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The controlled center dynamics of discrete time control bifurcations

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Abstract

In this paper, we introduce the *Controlled center dynamics* for nonlinear discrete time systems with uncontrollable linearization. This is a reduced order control system whose dimension is the number of uncontrollable modes and whose stabilizability properties determine the stabilizability properties of the full order system. After reducing the order of the system, the synthesis of a stabilizing controller is performed based on the reduced order control system. By changing the feedback, the stability properties of the controlled center dynamics will change, and thus the stability properties of the full order system will change too. Thus, choosing a feedback that stabilizes the controlled center dynamics allows stabilizing the full order system. This approach is a reduction technique for some classes of controlled differential equations.

Keywords: Nonlinear control systems; Model reduction; Control bifurcations; Stabilization; Discrete time

1. Introduction

Center manifold theory plays an important role in the study of the stability of nonlinear systems when the equilibrium point is not hyperbolic. The center manifold is an invariant manifold of the differential (difference) equation which is tangent at the equilibrium point to the eigenspace of the neutrally stable eigenvalues. In practice, one does not compute the center manifold and its dynamics exactly, since this requires the resolution of a quasilinear partial differential (nonlinear functional) equation which is not easily solvable. In most cases of interest, an approximation of degree two or three of the solution is sufficient. Then, we determine the reduced dynamics on the center manifold, study its stability and then conclude about the stability of the original system [29,22,6,19]. This theory combined with the normal form approach of Poincaré [30] was used extensively to study parameterized dynamical systems exhibiting bifurcations (see [33] and references therein).

For continuous-time nonlinear systems with uncontrollable linearization, a similar approach was used for the analysis and stabilization of systems with one or two uncontrollable modes [4,1,2,5,8,21,13,17]. The procedure to stabilize these systems is based on using a quadratic feedback where the linear part is used to asymptotically stabilize the linearly controllable part, and the quadratic part is used to change the stability properties of the restriction of the original control system on the center manifold. This approach was, then, generalized to the general class of nonlinear systems with any number of uncontrollable modes in [14] by introducing the *controlled center dynamics*. The controlled center dynamics is a reduced order control system whose dimension is the number of uncontrollable modes and whose stabilizability properties determine the stabilizability properties of the full order system. After reducing the order of the system, the synthesis of a stabilizing controller is performed based on the reduced order control system. By changing the feedback, the stability properties of the controlled center dynamics will change, and thus the stability properties of the full order system will change too. Thus, choosing a feedback that stabilizes the controlled center dynamics allows stabilizing the full order system. Thus, this approach can also be viewed as a reduction technique for some classes of controlled differential equations.

For discrete-time systems, a similar approach was used for one real or complex uncontrollable mode in [10,11,26,12,18].

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The object of this paper is to generalize this methodology to the case of discrete-time systems with any number of uncontrollable modes. We will focus on the case of unparameterized systems, as the methodology generalizes easily to the case of parameterized systems with any number of uncontrollable modes by considering the parameters as an extra-state, i.e. satisfying the equation $\mu^+ = \mu$, where μ denotes the parameter. Let us also denote that when dealing with controlled dynamical systems, it becomes difficult to parallel both the studies referred either to differential or difference nonlinear equations even if many analogies can be set. This is due to the fact that the study of difference equations induces compositions of functions and also because some phenomena appear only in discrete-time. For example, the period-doubling bifurcation appears in monodimensional systems only in discrete-time.

The paper is organized as follows: In Section 2, we define the controlled center dynamics, and show how a feedback will affect it, then, in Sections 3 and 4 we apply this technique to stabilize systems with a transcontrollable bifurcation, fold, and period-doubling control bifurcations. We shall treat the bird foot bifurcation for maps in the appendix. Preliminary results of this work have been published in [16,15].

2. The controlled center dynamics

Consider the following nonlinear system

$$\zeta^+ = f(\zeta, v) \tag{2.1}$$

the variable $\zeta \in \mathbb{R}^n$ is the state, $v \in \mathbb{R}$ is the input variable, and $\zeta^+ = \zeta(k+1)$, for $k \in \mathbb{N}$. The vectorfield $f(\zeta)$ is assumed to be C^k for some sufficiently large k.

Assume $f(0,0) = 0$, and suppose that the linearization of the system at the origin is (A, B),

$$A = \frac{\partial f}{\partial \zeta}(0,0), \quad B = \frac{\partial f}{\partial v}(0,0),$$

with

$$\operatorname{rank}([B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{n-1}B]) = n - r, \tag{2.2}$$

and $r > 0$. Assume also that the system has $n - r$ eigenvalues strictly inside the unit disk, and r eigenvalues on the unit circle. Let us denote by $\Sigma_{\mathscr{U}}$ the system (2.1) under the above assumptions.

The system $\Sigma_{\mathscr{U}}$ is not linearly controllable at the origin, and a change of some control properties may occur around this equilibrium point, this is called a control bifurcation if it is linearly controllable at other equilibria [25].

From linear control theory, we know that there exist a linear change of coordinates and a linear feedback transforming the system $\Sigma_{\mathscr{U}}$ to

$$x_1^+ = A_1 x_1 + \bar{f}_1(x_1, x_2, u),$$
$$x_2^+ = A_2 x_2 + B_2 u + \bar{f}_2(x_1, x_2, u), \tag{2.3}$$

with $x_1 \in \mathbb{R}^r$, $x_2 \in \mathbb{R}^{n-r}$, $u \in \mathbb{R}$, $A_1 \in \mathbb{R}^{r\times r}$ is in the Jordan form and its eigenvalues are on the unit circle, $A_2 \in \mathbb{R}^{(n-r)\times(n-r)}$, $B_2 \in \mathbb{R}^{(n-r)\times 1}$ are in the Brunovskỳ form, i.e.

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

and $\bar{f}_k(x_1, x_2, u)$, for $k = 1, 2$, designates a vectorfield which is a polynomial of degree greater or equal to two.

Now, consider the feedback given by

$$u(x_1, x_2) = \kappa(x_1) + K_2 x_2, \tag{2.4}$$

with κ a smooth function and $K_2 = [k_{2,1} \;\; \cdots \;\; k_{2,n-r}]$.

Because (A_2, B_2) is controllable, the eigenvalues in the closed-loop system associated with the equation of x_2 can be placed at arbitrary points in the complex plane by selecting the appropriate values for K_2. If one of these controllable eigenvalues is placed outside the unit disk, the closed-loop system is unstable around the origin. Therefore, we assume that K_2 has the following property.

Property $\mathscr{P}$. The eigenvalues of the matrix $\bar{A}_2 = A_2 + B_2 K_2$ are strictly inside the unit disk.

Let us denote by $\mathscr{F}$ the feedback (2.4) with the property $\mathscr{P}$.

The closed loop system (2.3)–(2.4) given by

$$x_1^+ = A_1 x_1 + \bar{f}_1(x_1, x_2, \kappa(x_1) + K_2 x_2),$$
$$x_2^+ = A_2 x_2 + B_2(K_2 x_2 + \kappa(x_1))$$
$$\quad + \bar{f}_2(x_1, x_2, \kappa(x_1) + K_2 x_2) \tag{2.5}$$

possesses r eigenvalues on the unit circle, and $n-r$ eigenvalues strictly inside the unit disk. Thus, a center manifold exists [33]. It is represented locally around the origin as

$$W^c = \{(x_1, x_2) \in \mathbb{R}^r \times \mathbb{R}^{n-r} \mid x_2 = \pi(x_1),$$
$$|x_1| < \delta, \pi(0) = 0\} \tag{2.6}$$

for sufficiently small positive real number δ. This means that π and κ satisfy the nonlinear functional equation [33]

$$\bar{A}_2 \pi(x_1) + B_2 \kappa(x_1) + \bar{f}_2(x_1, \pi(x_1), \kappa(x_1) + K_2 \pi(x_1))$$
$$= \pi(A_1 x_1 + \bar{f}_1(x_1, \pi(x_1), \kappa(x_1) + K_2 \pi(x_1))). \tag{2.7}$$

The center manifold theorem ensures that this equation has a local solution for any smooth $\kappa(x_1)$. The reduced dynamics of the closed loop system (2.3)–(2.4) on the center manifold is given by

$$x_1^+ = f_1(x_1; \kappa), \tag{2.8}$$

where $f_1(x_1; \kappa) = A_1 x_1 + \bar{f}_1(x_1, \pi(x_1), \kappa(x_1) + K_2 \pi(x_1))$. According to the center manifold theorem, we know that if the dynamics (2.8) is locally asymptotically stable then the closed loop system (2.3)–(2.4) is locally asymptotically stable.

The part of the feedback $\mathscr{F}$ given by $\kappa(x_1)$ determines the controlled center manifold $x_2=\pi(x_1)$ which in turn determines the dynamics (2.8). Hence the problem of stabilization of the system (2.3) reduces the problem to stabilizing the system (2.8) after solving Eq. (2.7), i.e. finding $\kappa(x_1)$ such that the origin of the dynamics (2.8) is asymptotically stable. Thus we can view $\kappa(x_1)$ as an input for the controlled dynamics (2.8).

