ON THE CONVERGENCE AND BEHAVIOR OF THREE DIMENSIONAL NORMAL FORMS

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Abstract: Low codimensional bifurcations of control systems have been classified using the theory of normal forms. However, the normal form theory was developed as local results. To understand the behavior of control systems outside a local area of a bifurcation point, it is necessary to study the convergence of the normal forms if the vector fields are analytic. The convergence is a long time open problem. In this paper, we address the convergence problem of normal forms for three dimensional control systems. Bounded attractors are found for a family of such normal forms, which include the controlled Lorenz system as a special case.

Keywords: Normal forms, convergence, bifurcation and chaos.

1. INTRODUCTION

Since early 1990's, normal forms of nonlinear control systems have become an increasingly active research subject with applications to bifurcation control and engineering systems. The transformation group used in normal form theory consists of formal power series (Kang and Krener, 2006). This approach is useful for the analysis of the local behavior around equilibria with control bifurcations (Kang, 1998*b*; Kang, 1998*a*; Krener *et al.*, 2004). However, qualitative properties of analytic systems in a given neighborhood cannot be analyzed using these normal forms because there is no guarantee on the convergence of the normal forms. The problem of convergence goes back to Poincaré. For classical dynamical systems without control, the convergence of the Poincaré normal form had been a problem of active research for many years. It was proved that the distribution of the eigenvalues of the linearized dynamical system provides critical information about the convergence. The well known Poincar-Dulac Theorem and Siegels Theorem provide sufficient conditions for the convergence of the Poincaré normal form (Arnold, 1983).

The eigenvalues in the linearization of control systems can be changed by state feedback, therefore the convergence results on the Poincaré normal form do not apply to control systems. The problem must be solved using new approaches. Thus far, very little is known about the convergence of control system normal forms, except for some very special cases such as the systems that are completely feedback linearizable. In 2004, the problem of convergence for control system normal forms is published in (Kang and Krener, 2004) as an open problem of mathematical systems and control theory. In the following, we focus on the convergence of the three dimensional normal forms of control systems. The systems are classified into three cases. In the case of one uncontrollable mode, the problem is completely solved. In the cases of two uncontrollable modes and no uncontrollable mode, sufficient conditions are proved for the convergence of the normal forms.

If a system is equivalent to its normal form under convergent transformations, the behavior of the normal form inside the region of convergence is the same as the original system. Therefore, studying the global behavior of the convergent normal forms helps to understand the behavior of general nonlinear systems. In this paper, a family of control feedbacks are derived under which a set of three dimensional normal forms has a bounded attractor. The system have a variety of different behavior inside the attractor, depending on the feedback, including stable equilibria, bifurcations, and chaos. In fact, it is proved that the controlled Lorenz equation is globally and analytically equivalent to its normal form. While the normal form of the controlled Lorenz system reveals the complex dynamics of a special set of normal forms, the global behavior of general normal forms is still a problem widely open. We hope that the nonlinear feedback and the bounded attractor studied in this paper will motivate and inspire more research in this subject of nonlinear control theory.

In the next section, the convergence of the normal form of three dimensional control systems are addressed. Three different types of systems are addressed in three subsections. Then, a Lyapunov function is constructed in Section 3 for a family of normal forms with a single uncontrollable mode. A family of feedbacks is derived under which the controlled normal form has a bounded attractor. In addition, it is proved that the controlled Lorenz system is globally equivalent to a special case of the normal forms addressed in Section 3.

2. THE CONVERGENCE OF NORMAL FORMS

Dynamical systems can be classified under transformation groups. A system is said to be equivalent to another if it can be transformed to the other system. Different types of transformations are used in the literature for the purpose of system classification. For the Poincaré normal form, formal transforms are used, i.e the changes of coordinates defined by formal power series. When using the formal transformation, the problem of convergence can be avoided. However, conclusions about the qualitative properties of the normal forms are local. If the analytic transformations are used, one must prove convergence of the transformation, which could be very difficult. For control systems, the normal form theory has been addressed using formal transformations. The classification and the normal form of control systems under analytic transformations are still open (Kang and Krener, 2004). In this paper, we study the normal forms of the three dimensional control systems using analytic transformations. Consider a control system with a single input

$$\dot{\xi} = f(\xi) + g(\xi)v, \; f(0) = 0, \; g(0) \neq 0 \qquad (2.1)$$

with $\xi \in \mathbb{R}^3$, and $v \in \mathbb{R}$. Throughout the paper, we assume that $f(\xi)$ and $g(\xi)$ are analytic vector fields around the origin. A transformation consists of a change of coordinates and an invertible feedback of the following form

$$\begin{aligned} x &= \phi(\xi), \qquad \left.\frac{\partial \phi}{\partial \xi}\right|_{\xi=0} \neq 0 \\ u &= \alpha(\xi) + \beta(\xi)v, \; \beta(0) \neq 0 \end{aligned} \qquad (2.2)$$

where $\phi(\xi)$, $\alpha(\xi)$, and $\beta(\xi)$ are analytic functions around the origin $(\xi, v) = (0, 0)$, x and ξ are state variables of the control system, u and v are the control inputs. The family of transformations defined by (2.2) is a group. Control systems can be classified based on this group of transformations.

