\beta(y(r))\ Dr$$

so

$$\dot{z}(t) = Az(t) + \beta(y(t))$$

□

Remark. It is interesting to note that the previous theorem does not require any assumption of differentiability. □

Corollary 2.3 *The Kreisselmeier-Engel observer (1.9) has linear error dynamics in transformed variables.*

Corollary 2.4 *Under the assumptions 1-4, if $\theta(x)$ is C^1 then $\theta(x)$ satisfies the KK PDE (1.3).*

Corollary 2.5 *Under the assumptions 1-4, the observer (1.5) has asymptotically stable, linear error dynamics in the z variables.*

Theorem 2.6 *In addition to the assumptions 1-4, assume that f, h, and β are C^1. Then θ is C^1.*

Proof: Proceeding formally from (1.8) we have

$$\frac{\partial\theta}{\partial x^0}(x^0) = \int_{-\infty}^{0} e^{-As}\frac{\partial\beta}{\partial x^0}(y(s,x^0))\ ds.$$

If this integral converges then it is the actual derivative. By the chain rule

$$\frac{\partial\theta}{\partial x^0}(x^0) = \int_{-\infty}^{0} e^{-As}\frac{\partial\beta}{\partial y}(y(s,x^0))\frac{\partial h}{\partial x}(x(s,x^0))\frac{\partial x}{\partial x^0}(s,x^0)\ ds.$$

Let

$$\begin{aligned}
\Phi(s,x^0) &= \frac{\partial x}{\partial x^0}(s,x^0) \\
F(s,x^0) &= \frac{\partial f}{\partial x}(x(s,x^0)) \\
H(s,x^0) &= \frac{\partial h}{\partial x}(x(s,x^0)) \\
B(s,x^0) &= \frac{\partial \beta}{\partial y}(y(s,x^0))
\end{aligned}$$

Now

$$\begin{aligned}
\frac{d}{ds}\,\Phi(s,x^0) &= F(s,x^0)\Phi(s,x^0) \\
\Phi(0,x^0) &= I
\end{aligned}$$

so

$$\begin{aligned}
\Phi(t,x^0) &= I + \int_0^t F(s,x^0)\Phi(s,x^0)\;ds \\
\left|\Phi(t,x^0)\right| &\leq |I| + \int_0^t \left|F(s,x^0)\right|\left|\Phi(s,x^0)\right|\;ds \\
\left|\Phi(t,x^0)\right| &\leq 1 + \int_0^t L_f\left|\Phi(s,x^0)\right|\;ds
\end{aligned}$$

and by Gronwall's inequality

$$\left|\Phi(t,x^0)\right| \leq e^{L_f|t|}.$$

Finally

$$\begin{aligned}
\int_{-\infty}^0 \left|e^{-As}B(s,x^0)H(s,x^0)\Phi(s,x^0)\right|\;ds &\leq \int_{-\infty}^0 ML_\beta L_h e^{(a-L_f)s}\;ds \\
&\leq L_\theta
\end{aligned}$$

so $\frac{\partial\theta}{\partial x}(x^0)$ exists for $x^0 \in \mathcal{X}$. □

Suppose f, h are C^1 on $\mathcal{X}$ with Lipschitz continuous derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial h}{\partial x}$ and β is C^1 on $\mathcal{Y}$ with Lipschitz continuous derivative $\frac{\partial\beta}{\partial y}$. One can show that if a is large enough then $\frac{\partial\theta}{\partial x}$ is Lipschitz continuous. Furthermore if f, h, β are C^2 then $\frac{\partial^2\theta}{\partial x^2}$ exists. Similar statements hold for the higher derivatives. For C^∞ functions f, h, β and compact $\mathcal{X}$, the larger a the more derivatives of θ that can be shown to exist.

Theorem 2.7 *In addition to the assumptions 1-4, assume that f, h, and β are C^1 and*

$$\begin{aligned}\frac{\partial f}{\partial x}(0) &= F \\ \frac{\partial h}{\partial x}(0) &= H \\ \frac{\partial \beta}{\partial y}(0) &= B.\end{aligned}$$

Then

$$\frac{\partial \theta}{\partial x}(0) = T$$

where T is the unique solution of

$$TF - AT = BH \tag{2.3}$$

Proof: If λ is an eigenvalue of F then $|\lambda| \leq L_f$ and if μ is an eigenvalue of A then $|\mu| \geq a$. Since $a > L_f$, the spectra of F and A are disjoint. Therefore (2.3) has a unique solution T. From the definition of θ, we have

$$\frac{\partial \theta}{\partial x}(0) = \int_{-\infty}^{0} e^{-As} B H e^{Fs} ds.$$

Let

$$S = \int_{-\infty}^{0} e^{-As} B H e^{Fs} ds$$

then

$$\begin{aligned}\int_{-\infty}^{0} \frac{d}{ds} e^{-As} B H e^{Fs} ds &= BH \\ SF - AS &= BH \qquad (2.4)\end{aligned}$$

Therefore T and S satisfy the same equation (2.3, 2.4) so $T = S$. □

Corollary 2.8 *In addition to the assumptions 1-4, assume that f, h, and β are C^1. If F, H, A, B are not related by (2.3) for some T then θ (2.2) is not differentiable at $x = 0$.*

Corollary 2.9 *In addition to the assumptions 1-4, assume that $k = n$ and f, h, and β are C^1 then θ is a local diffeomorphism iff the unique T satisfying (2.3) is invertible.*

Theorem 2.10 *Suppose the spectra of F and A are disjoint and T satisfies (2.3). If (H, F) is not observable then T is not invertible.*

Proof: Suppose (H, F) is not observable then there exist $\lambda \in \sigma(F)$ and a vector $x \in I\!R^n$, $x \neq 0$ such that $Hx = 0$ and $Fx = \lambda x$. We multiply (2.3) by x to obtain

$$\lambda TX - Tax = 0.$$

If T is invertible the $Tx \neq 0$ so λ is an eigenvalue of A which is a contradiction. □

Theorem 2.11 *Suppose the spectra of F and A are disjoint and T satisfies (2.3). If (A, B) is not controllable then T is not invertible.*

Proof: Suppose (A, B) is not controllable then there exist $\mu \in \sigma(A)$ and a vector $\xi \in I\!R^n$, $\xi \neq 0$ such that $\xi' B = 0$ and $\xi' A = \mu \xi'$. We multiply (2.3) by ξ' to obtain

$$\xi' TF - \mu \xi' T = 0.$$

If T is invertible the $\xi' T \neq 0$ so μ is an eigenvalue of F which is a contradiction. □

3 Conclusion

We have shown that the approaches of Kazantzis-Kravaris and Kreisselmeier-Engel to observer design are closely related. Both lead to observers with linear error error dynamics in transformed variables. The former requires the solution of a PDE and the latter requires multiple solutions to an ODE followed by quadratures. From a implementation point of view, the former is easier as the PDE can be solved approximately by a finite power series. But this solution is only local as is the resulting observer. Neither approach has been generalized to systems with inputs yet.

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