But since solving Eq. (2.7) is difficult, usually we do not need to solve it exactly, and frequently it suffices to compute the low degree terms of the Taylor series expansion of π and κ around $x_1 = 0$.

Because κ starts with linear terms

$$\kappa(x_1) = K_1x_1 + \kappa^{[2]}(x_1) + \cdots \,. \tag{2.9}$$

π starts with linear terms

$$\pi(x_1) = \pi^{[1]}x_1 + \pi^{[2]}(x_1) + \cdots \,. \tag{2.10}$$

Eq. (2.7) implies that

$$\bar{A}_2\pi^{[1]} + B_2K_1 = \pi^{[1]}A_1, \tag{2.11}$$

and

$$\begin{aligned}\bar{A}_2\pi^{[2]}(x_1) + B_2\kappa^{[2]}(x_1) + \bar{f}_2^{[2]}(x_1, \pi^{[1]}x_1, K_1x_1 + K_2\pi^{[1]}x_1)\\ = \pi^{[2]}(A_1x_1) + \pi^{[1]}\bar{f}_1^{[2]}(x_1, \pi^{[1]}x_1, K_1x_1 + K_2\pi^{[1]}x_1),\end{aligned} \tag{2.12}$$

and so on.

The degree k equations are

$$\begin{aligned}\bar{A}_2\pi^{[k]}(x_1) + B_2\kappa^{[k]}(x_1) + \tilde{f}_2^{[k]}(x_1)\\ = \pi^{[1]}\tilde{f}_1^{[k]}(x_1) + \zeta^{[k]}(x_1) + \pi^{[k]}(A_1x_1),\end{aligned} \tag{2.13}$$

where $\tilde{f}_i(x_1) = \bar{f}_i(x_1, \pi(x_1), \kappa(x_1) + K_2\pi(x_1))$, and $\zeta(x_1) = \sum_{i=2}^{k-1} \pi^{[i]}(A_1x_1 + \tilde{f}_1(x_1))$.

For any $\kappa^{[k]}(x_1)$, these linear equations are solvable for $\pi^{[k]}(x_1)$ because the eigenvalues of $\bar{A}_2$ do not coincide with the eigenvalues of A_1. Note that $\tilde{f}_i^{[j]}(x_1)$ only depends on $\pi^{[1]}(x_1), \ldots, \pi^{[j-1]}(x_1)$ and $\kappa^{[1]}(x_1), \ldots, \kappa^{[j-1]}(x_1)$.

For $1 \leqslant i \leqslant n-r-1$, the ith row of these equations is

$$\begin{aligned}\pi_{i+1}^{[k]}(x_1) = \pi_i^{[k]}(A_1x_1) + \zeta_i^{[k]}(x_1) - \tilde{f}_{2,i}^{[k]}(x_1)\\ + \sum_{j=1}^{r} \pi_{i,j}^{[1]}(x_1)\tilde{f}_{1,j}^{[k]}(x_1).\end{aligned} \tag{2.14}$$

The $(n-r)$th row is

$$\begin{aligned}\kappa^{[k]}(x_1) = \pi_{n-r}^{[k]}(A_1x_1) + \zeta_{n-r}^{[k]}(x_1) - \tilde{f}_{2,n-r}^{[k]}(x_1)\\ + \sum_{j=1}^{r} \pi_{n-r,j}^{[1]}(x_1)\tilde{f}_{1,j}^{[k]}(x_1) - \sum_{i=1}^{n-r} k_{2,i}\pi_i^{[k]}(x_1).\end{aligned} \tag{2.15}$$

Note that $\pi_1^{[k]}(x_1)$ determines $\pi_2^{[k]}(x_1), \ldots, \pi_r^{[k]}(x_1)$ and $\kappa^{[k]}(x_1)$. Therefore we may change our point of view. Instead of viewing $\kappa^{[k]}(x_1)$ as determining $\pi_1^{[k]}(x_1), \ldots, \pi_r^{[k]}(x_1)$, we can view $\pi_1^{[k]}(x_1)$ as determining $\pi_2^{[k]}(x_1), \ldots, \pi_r^{[k]}(x_1)$ and $\kappa^{[k]}(x_1)$. In other words, instead of viewing the feedback as determining the center manifold, we can view the first coordinate function of the center manifold as determining the other coordinate functions and the feedback. Alternatively we can view π_1 as a pseudo control and write the dynamics as

$$x_1^+ = A_1x_1 + \bar{f}_1(x_1; \pi_1). \tag{2.16}$$

Definition 2.1. The controlled center dynamics of the system $\Sigma_{\mathscr{U}}$ subject to the feedback $\mathscr{F}$ is the control system (2.16) given by the reduction of the system (2.5) on the center manifold (2.6) where the first component of the center manifold plays the role of the input.

2.1. Linear center manifold

In this section, we give an explicit solution to Eq. (2.11) defining the linear part of the center manifold. Suppose the entries in K_2 are $k_{2,1}, k_{2,2}, \ldots, k_{2,n-r}$. Then the characteristic polynomial, $p(\lambda)$, of the matrix $A_2 + B_2K_2$ is defined by

$$\begin{aligned}p(\lambda) &= \det(\lambda I_{(n-r)\times(n-r)} - A_2 - B_2K_2)\\ &= \lambda^{n-r} - k_{2,n-r}\lambda^{n-r-1} - \cdots, k_{2,2}\lambda - k_{2,1}.\end{aligned} \tag{2.17}$$

The linear part of the feedback (2.4) is given by

$$u(x_1, x_2) = K_1x_1 + K_2x_2. \tag{2.18}$$

From (2.10), the linear part of the center manifold is given by

$$\pi^{[1]}(x_1) = \pi^{[1]}x_1$$

and (2.11) is equivalent to the following system of equations:

$$\begin{aligned}\pi_2^{[1]} &= \pi_1^{[1]}A_1,\\ \pi_3^{[1]} &= \pi_2^{[1]}A_1,\\ &\vdots\\ \pi_{n-r}^{[1]} &= \pi_{r-1}^{[1]}A_1\end{aligned}$$

and

$$0 = \pi_{n-r}^{[1]}A_1 - K_1 - k_{2,1}\pi_1^{[1]} - \cdots - k_{2,n-r}\pi_{n-r}^{[1]},$$

where $\pi_i^{[1]}$ is the ith row vector in $\pi^{[1]}$. Therefore,

$$\begin{aligned}\pi_2^{[1]} &= \pi_1^{[1]}A_1,\\ \pi_3^{[1]} &= \pi_1^{[1]}A_1^2,\\ &\vdots\\ \pi_{n-r}^{[1]} &= \pi_1^{[1]}A_1^{n-r-1}\end{aligned}$$

and

$$0 = -K_1 + \pi_1^{[1]} A_1^{n-r} - \sum_{i=1}^{n-r} k_{2,i} \pi_1^{[1]} A_1^{i-1}$$
$$= -K_1 + \pi_1^{[1]} \left(A_1^{n-r} - \sum_{i=1}^{n-r} k_{2,i} A_1^{i-1} \right).$$

The last equation has the form of the characteristic polynomial defined by (2.17).

To summarize, the linear part of the center manifold is defined by the following equations:

$$\pi_1^{[1]} = K_1 p(A_1)^{-1},$$
$$\pi_i^{[1]} = \pi_1^{[1]} A_1^{i-1} \quad \text{for } i = 2, \dots, n-r. \tag{2.19}$$

The matrix $p(A_1)$ is always invertible. Indeed, since the eigenvalues of $p(A_1)$ equal the values of $p(\lambda)$ evaluated at the eigenvalues of A_1, and since $\bar{A}_2 = A_2 + B_2 K_2$ has all its eigenvalues strictly inside the unit disk, the roots of the characteristic polynomial (2.17) are all strictly inside the unit disk. Since the eigenvalues of A_1 are all on the unit circle, and they are different from the roots of $p(\lambda)$, we deduce that $p(A_1)$ has no zero eigenvalue. Thus, the matrix $p(A_1)$ is invertible.

Theorem 2.1. *Given the feedback* $\mathscr{F}$, *the center manifold is given by*

$$x_2 = \pi^{[1]} x_1 + O(x_1^2)$$

with the components of $\pi^{[1]}$ *uniquely determined by* (2.19).

Now, consider the following change of coordinates

$$\tilde{x}_{2,i} = x_{2,i} - \pi_1^{[1]} A_1^{i-1} x_1, \quad i = 1, \dots, n-r-1 \tag{2.20}$$

then,

$$\tilde{x}_{2,i}^+ = \tilde{x}_{2,i+1} \quad \text{for } i = 1, \dots, n-r,$$
$$\tilde{x}_{2,n-r}^+ = \sum_{i=1}^{n-r} k_{2,i} \tilde{x}_{2,i}.$$

Hence the coefficient K_1 has been removed from the x_2-part of the dynamics (2.3)–(2.18) by a change of coordinates. With $K_1 = 0$, we deduce from (2.19) that $\pi^{[1]} = 0$. So, in the new coordinates system, the linear terms of the center manifold are null.

Proposition 2.1. *Given any feedback* (2.18) *satisfying Property* $\mathscr{P}$, *and the change of coordinates* (2.20), *then the center manifold is given by*

$$\tilde{x}_2 = O(x_1^2). \tag{2.21}$$

2.2. Quadratic approximation of the center manifold

In this section, we derive the quadratic approximation of the center manifold. Under the linear change of coordinates (2.20), the closed-loop system (2.5) is transformed into the following system

$$x_1^+ = A_1 x_1 + f_1^{[2]}(x_1, \tilde{x}_2 + \pi^{[1]} x_1, \kappa^{[2]}(x_1)) + O(x_1, \tilde{x}_2)^3,$$

$$\tilde{x}_2^+ = A_2(\tilde{x}_2 + \pi^{[1]} x_1) - \pi^{[1]} A_1 x_1$$
$$+ B_2(K_1 x_1 + K_2 \tilde{x}_2 + K_2 \pi^{[1]} x_1 + \kappa^{[2]}(x_1))$$
$$+ f_2^{[2]}(x_1, \tilde{x}_2 + \pi^{[1]} x_1, u(x_1, \tilde{x}_2 + \pi^{[1]} x_1))$$
$$- \pi^{[1]} f_1^{[2]}(x_1, \tilde{x}_2 + \pi^{[1]} x_1, u(x_1, \tilde{x}_2 + \pi^{[1]} x_1))$$
$$+ O(x_1, \tilde{x}_2)^3.$$

Define a quadratic vector field $\bar{f}_2^{[2]}(x_1, \tilde{x}_2)$ by

$$\bar{f}_2^{[2]}(x_1, \tilde{x}_2) = f_2^{[2]}(x_1, \tilde{x}_2 + \pi^{[1]} x_1, K_1 x_1 + K_2 \tilde{x}_2 + K_2 \pi^{[1]} x_1)$$
$$- \pi^{[1]} f_1^{[2]}(x_1, \tilde{x}_2 + \pi^{[1]} x_1, K_1 x_1$$
$$+ K_2 \tilde{x}_2 + K_2 \pi^{[1]} x_1). \tag{2.22}$$

Then from (2.20) and (2.22), Eq. (2.3) is equivalent to

$$x_1^+ = A_1 x_1 + f_1^{[2]}(x_1, \tilde{x}_2 + \pi^{[1]} x_1, u(x_1, \tilde{x}_2 + \pi^{[1]} x_1))$$
$$+ O(x_1, \tilde{x}_2)^3,$$

$$\tilde{x}_2^+ = A_2 \tilde{x}_2 + B_2(K_2 z_2 + \kappa^{[2]}(x_1, \tilde{x}_2 + \pi^{[1]} x_1))$$
$$+ \bar{f}_2^{[2]}(x_1, \tilde{x}_2) + O(x_1, \tilde{x}_2)^3. \tag{2.23}$$

In the $(x_1, \tilde{x}_2)$ coordinates, the center manifold has the form (2.21). It satisfies the center manifold equation

$$\bar{A}_2 \pi^{[2]}(x_1) + B_2 \kappa^{[2]}(x_1) + \bar{f}_2^{[2]}(x_1, 0) = \pi^{[2]}(A_1 x_1).$$

This equation can be written as

$$\pi_{i+1}^{[2]}(x_1) = \pi_i^{[2]}(A_1 x_1) - \bar{f}_{2,i}^{[2]}(x_1, 0)$$
$$\text{for } i = 1, \dots, n-r-1, \tag{2.24}$$

$$\sum_{i=1}^{n-r} k_{2,i} \pi_i^{[2]}(x_1) + \kappa^{[2]}(x_1) = \pi_{n-r}^{[2]}(A_1 x_1) - \bar{f}_{2,n-r}(x_1, 0). \tag{2.25}$$

Solving these equations, we obtain

$$\pi_i^{[2]}(x_1) = \pi_1^{[2]}(A_1^{i-1} x_1)$$
$$- \sum_{j=1}^{i-1} \bar{f}_{2,j}^{[2]}(A_1^{i-j-1} x_1, 0) \quad \text{for } i = 2, \dots, n-r,$$

$$\pi_1^{[2]}(A_1^{n-r}x_1) - \sum_{i=1}^{n-r} k_{2,i}\pi_1^{[2]}(A_1^{i-1}x_1)$$
$$= \kappa^{[2]}(x_1) + \sum_{j=1}^{n-r} \bar{f}_{2,j}^{[2]}(A_1^{n-r-j}x_1, 0)$$
$$- \sum_{i=2}^{n-r}\sum_{j=1}^{i-1} k_{2,i}\bar{f}_{2,j}^{[2]}(A_1^{i-j-1}x_1, 0). \quad (2.26)$$

If we adopt the matrix notation

$$\pi_i^{[2]}(x_1) = x_1^{\mathrm{T}} Q_i x_1,$$
$$\bar{f}_{2,i}^{[2]}(x_1, 0) = x_1^{\mathrm{T}} R_i x_1,$$
$$\kappa(x_1) = x_1^{\mathrm{T}} L x_1, \quad (2.27)$$

where Q_i, R and L are symmetric $r \times r$ matrices, and by defining $\mathscr{S}$ as the operator given by

$$\mathscr{S}_{A_1}(Q) = A_1^{\mathrm{T}} Q A_1 \quad (2.28)$$

for all symmetric $r \times r$ matrices Q. Then, we can write (2.26) as

$$Q_i = \mathscr{S}_{A_1}^{i-1}(Q_1) - \sum_{j=1}^{i-1} \mathscr{S}_{A_1}^{j}(R_{i-j-1})$$

for $i = 2, \ldots, n-r$,

$$p(\mathscr{S}_{A_1})Q_1 = L + \sum_{j=1}^{n-r} \mathscr{S}_{A_1}^{n-r-j}(R_j)$$
$$- \sum_{i=2}^{n-r}\sum_{j=1}^{i-1} k_{2,i}\mathscr{S}_{A_1}^{i-j-1}(R_j). \quad (2.29)$$

To summarize, Eqs. (2.29) imply the following result on quadratic center manifold.

Theorem 2.2. *If*

$$x_2 = \pi^{[1]}x_1 + \pi^{[2]}(x_1) + \mathrm{O}(x_1)^3$$

is the center manifold of (2.3), *then* $\pi^{[2]}(x_1)$ *is uniquely determined by the following equations*:

$$\pi_i^{[2]}(x_1) = x_1^T Q_i x_1 \quad \textit{for } i = 1, 2, \ldots, n-r$$

where

$$Q_1 = p(\mathscr{S}_{A_1})^{-1}\left(L + \sum_{j=1}^{n-r} \mathscr{S}_{A_1}^{n-r-j}(R_j) - \sum_{i=2}^{n-r}\sum_{j=1}^{i-1} k_{2,i}\mathscr{S}_{A_1}^{i-j-1}(R_j)\right),$$

$$Q_i = \mathscr{S}_{A_1}^{i-1}(Q_1) - \sum_{j=1}^{i-1} \mathscr{S}_{A_1}^{j}(R_{i-j-1}) \quad \textit{for } i = 2, \ldots, n-r,$$

in which $\mathscr{S}_{A_1}$ *is the operator defined by* (2.28); R_i *is from the quadratic dynamics and it is defined by* (2.27) *and* (2.22); L *is from the quadratic feedback and it is defined by* (2.27); *and* p *is the characteristic polynomial of* $\bar{A}_2$ *given in* (2.17).

Similar to the derivation of the linear part of the center manifold, the operator $p(\mathscr{S}_{A_1})$ is always invertible. The set of eigenvalues of the operator $\mathscr{S}_{A_1}$ is $\{\lambda_i\lambda_j \,:\, \text{for } i, j = 1, \ldots, r\}$ with $\lambda_\ell, \ell = 1, \ldots, r$, being the eigenvalues of A_1. Therefore, $|\lambda_i| = 1$ implies that all the eigenvalues of $\mathscr{S}_{A_1}$ have a modulus equal to one. Since $\bar{A}_2$ has all its eigenvalues strictly inside the unit disk, all the roots of $p(\lambda)$ has modulus strictly less than one. They do not coincide with the eigenvalues of $\mathscr{S}_{A_1}$. Thus the eigenvalues of $p(\mathscr{S}_{A_1})$ given by $p(\lambda_i\lambda_j)$, $i, j = 1, \ldots, r$, are nonzero. We deduce that the operator $p(\mathscr{S}_{A_1})$, from $\mathbb{R}^{r\times r}$ to $\mathbb{R}^{r\times r}$, is invertible.

There are some special cases in which the center manifold is simpler. For instance, if (2.23) is in quadratic normal form (see [10,26]), then $\bar{f}_2^{[2]}$ is independent of x_1. In this case, $\bar{f}_2^{[2]}(x_1, 0) = 0$. Therefore, $R_i = 0$. Under the feedback

$$u = K_2 x_2 + x_1^{\mathrm{T}} L x_1$$

the center manifold of (2.23) is

$$x_2 = \pi^{[2]}(x_1),$$

where

$$\pi_i^{[2]}(x_1) = x_1^{\mathrm{T}} Q_i x_1,$$
$$Q_1 = p(\mathscr{S}_{A_1})^{-1}(L),$$
$$Q_i = \mathscr{S}_{A_1}^{i-1}(Q_1).$$

Remark.

1. Similarly to the procedure above, we can explicit the kth order part of the center manifold by using the change of variable $\tilde{x}_2 = x_2 - \pi^{[1]}x_1 - \sum_{j=2}^{k-1}\pi^{[j]}(x_1)$, with $k \geqslant 3$. Moreover, we can show that the mapping relating the kth order part of the feedback and the kth order of the center manifold is a bijection provided $p(\lambda_{i_1}\cdots\lambda_{i_k}) \neq 0$, with $i_1, \ldots, i_k = 1, \ldots, r$ and $\lambda_\ell, \ell = 1, \ldots, r$, being the eigenvalues of A_1. This condition is satisfied since the eigenvalues of A_1 and $\bar{A}_2$ do not coincide as above.
2. As in the center manifold theorem for dynamical systems, it will not be necessary to find the kth order approximation of the controlled center dynamics for any $k \in \mathbb{N}$. We will use the lowest degree of approximation of the center manifold (2.11) and the feedback (2.9) that allows to prove asymptotic stability of the controlled center dynamics. In fact, the procedure is very similar to the one used in the center manifold theorem, we start by degree $k = 1$ and if we are able to find K_1 in (2.9) such that the controlled center dynamics is asymptotically stable then we deduce an asymptotically stabilizing controlled for $\Sigma_{\mathscr{U}}$ from the expression of the feedback $\mathscr{F}$. If we are not able to find such a K_1 using

an approximation of degree $k=1$, then we use an approximation of degree 2 and try to find$\kappa^{[2]}(x)$ for which we have asymptotic stability of the controlled center dynamics and so on.

3. We note that it is not necessary to use the normal forms in order to find the controlled center dynamics, but their use simplify finding explicit solutions to the equations defining the controlled center dynamics.
4. As pointed out to the authors by a reviewer, there are similarities in the algebra between our technique and the one in [7]. In [7], a term by term approach was used to compute the approximated center manifold solutions in order to deal with the problem of output regulation.

3. Stabilization of systems with transcontrollable bifurcation

In this section, we use the preceding results to stabilize systems with a transcontrollable bifurcation, i.e. those where $A_1=1$ in (2.3).

From [10,26], we know that there exist a quadratic change of coordinates and a feedback bringing the system (2.3) to a quadratic normal form

$$z_1^+ = z_1 + \beta z_1^2 + \gamma z_1 z_{21} + \sum_{i=1}^{n-r+1} \delta_i z_{2i}^2 + \mathrm{O}(z_1, z_2, v)^3,$$

$$z_2^+ = A_2 z_2 + B_2 v + \mathrm{O}(z_1, z_2, v)^2, \tag{3.30}$$

with $z_{2,r+1}=v$, and $\beta, \gamma, \delta_1, \ldots, \delta_{n-r}$ are real numbers. Suppose we use the linear feedback

$$v = K_1 z_1 + K_2 z_2$$

and assume that the linear part of the center manifold is given by

$$z_2 = \pi^{[1]} z_1. \tag{3.31}$$

Since $A_1 = 1$, we deduce from (2.19) that

$$\pi_i^{[1]} = \pi_1^{[1]}, \quad i = 2, \ldots, n-r,$$

$$K_1 = -K_{21}\pi_1^{[1]} \tag{3.32}$$

so $\pi_2^{[1]}, \ldots, \pi_r^{[1]}$, K_1 depend on $\pi_1^{[1]}$.

First, suppose that we use the piecewise linear feedback

$$v = K_1 z_1 + K_2 z_2, \tag{3.33}$$

with

$$K_1 = \begin{cases} \overline{k}_1, & z \geqslant 0, \\ \widetilde{k}_1, & z < 0. \end{cases}$$

Proposition 3.1. *The closed-loop system* (3.30)–(3.33) *possesses a piecewise smooth center manifold.*

Proof. The linear part of the dynamics (3.30)–(3.33) is given by

$$z_1^+ = z_1 + \mathrm{O}(z_1, z_2)^2,$$

$$z_2^+ = B_2 K_1 z_1 + \bar{A}_2 z_2 + \mathrm{O}(z_1, z_2)^2. \tag{3.34}$$

Let $\Sigma_{\overline{k}_1}$ (resp. $\Sigma_{\widetilde{k}_1}$) be the system (3.34) when $K_1=\overline{k}_1$ (resp. $K_1=\widetilde{k}_1$) for all z_1. Since the system $\Sigma_{\overline{k}_1}$ (resp. $\Sigma_{\widetilde{k}_1}$) is smooth, and possesses one eigenvalue on the unit circle and $n-1$ eigenvalues strictly inside the unit disk; then, from the center manifold theorem, in a neighborhood of the origin, $\Sigma_{\overline{k}_1}$ (resp. $\Sigma_{\widetilde{k}_1}$) has a center manifold $\overline{W}^c$ (resp. $\widetilde{W}^c$).

For $\Sigma_{\overline{k}_1}$, the center manifold is represented by $z_2 = \overline{\pi}(z_1)$, for z_1 sufficiently small. The ith component of the linear part of the center manifold, $z_{2,i} = \overline{\pi}_i^{[1]} z_1$, for $i = 1, \ldots, n-1$ is given by (3.32) with $K_1 = \bar{k}_1$.

Similarly for $\Sigma_{\widetilde{k}_1}$, the center manifold is represented by $z_2 = \widetilde{\pi}(z_1)$, and its linear part $z_{2,i} = \widetilde{\pi}_i^{[1]} z_1$, for $i = 1, \ldots, n-1$ is given by (3.32) with $K_1 = \tilde{k}_1$.

The center manifolds $\overline{W}^c$ and $\widetilde{W}^c$ intersect along the line $z_1 = 0$. Hence, if we slice them along the line $z_1 = 0$ and then glue the part of $\overline{W}^c$ for which $z_1 > 0$ with the part of $\widetilde{W}^c$ for which $z_1 < 0$, along this line, we deduce that in an open neighborhood of the origin, $\overline{\mathscr{D}}$, the piecewise smooth system (3.34) has a piecewise smooth center manifold W_c. The linear part of the center manifold W_c is represented by $z_2 = \pi^{[1]} z_1$. The ith component of z_2, $z_{2,i}$, is given by $z_{2,i} = \pi_i^{[1]} z_1$, with $\pi_i^{[1]} = K_1/p(1)$, for $1 \leqslant i \leqslant n-1$. $\square$

Using (3.31), (3.32), and (3.33) we deduce that the controlled center dynamics is given by

$$z_1^+ = z_1 + \beta z_1^2 + \gamma z_1 \cdot \pi_1^{[1]} z_1 + \sum_{i=1}^{n-r} \delta_i (\pi_1^{[1]} z_1)^2 + \mathrm{O}(z_1^3), \tag{3.35}$$

with $\pi_1^{[1]}=\overline{\pi}_1^{[1]}=-\overline{k}_1/k_{2,1}$ when $z \geqslant 0$, and $\pi_1^{[1]}=\widetilde{\pi}_1^{[1]}=-\widetilde{k}_1/k_{2,1}$ when $z<0$. Now, let $\Phi(X) = \beta + \gamma X + \sum_{i=1}^{r} \delta_i X^2$, then the controlled center dynamics (3.35) can be written as

$$z_1^+ = \begin{cases} z_1 + \Phi(\overline{\pi}_1^{[1]}) z_1^2 + \mathrm{O}(z_1^3), & z_1 \geqslant 0, \\ z_1 + \Phi(\widetilde{\pi}_1^{[1]}) z_1^2 + \mathrm{O}(z_1^3), & z_1 < 0. \end{cases} \tag{3.36}$$

From [10,26], we know that in order to have a trancontrollable bifurcation, the condition $\gamma^2 - 4\beta \sum_{i=1}^{r} \delta_i > 0$ has to be satisfied. Thus, the polynomial $\Phi(X)$ changes its sign. So, it is possible to find $\overline{\pi}_i^{[1]}$ and $\widetilde{\pi}_i^{[1]}$ such that $\Phi(\overline{\pi}_1^{[1]}) = -\Phi(\widetilde{\pi}_1^{[1]}) = -\Phi_0$, for some $\Phi_0 > 0$. Thus the controlled center dynamics can be written as

$$z_1^+ = z_1 - \Phi_0 z_1 |z_1| + \mathrm{O}(z_1^3). \tag{3.37}$$

If we choose $\Phi_0 > 0$, the origin of this dynamics is asymptotically stable. Thus, using a similar approach[1] to the one in B.2, we deduce that the closed-loop system (3.30)–(3.33) is asymptotically stable. Hence, the controller (3.33) asymptotically stabilizes the system (3.30).

Remark. Let us note here that we cannot apply the center manifold theorem to this case in order to deduce that the full order dynamics is asymptotically stable, since the center manifold theorem applies only to the case where the center manifold is smooth, and in our case the center manifold is piecewise smooth. This is why we have to use a similar argument to the one in Appendix B.2 where a Lyapunov function is used to prove that when the reduced order dynamics on a piecewise smooth center manifold is locally asymptotically/practically stable then the full order dynamics is locally asymptotically/practically stable.

Now, let us consider the case of a quadratic feedback

$$v = K_1 z_1 + K_2 z_2 + \kappa^{[2]}(z_1) \tag{3.38}$$

in order to asymptotically stabilize the system (3.30).

Since $\gamma^2 - 4\beta\sum_{i=1}^{r}\delta_i > 0$, there are two choices of $\pi_1^{[1]}$ such that $\Phi(\pi_1^{[1]}) = 0$. After such a choice, the stability of the controlled center dynamics depends on cubic terms.

Let us consider quadratic and cubic change of state coordinates and invertible quadratic and cubic feedback

$$x = z + T^{[2]}(z) + T^{[3]}(z),$$

$$u = v + \alpha^{[2]}(z, v) + \alpha^{[3]}(z, v)$$

to bring the system from linear normal form to quadratic and cubic normal form (see [26]),

$$\begin{aligned} z_1^+ &= z_1 + \beta z_1^2 + \gamma z_1 z_{21} + \sum_{i=1}^{n-r+1} \delta_i z_{2i}^2 + \bar\beta z_1^3 + \bar\gamma z_1^2 z_{21} \\ &\quad + \sum_{i=1}^{n-r+1} \bar\delta_i z_1 z_{2i}^2 + \sum_{i=1}^{n-r+1}\sum_{j=i}^{n-r+1} \bar\varepsilon_{ij} z_{21} z_{2j} z_{2i} \\ &\quad + \mathrm{O}(z_1, z_2, v)^4, \\ z_2^+ &= A_2 z_2 + B_2 v + \mathrm{O}(z_1, z_2, v)^2, \end{aligned} \tag{3.39}$$

with $\beta, \gamma, \bar\beta, \bar\gamma, \delta_i, \bar\delta_i, \bar\varepsilon_{ij}$ (for $i = 1, \ldots, n-r+1$, $j = i, \ldots, n-r+1$) are real numbers. Because z_2 is linearly stabilizable, the quadratic and cubic terms will not affect the local stability properties of the z_2-dynamics.

The procedure to choose the parameters of the feedback (3.38) is as follows: from Property $\mathscr{P}$, we know that $K_2 = [k_{2,1} \ \cdots \ k_{2,n-r}]$ is such that the eigenvalues of $A + B_2 K_2$ are strictly inside the unit disk. Moreover, we choose $\pi_1^{[1]}$ so the quadratic part of the controlled center dynamics is zero, then we deduce K_1 from (3.32). For the quadratic part of (3.38), we can choose $\pi_1^{[2]}(z_1) = c z_1^2$ arbitrarily, and the controlled center dynamics is

$$z_1^+ = z_1 + \sigma z_1^3 + \mathrm{O}(z_1^4),$$

with

$$\sigma = \left(\gamma + 2\sum_{i=1}^{n-r} \delta_i \pi_1^{[1]}\right) c + \bar\beta + \bar\gamma \pi_1^{[1]} + \sum_{i=1}^{n-r} \bar\delta_i (\pi_1^{[1]})^2 + \sum_{i=1}^{n-r}\sum_{j=i}^{n-r} \bar\varepsilon_{ij} (\pi_1^{[1]})^3.$$

There were two possible choices of $\pi_1^{[1]}$ that canceled the quadratic part of controlled center dynamics. Since $\gamma^2 - 4\beta\sum_{i=1}^{r}\delta_i > 0$, there is at least one such $\pi_1^{[1]}$ so that $\gamma + 2\sum_{i=1}^{r+1}\delta_i \pi_1^{[1]} \neq 0$.

By choosing c so that $\sigma < 0$, the origin of controlled center dynamics will be locally asymptotically stable. Thus, we deduce that the origin of the closed loop system (3.30)–(3.38) is locally asymptotically stable by applying the center manifold theorem.

We can summarize the results of this section in the following theorem.

Theorem 3.1. *Consider the system* (3.30) *with* $\gamma^2 - 4\beta \sum_{i=1}^{r}\delta_i > 0$. *Then, the feedbacks* (3.33) *and* (3.38) *locally asymptotically stabilize the system around the origin.*

4. Stabilization of systems with a fold or period doubling control bifurcation

In this section, we use the preceding results to stabilize systems with a fold or period doubling control bifurcation i.e. those where the system (2.3) has a single uncontrollable mode $\lambda \in \mathbb{R}$, such that, $|\lambda| > 1$ or $\lambda = -1$, respectively.

When there is only one uncontrollable mode $\lambda \notin \{0, 1\}$ in (2.3), we know, from [10,18,26], that there exist a cubic change of coordinates and a feedback bringing the system to its cubic normal form

$$\begin{aligned} z_1^+ &= \lambda z_1 + \gamma z_1 z_{21} + \sum_{i=1}^{n-r+1} \delta_i z_{2i}^2 + \bar\gamma z_1^2 z_{21} + \sum_{i=1}^{n-r+1} \bar\delta_i z_1 z_{2i}^2 \\ &\quad + \sum_{i=1}^{n-r+1}\sum_{j=i}^{n-r+1} \bar\varepsilon_{ij} z_{21} z_{2j} z_{2i} + \mathrm{O}(z_1, z_2, v)^4, \\ z_2^+ &= A_2 z_2 + B_2 v + \mathrm{O}(z_1, z_2, v)^2, \end{aligned} \tag{4.40}$$

with $z_{2,n-r+1} = v$, and $\beta, \gamma, \bar\beta, \bar\gamma, \delta_i, \bar\delta_i, \bar\varepsilon_{ij}$ (for $i = 1, \ldots, n-r+1$, $j = i, \ldots, n-r+1$) are real numbers. We know also that this

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[1] We consider the Lyapunov function $V(z_1) = z_1^2$, then, from (3.36), we have

$$\Delta V = V(z_1^+) - V(z_1) = \begin{cases} 2\Phi(\bar\pi_1^{[1]}) z_1^3 + \mathrm{O}(z_1^4), & z_1 \geqslant 0, \\ 2\Phi(\tilde\pi_1^{[1]}) z_1^3 + \mathrm{O}(z_1^4), & z_1 < 0, \end{cases}$$

and the proof follows the same steps as in the case $\lambda = -1$ in Appendix B.2.

system exhibits a control bifurcation provided the transversality condition $\widetilde{\chi} = \sum_{i=1}^{n-r+1} (1+\lambda^{i-1})\delta_i \neq 0$ is satisfied [26]. Let $\widehat{\chi} = \sum_{i=1}^{n-r+1} \delta_i$.

Theorem 4.1. *Consider the system* (4.40). *If* $\gamma\widetilde{\chi}\widehat{\chi} \neq 0$, *then the piecewise linear feedback* (3.33) *practically stabilizes the system* (4.40) *around the origin when* $\lambda > 1$ *or* $\lambda < -1$. *The feedback asymptotically stabilizes the system around the origin when* $\lambda = -1$.

The procedure to choose the parameters of the feedback (3.33), $\bar{k}_1$ and $\tilde{k}_1$, is as follows: let $\eta(X) = X(\gamma + \widehat{\chi}X)$ with $X \in \mathbb{R}$, and let

$$\overline{\pi}_1^{[1]} = \frac{\bar{k}_1}{p(\mathrm{sign}(\lambda))} \quad \text{and} \quad \widetilde{\pi}_1^{[1]} = \frac{\tilde{k}_1}{p(\mathrm{sign}(\lambda))}, \tag{4.41}$$

with p the characteristic polynomial of $\bar{A}_2$. Since it is always possible to choose $\bar{k}_1$ and $\tilde{k}_1$ such that $\eta(\overline{\pi}_1^{[1]}) = -\eta(\widetilde{\pi}_1^{[1]}) = \eta_0$ then we will choose K_1 such that $\eta_0 > 0$ when $\lambda > 1$, $\eta_0 < 0$ when $|\lambda| < 1$ or $\lambda = -1$. Moreover, K_2 is chosen such that $A_2 + B_2K_2$ has all its eigenvalues strictly inside the unit disc.

Proof. The linear part of the closed-loop dynamics (4.40)–(3.33) can be written as

$$z_1^+ = \lambda z_1 + \mathrm{O}(z_1, z_2)^2,$$

$$z_2^+ = \bar{A}_2 z_2 + \mathrm{O}(z_1, z_2)^2. \tag{4.42}$$

Let us write λ as $\lambda = (1+\varepsilon)\,\mathrm{sign}(\lambda)$, with ε is a slightly positive number. If we consider ε as an extra state whose equation is $\varepsilon^+ = \varepsilon$, the term εz_1 will be considered of order two. Then, the dynamics (4.42) can be written as

$$\varepsilon^+ = \varepsilon,$$

$$z_1^+ = \mathrm{sign}(\lambda) z_1 + \mathrm{O}(z_1, z_2, \varepsilon)^2,$$

$$z_2^+ = \bar{A}_2 z_2 + \mathrm{O}(z_1, z_2)^2. \tag{4.43}$$

Using the same kind of arguments as in Proposition 3.1, we can show that for the closed loop system (4.40)–(3.33), a piecewise smooth center manifold exists. It is defined by $z_2 = \pi(\varepsilon, z_1)$. Since there is no linear term in ε in the z_1-subdynamics of the system (4.43), the linear part of the center manifold can be written as

$$z_2 = \pi^{[1]} z_1.$$

The components of $\pi^{[1]}$ are given by (2.19), with

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & \mathrm{sign}(\lambda) \end{bmatrix}$$

for the dynamics in the (ε, z_1, z_2) space. Thus, the controlled center dynamics is

$$z_1^+ = \begin{cases} \lambda z_1 + \eta(\overline{\pi}_1^{[1]}) z_1^2 + \mathrm{O}(z_1^3), & z_1 \geqslant 0, \\ \lambda z_1 + \eta(\widetilde{\pi}_1^{[1]}) z_1^2 + \mathrm{O}(z_1^3), & z_1 < 0. \end{cases} \tag{4.44}$$

Since $\gamma \neq 0$ and $\widehat{\chi} \neq 0$, by the assumption in the theorem, there are two distinct solutions for the equation $\eta(\pi_1^{[1]}) = 0$, hence $\eta(\pi_1^{[1]})$ changes its sign. So we can choose $\overline{\pi}_1^{[1]}$ and $\widetilde{\pi}_1^{[1]}$ such that $\eta(\overline{\pi}_1^{[1]}) = -\eta(\widetilde{\pi}_1^{[1]}) = -\eta_0$, with $\eta_0 > 0$ if $\lambda > 1$, and $\eta_0 < 0$ if $\lambda < 1$. In this case, the controlled center dynamics will have the form

$$z_1^+ = \lambda z_1 - \eta_0 |z_1| z_1 + \mathrm{O}(z_1^3), \tag{4.45}$$

which is the normal form of the supercritical "bird foot bifurcation for maps" (see Appendix A).

For λ such that $\lambda \notin \{0, 1\}$, and in a sufficiently small neighborhood of the origin (in the case, for example, where we choose η_0 sufficiently large or λ sufficiently close to one) three equilibrium points exist: the origin and $\bar{z}^* = (\lambda - 1)/\eta_0$, $\bar{z}^{**} = -(\lambda - 1)/\eta_0 = -\bar{z}^*$. The origin is unstable for $\lambda > 1$ or $\lambda < -1$, and the two other equilibrium points are stable. Thus, the solution converges to $\bar{z}^*$ or $\bar{z}^{**}$. Hence, by making $\bar{z}^*$ sufficiently close to the origin, i.e. by choosing η_0 sufficiently large, we shall have practical stability for the origin of the controlled center dynamics. Using a similar methodology to the one in [17], we can show that this implies practical stability of the origin of the system (4.40) (see Appendix B.2).

When $\lambda = -1$, the controlled center dynamics (4.45) reduces to

$$z_1^+ = -z_1 - \eta_0 |z_1| z_1 + \mathrm{O}(z_1^3).$$

If we use the Lyapunov function $V(z_1) = z_1^2$, then

$$\Delta V = V(z^+) - V(z) = 2\eta_0 |z_1| z_1^2 + \mathrm{O}(z_1^3).$$

Hence choosing $\eta_0 < 0$, permits to ensure that the origin is asymptotically stable. □

Now let us consider the quadratic feedback

$$v = K_1 z_1 + K_2 z_2 + \kappa^{[2]}(z_1) \tag{4.46}$$

instead of the feedback (3.33). The coefficient K_2 is such that the eigenvalues of $A + B_2K_2$ are all strictly inside the unit disk.

Theorem 4.2. *Consider the system* (4.40). If $\gamma\widetilde{\chi} \neq 0$, *then the feedback* (4.46) *with* $K_1 = 0$ *practically stabilizes the system* (4.40) *around the origin when* $\lambda > 1$ *or* $\lambda < -1$. *It asymptotically stabilizes the system around the origin when* $\lambda = -1$.[2]

Proof. Adopting the same approach as precedently we show the existence of a center manifold in the (ε, z_1) plane. The feedback (4.46) shapes the linear and quadratic parts of the approximation of the center manifold

$$z_2 = \pi^{[1]} z_1 + \pi^{[2]}(z_1),$$

[2] One should note that, similarly to the period-doubling bifurcation for dynamical systems, the period-doubling control bifurcation for one dimensional systems appears only in the case of discrete-time systems: there is no period-doubling (control) bifurcation for one dimensional continuous-time (controlled) dynamical systems. To control systems where the bifurcation is due to the parameter different techniques have been used, see for example [3,31,32]. It is important to note that in our work the bifurcation is due to the control and not to the parameter.

which in turn shape the quadratic and cubic parts of the controlled center dynamics given by

$$z_1^+ = \lambda z_1 + \eta(\pi_1^{[1]}) z_1^2 + O(z_1^3).$$

Since the equation $\eta(X) = 0$ admits zero as a solution, we can choose the solution $\pi_1^{[1]} = 0$, which gives $K_1 = 0$ from (3.32). Then, by choosing $\pi_1^{[2]}(z_1) = cz_1^2$ arbitrarily, we deduce that the controlled center dynamics is given by

$$z_1^+ = \lambda z_1 + \gamma c z_1^3 + O(z_1^4). \tag{4.47}$$

Since $|\lambda| > 1$, the origin is unstable. If we choose c such that $(1-\lambda)\gamma c > 0$, the two equilibrium points $\hat{z}^* = \sqrt{(1-\lambda)/\gamma c}$ and $\hat{z}^{**} = -\sqrt{(1-\lambda)/\gamma c}$, when they exist,[3] are stable. The controlled center dynamics (4.47) has the form of a system with a supercritical pitchfork bifurcation. Since the solution converges to one of the equilibrium points $\hat{z}^*$ or $\hat{z}^{**}$, the origin of the controlled center dynamics can be made practically stable by having the equilibrium points $\hat{z}^*$and $\hat{z}^{**}$ sufficiently close to the origin. We can show that this implies practical stability of the origin of the system (4.40) (by adopting the same approach as in Appendix B).

When $\lambda = -1$, the controlled center dynamics (4.47) reduces to

$$z_1^+ = -z_1 + \gamma c z_1^3 + O(z_1^4).$$

We see that choosing c such that $\gamma c > 0$ permits to ensure that the origin is asymptotically stable. □

The piecewise linear feedback (3.33) is more robust than the quadratic feedback (4.46). Indeed, using the quadratic feedback (4.46) requires finding the exact of $\pi^{[1]}$ of the equation $\eta(\pi_1^{[1]}) = (\sum_{i=1}^{r} \delta_i)(\pi_1^{[1]})^2 + \gamma \pi_1^{[1]} + \beta = 0$. If there exists a small uncertainty on the invariants γ and δ_i (with $i = 1, \ldots, r+1$), the quadratic terms generated by the uncertainty in the controlled center dynamics (4.47) will be a source of instability of the system. Using the piecewise linear feedback (3.33) does not necessitate the exact solutions of the equation $\eta(\pi_1^{[1]}) = 0$, as we just have to find $\overline{\pi}_1^{[1]}$ and $\widetilde{\pi}_1^{[1]}$ such that $\eta(\overline{\pi}_1^{[1]})\eta(\widetilde{\pi}_1^{[1]}) < 0$. Thus the piecewise linear feedback is more robust.

Appendix A. The birdfoot bifurcation for maps

In this section, we analyze the discrete-time version of the "bird foot bifurcation" (see [24] for a treatment of the continuous-time case).

[3] In order for the two equilibria to exist, $1-\lambda$ has to be sufficiently small, i.e. λ has to be in a small neighborhood of one in order to be able to choose c such that $(1-\lambda)\gamma c > 0$. The size of that neighborhood, around one, in which λ lies, and for which the two equilibria $\hat{z}^*$ and $\hat{z}^{**}$ exist, depends on the value of c as well as on the terms in $O(z^4)$ in Eq. (4.47).

Consider a dynamical system

$$x^+ = \mu x - \eta_0 x|x| + O(x^3), \tag{A.48}$$

with $x \in \mathbb{R}$, $\mu \in \mathbb{R}$ a parameter, and $\eta_0 \in \mathbb{R}\backslash\{0\}$ a constant. The fixed points of the system are the solutions of the equation

$$((1-\mu) + \eta_0 |x|)x = 0.$$

Provided μ sufficiently close to one or η_0 sufficiently large, and that $(\mu - 1)\eta_0 > 0$, the dynamical system has three fixed points: the origin, $x^* = (\mu-1)/\eta_0$, and $x^{**} = -(\mu-1)/\eta_0 = -x^*$. If $\mu = 1$, the dynamical system has the origin as the only fixed point.

Let us consider the Lyapunov function $V(x) = x^2$, then

$$\Delta V = V(x^+) - V(x) = (\mu^2 - 1)x^2 - 2\eta_0 \mu |x| x^2 + O(x^4).$$

If $|\mu| < 1$, then $\Delta V < 0$ and the origin is an asymptotically stable equilibrium point. If $|\mu| > 1$, then $\Delta V > 0$ and the origin is an unstable equilibrium point.

When $\eta_0 > 0$ (resp. $\eta_0 < 0$), the equilibrium points x^* and x^{**} appear when $\mu > 1$ (resp. $\mu < 1$). For μ sufficiently close to one, the equilibrium points x^* and x^{**} are unstable when the origin is asymptotically stable, and are asymptotically stable when the origin is unstable. As for the pitchfork bifurcation, we have an exchange of the stability properties, at $\mu = 1$, between the origin and the two equilibrium points x^* and x^{**}. If $\mu = 1$, the origin is the only equilibrium point. It is asymptotically stable when $\eta_0 > 0$, and unstable when $\eta_0 < 0$.

When $\eta_0 > 0$, we shall call the bifurcation a *supercritical birdfoot bifurcation*. When $\eta_0 < 0$, we shall call the bifurcation *subcritical birdfoot bifurcation*.

When $\eta_0 > 0$ (resp. $\eta_0 < 0$), and $\mu > 1$ is sufficiently large, the three fixed points become unstable (resp. stable), and stable (resp. unstable) cycles appear (see [9,28]).

One of the properties of the birdfoot bifurcation is that a system with a birdfoot bifurcation is robust to small quadratic perturbations. Indeed, a system in a normal form (A.48) exhibits a birdfoot bifurcation if we perturb it by a small quadratic term vx^2; while the same perturbation will make a system with a pitchfork bifurcation exhibit a transcritical bifurcation.

Appendix B

B.1. Preliminaries

Let us first review the definition of class $\mathscr{K}$, $\mathscr{K}_\infty$ and $\mathscr{KL}$ functions.

Definition B.1 (*Khalil [23, Definitions 3.3, 3.4]*).

- A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathscr{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathscr{K}_\infty$ if $a = \infty$ and $\lim_{r\to\infty} \alpha(r) = \infty$.
- A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathscr{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathscr{K}$ with respect to r; and, for each

fixed r, the mapping $\beta(r,s)$ is decreasing with respect to s and $\lim_{r\to\infty}\beta(r,s)=0$.

Now, consider the dynamical system

$$x^{+}=f(x), \tag{B.49}$$

with $f:\mathscr{D}\to\mathbb{R}^n$ a continuous function such that $f(0)=0$.

Definition B.2. Let $\mathscr{D}\subset\mathbb{R}^n$ be an open set, and let V be a function $V:\mathscr{D}\to\mathbb{R}^+$, such that V is smooth on $\mathscr{D}$, and

$$x\in\mathscr{D}\Longrightarrow \alpha_1(\|x(k)\|)\leqslant V(x)\leqslant\alpha_2(\|x(k)\|), \tag{B.50}$$

with α_1 and α_2 class $\mathscr{K}$ functions. Then, V is a Lyapunov function if there exists a class $\mathscr{K}$ function α_3 such that

$$\Delta V(x)=V(f(x))-V(x)\leqslant-\alpha_3(\|x\|)\quad\text{for } x\in\mathscr{D}. \tag{B.51}$$

Let $\overline{\mathbb{B}}_\varepsilon$ be the closed ball, around the origin, of radius ε.

Definition B.3 (*ε-Practical stability*). The origin of the dynamical system $x^{+}=f(x)$, with $f(0)=0$, is said to be locally ε-practically stable, if there exists an open set $\mathscr{D}$ containing the closed ball $\overline{\mathbb{B}}_\varepsilon$, a class $\mathscr{KL}$ function ζ and a positive constant δ, such that for any initial condition $x(0)$ with $\|x(0)\|<\delta$, the solution $x(k)$ of (B.49) exists and satisfies

$$d_{\overline{\mathbb{B}}_\varepsilon}(x(k))\leqslant\zeta(d_{\overline{\mathbb{B}}_\varepsilon}(x(0)),k),\qquad\forall k\in\mathbb{N}, \tag{B.52}$$

with $d_{\overline{\mathbb{B}}_\varepsilon}(x(k))=\inf_{\rho\in\overline{\mathbb{B}}_\varepsilon}d(x(k),\rho)$, the usual point to set distance.

B.2. Proof of the practical stability of the whole closed-loop dynamics

Consider the Lyapunov function $V(z_1)=z_1^2$, and let $\varepsilon_1\triangleq-(\lambda^2-1)/2\,\lambda\eta(\overline{\pi}_1^{[1]})$ and $\varepsilon_2\triangleq-(\lambda^2-1)/2\,\lambda\eta(\widetilde{\pi}_1^{[1]})$. Then, from (4.44), we have

$$\Delta V=V(z_1^{+})-V(z_1)=\begin{cases}2\,\lambda\eta(\overline{\pi}_1^{[1]})(z_1-\varepsilon_1)z_1^2+\mathrm{O}(z_1^4), & z_1\geqslant0,\\ 2\,\lambda\eta(\widetilde{\pi}_1^{[1]})(z_1-\varepsilon_2)z_1^2+\mathrm{O}(z_1^4), & z_1<0,\end{cases} \tag{B.53}$$

- Practical stability for $\lambda>1$ or $\lambda<-1$.

By choosing[4] $\overline{\pi}_1^{[1]}$ and $\widetilde{\pi}_1^{[1]}$ such that $\eta(\overline{\pi}_1^{[1]})<0$ and $\eta(\widetilde{\pi}_1^{[1]})>0$, we get $\varepsilon_1>0$ and $\varepsilon_2<0$. This choice is always possible since the equation $\eta(X)=0$ admits two solutions $X^{*}=0$ and $X^{**}=-(\gamma/\widehat{\chi})\neq0$ (by the assumption in Theorem 4.1); so, η takes both positive and negative values. In this case, $\Delta V<0$ for $z_1>\varepsilon_1$ and $z_1<\varepsilon_2$, and $\Delta V=0$ for $z_1=\varepsilon_1$ or $z_1=\varepsilon_2$.

In the following, and without loss of generality, we choose $\overline{\pi}_1^{[1]}$ and $\widetilde{\pi}_1^{[1]}$ such that $\eta(\overline{\pi}_1^{[1]})=-\eta(\widetilde{\pi}_1^{[1]})$, so $\varepsilon_1=-\varepsilon_2\triangleq\varepsilon$, with $0\leqslant\varepsilon\leqslant r$, and r is the radius of $\overline{\mathbb{B}}_r$, the largest closed ball contained in the largest open neighborhood of the origin for which a center manifold exist for the system (4.40)–(3.33).

Let Ω_1 and Ω_2 be two sets defined by $\Omega_1=(\varepsilon,+r]$ and $\Omega_2=[-r,-\varepsilon)$. If $z_1(0)\in\Omega_1\cup\Omega_2$, then $\Delta V<0$ on $\Omega_1\cup\Omega_2$. From (B.50), (B.51), we have

$$\Delta V\leqslant-\alpha_3(\|z_1\|)\leqslant-\alpha_3(\alpha_2^{-1}(V)). \tag{B.54}$$

Since α_2 and α_3 are a class $\mathscr{K}$ functions, then $\alpha_3(\alpha_2^{-1})$ is also a class $\mathscr{K}$ function. Hence, using the comparison principle in [20, lemma 4.3] (this work is the discrete-time version of a result in [27]), there exists a class $\mathscr{KL}$ function Υ such that

$$V(z_1(k))\leqslant\Upsilon(V(z_1(0),k)). \tag{B.55}$$

The sets $\overline{\Omega}_1=[0,\varepsilon]$ and $\overline{\Omega}_2=[-\varepsilon,0]$ have the property that when a solution enters either set, it remains in it. This is due to the fact that ΔV is negative definite on the boundary of these two sets. For the same reason, if $z_1(0)\in\overline{\Omega}_1$ (resp. $z_1(0)\in\overline{\Omega}_2$), then $z_1(k)\in\overline{\Omega}_1$ (resp. $z_1(k)\in\overline{\Omega}_2$), for $k\in\mathbb{N}$.

Let k_ε be the first time such that the solution enters $\overline{\Omega}_1\cup\overline{\Omega}_2=\overline{\mathbb{B}}_\varepsilon$. Using (B.50) and (B.55), we get that for $0\leqslant k\leqslant k_\varepsilon$,

$$\varepsilon\leqslant\|z_1(k)\|\leqslant\alpha_1^{-1}(V(z_1(k))\leqslant\alpha_1^{-1}(\Upsilon(V(z_1(0),k)))$$
$$\triangleq\zeta(z_1(0),k)).$$

The function ζ is a class $\mathscr{KL}$ function, since α_1 is a class $\mathscr{K}$ function and Υ a class $\mathscr{KL}$ function. Since ζ is a class $\mathscr{KL}$ function, then k_ε is finite. Hence, $z_1(k)\in\overline{\Omega}_1\cup\overline{\Omega}_2$, for $k\geqslant k_\varepsilon$.

Thus, for $z_1\in\overline{\mathbb{B}}_r$, the solution satisfies

$$d_{\overline{\mathbb{B}}_\varepsilon}(z_1(k))\leqslant\zeta(d_{\overline{\mathbb{B}}_\varepsilon}(z_1(0)),k). \tag{B.56}$$

So, in $\overline{\mathbb{B}}_r$, the origin is locally ε-practically stable.

In order to prove the stability of the whole closed-loop dynamics we adapt, to the present problem, the proof in [23, Theorem 4.2], where the author proved the center manifold theorem for continuous-time systems using a Lyapunov argument.

The closed-loop dynamics (4.40)–(3.33) can be written as

$$z_1^{+}=\lambda z_1+\beta z_1^2+\gamma z_1 z_{2,1}+\sum_{i=1}^{n-1}\delta_i z_{2,i}^2+\mathrm{O}(z_1,z_2)^3,$$

$$z_2^{+}=B_2K_1z_1+\bar{A}_2z_2+\sum_{i=1}^{n-1}\sum_{j=i+2}^{n-1}\theta_i^j z_{2,j}^2 e_2^i+\mathrm{O}(z_1,z_2)^3. \tag{B.57}$$

Let $w_1=z_1$, $w_2=z_2-\pi(z_1)$, and $w=(w_1,w_2)^{\mathrm{T}}$. Then, the dynamics (B.57) is given by

$$w_1^{+}=\lambda w_1+\Psi(w_1,\pi_1(w_1))+\mathscr{N}_1(w_1,w_2).$$

$$w_2^{+}=\bar{A}_2w_2+\mathscr{N}_2(w_1,w_2),$$

[4] This choice will give us the parameters $\overline{k}_1$ and $\widetilde{k}_1$ of the feedback (2.18) using Eq. (4.41).

with

$$\mathcal{N}_i(w_1,w_2)=\begin{cases}\overline{\mathcal{N}}_i(w_1,w_2), & w_1\geqslant 0,\\ \widetilde{\mathcal{N}}_i(w_1,w_2), & w_1<0,\end{cases}\quad \text{for } i=1,2,$$

and

$$\pi_1(w_1)=\begin{cases}\bar{\pi}_1(w_1), & w_1\geqslant 0\\ \tilde{\pi}_1(w_1), & w_1<0.\end{cases}$$

The functions Ψ and $\mathcal{N}$ are such that $\Psi(w_1,\pi(w_1))=\mathrm{O}(w_1^2)$ as $w_1\to 0$, $\mathcal{N}_i(w_1,0)=0$, and $(\partial\mathcal{N}_i/\partial w_2)(0,0)=0$.

Since $\mathcal{N}_i(w_1,0)=0$ and $(\partial\mathcal{N}_i/\partial w_2)(0,0)=0$ $(i=1,2)$, then in a domain $\|w\|<\sigma$, $\mathcal{N}_1$ and $\mathcal{N}_2$ satisfy

$$\mathcal{N}_i(w_1,w_2)\leqslant\kappa_i\|w_2\|,\quad i=1,2,$$

where κ_1 and κ_2 can be arbitrarily small by making σ sufficiently small.

Since $\bar{A}_2$ has all its eigenvalues strictly inside the unit disk, there exists a unique P such that $\bar{A}_2^{\mathrm{T}}P+P\bar{A}_2=-I$. Let $\mathcal{V}$ be the following composite Lyapunov function

$$\mathcal{V}(w_1,w_2)=w_1^2+\sqrt{w_2^{\mathrm{T}}Pw_2}.$$

Then $\Delta\mathcal{V}$ is given by

$$\begin{aligned}\Delta\mathcal{V}(w_1,w_2)&=\mathcal{V}(w_1^+,w_2^+)-\mathcal{V}(w_1,w_2)\\ &=(w_1^+)^2-w_1^2+\sqrt{(w_2^+)^{\mathrm{T}}Pw_2^+}-\sqrt{w_2^{\mathrm{T}}Pw_2}.\end{aligned}$$

For $w_1\in\Omega_1\cup\Omega_2$, and using (B.54), we obtain

$$\begin{aligned}&(w_1^+)^2-w_1^2\\ &\quad=\Delta V_1+\mathcal{N}_1^2(w_1,w_2)+2\lambda\Psi(w_1,\pi(w_1))w_2\\ &\qquad+2\lambda\mathcal{N}_1(w_1,w_2)w_1+\Psi(w_1,\pi(w_1))\mathcal{N}_1(w_1,w_2),\\ &\quad\leqslant-\alpha_3(\|w_1\|)+\mathcal{N}_1^2(w_1,w_2)+2\lambda\Psi(w_1,\pi(w_1))w_2\\ &\qquad+2\lambda\mathcal{N}_1(w_1,w_2)w_1+\Psi(w_1,\pi(w_1))\mathcal{N}_1(w_1,w_2),\\ &\quad\leqslant-\alpha_3(\|w_1\|)+2\lambda\kappa_1\|w_1\|\|w_2\|+\|\mathrm{O}(w_1,w_2)^2\|.\end{aligned}$$

Using the fact that $\lambda_{\min}(P)\|w_2\|^2\leqslant w_2^{\mathrm{T}}Pw_2\leqslant\lambda_{\max}(P)\|w_2\|^2$, we obtain

$$\begin{aligned}&\sqrt{(w_2^+)^{\mathrm{T}}Pw_2^+}-\sqrt{w_2^{\mathrm{T}}Pw_2}\\ &\quad\leqslant\sqrt{\lambda_{\max}(\bar{A}_2^{\mathrm{T}}P\bar{A}_2)+2\lambda_{\max}(\bar{A}_2^{\mathrm{T}}P)\kappa_2+\lambda_{\max}(P)\kappa_2^2}\\ &\qquad\times\|w_2\|-\sqrt{\lambda_{\min}(P)}\|w_2\|.\end{aligned}$$

Hence

$$\begin{aligned}&\Delta\mathcal{V}(w_1,w_2)\\ &\quad\leqslant-\alpha_3(\|w_1\|)+\Bigg(2\lambda\kappa_1 v\\ &\qquad+\sqrt{\lambda_{\max}(\bar{A}_2^{\mathrm{T}}P\bar{A}_2)+2\lambda_{\max}(\bar{A}_2^{\mathrm{T}}P)\kappa_2+\lambda_{\max}(P)\kappa_2^2}\\ &\qquad-\sqrt{\lambda_{\min}(P)}\Bigg)\|w_2\|,\end{aligned}$$

with $v=\max_{\{w_1:w_1\in\Omega_1\cup\Omega_2\}}\|w_1\|$.

By choosing κ_1 and κ_2 such that

$$\begin{aligned}&2\lambda\kappa_1 v+\sqrt{\lambda_{\max}(\bar{A}_2^{\mathrm{T}}P\bar{A}_2)+2\lambda_{\max}(\bar{A}_2^{\mathrm{T}}P)\kappa_2+\lambda_{\max}(P)\kappa_2^2}\\ &\quad-\sqrt{\lambda_{\min}(P)}<0,\end{aligned}$$

we shall have

$$\Delta\mathcal{V}(w_1,w_2)<0.$$

Hence, for $w_1\in\Omega_1\cup\Omega_2$, $\Delta\mathcal{V}(w_1,w_2)<0$. So, there exists a class $\mathcal{KL}$ function $\overline{\Upsilon}$ such that

$$\|w(k)\|\leqslant\overline{\Upsilon}(\|w(0)\|,k).\qquad\text{(B.58)}$$

When $w_1\in\overline{\Omega}_1\cup\overline{\Omega}_2$, and by considering w_1 as an input of the system

$$w_2^+=\bar{A}_2w_2+\mathcal{N}_2(w_1,w_2),$$

we deduce that $\|w_2\|$ is bounded, since $\bar{A}_2$ has all its eigenvalues strictly inside the unit disk. Hence, for $w_1\in\overline{\Omega}_1\cup\overline{\Omega}_2$, there exists $\bar{\varepsilon}$ such that

$$\|w(k)\|\leqslant\bar{\varepsilon}.\qquad\text{(B.59)}$$

From (B.58)–(B.59) we obtain

$$d_{\overline{\mathbb{B}}_{\bar{\varepsilon}}}(w(k))\leqslant\overline{\Upsilon}(d_{\overline{\mathbb{B}}_{\bar{\varepsilon}}}(w(0)),k).\qquad\text{(B.60)}$$

So the origin of the whole dynamics is locally $\bar{\varepsilon}$-practically stable.

- Asymptotic stability for $\lambda=-1$.

In this case $\varepsilon_1=\varepsilon_2=0$, and the sets $\overline{\Omega}_1$ and $\overline{\Omega}_2$ reduce to the origin. Hence, the origin of the reduced closed-loop system is asymptotically stable, since the solution converges to $\overline{\Omega}_1\cup\overline{\Omega}_2=\{0\}$. We deduce that the origin of the whole closed-loop dynamics is asymptotically stable since $\Delta\mathcal{V}(w_1,w_2)<0$ for $w_1\in\Omega_1\cup\Omega_2=[-r,0)\cup(0,r]$. When $w_1=0$, then $w_2\to 0$ since the system $w_2^+=\bar{A}_2w_2+\mathcal{N}_2(0,w_2)$ is locally asymptotically stable because $\bar{A}_2$ has all its eigenvalues inside the unit disk.

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