Definition 2.1. Given two analytic control systems. They are analytically equivalent to each other if one system can be transformed into another by an analytic change of coordinates and an analytic state feedback.

In this paper, we divide systems defined by (2.1) into three subsets based on the number of uncontrollable modes. More specifically, define the matrices A, B, and the integer n_0 by

$$\begin{aligned} A &= \frac{\partial f}{\partial \xi}(0), \quad B = g(0) \\ n_0 &= 3 - \mathrm{rank}[B, AB, A^2B] \end{aligned} \qquad (2.3)$$

If $n_0 = 0$, the system is linearly controllable. It has no uncontrollable mode. If $n_0 = 1$, the system has one uncontrollable mode. If $n_0 = 2$, the system has two uncontrollable modes. Because $g(0) \neq 0$, then $n_0 \neq 3$. In the following, the three cases of $n_0 = 0, 1, 2$ are addressed in three subsections. The convergence problem is partially solved for $n_0 = 0$, and completely solved for $n_0 = 1$. For $n_0 = 2$, it is proved that the normal form converges provided that the associated Poincaré normal form converges.

2.1 The normal form of linearly controllable systems

Consider a linearly controllable analytic system $(n_0 = 0)$

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 + f_1^{[2+]}(\xi) \\ \dot{\xi}_2 &= \xi_3 + f_2^{[2+]}(\xi) \\ \dot{\xi}_3 &= v + f_3^{[2+]}(\xi) \end{aligned} \qquad (2.4)$$

The convergence of the normal form for systems defined by (2.4) is still an open problem. In this section, we can solve the convergence problem for those systems satisfying the following assumption

$$\left.\frac{\partial f_1^{[2+]}}{\partial \xi_1}\right|_{\xi_3=0} = 0, \ \left.\frac{\partial f_1^{[2+]}}{\partial \xi_1 \partial \xi_3}\right|_{\xi_3=0} = 0 \qquad (2.5)$$
$$\frac{\partial f_2^{[2+]}}{\partial \xi_1} = 0, \quad f_2^{[2+]}|_{\xi_3=0} = 0$$

More specifically, (2.4) and (2.5) imply

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 + f_{1I}^{[2+]}(\xi_2) \\ &\quad + f_{1II}^{[1+]}(\xi_2)\xi_3 + f_{1III}(\xi_1, \xi_2, \xi_3)\xi_3^2 \\ \dot{\xi}_2 &= \xi_3 + f_2^{[1+]}(\xi_2, \xi_3)\xi_3 \\ \dot{\xi}_3 &= v + f_3^{[2+]} \end{aligned} \qquad (2.6)$$

Theorem 2.1. A system defined by (2.6) is analytically equivalent to its normal form

$$\begin{aligned} \dot{x}_1 &= x_2 + \epsilon(x_1, x_2, x_3)x_3^2 \\ \dot{x}_2 &= x_3, \ \ \dot{x}_3 = u \end{aligned} \qquad (2.7)$$

Proof. This theorem is a slightly more general version of a result proved in (Kang, 1991) and (Kang, 1996). It is simple to remove $f_3^{[2+]}$ by a feedback. So, we can assume $f_3^{[2+]} = 0$. To remove $f_2^{[1+]}(\xi_2, \xi_3)\xi_3$ we use the following transformation called "pushing down"

$$\bar{\xi}_3 = \xi_3 + f_2^{[1+]}(\xi_2, \xi_3)\xi_3, \ \ \bar{v} = \dot{\bar{\xi}}_3$$

The inverse mapping is analytic and it satisfies

$$\xi_1 = \bar{\xi}_1, \ \xi_2 = \bar{\xi}_2, \ \xi_3 = \bar{\xi}_3 - \bar{f}_2^{[1+]}(\bar{\xi}_2, \bar{\xi}_3)\bar{\xi}_3$$

The transformation removes $f_2^{[1+]}(\xi_2, \xi_3)\xi_3$ from the second equation in (2.6). Meanwhile, the formality of

$$f_{1I}^{[2+]}(\xi_2) + f_{1II}^{[1+]}(\xi_2)\xi_3 + f_{1III}(\xi_1, \xi_2, \xi_3)\xi_3^2$$

in the first equation of (2.6) remains unchanged under $\bar{\xi}$. From now, we assume $f_2^{[1+]}(\xi_2, \xi_3)\xi_3 = 0$ and $f_3^{[2+]} = 0$ in (2.6). For the rest of the proof, we use the transformation in (Kang, 1991) and (Kang, 1996). Define

$$\begin{aligned} \tilde{f}_{1II}(\xi_2) &= \int_0^{\xi_2} f_{1II}(t)dt \\ x_1 &= \xi_1 - \tilde{f}_{1II}(\xi_2) \\ x_2 &= \xi_2 + f_{1I}(\xi_2) \\ x_3 &= \xi_3 + \frac{\partial f_{1I}(\xi_2)}{\partial \xi_2}\xi_3 \\ u &= v + \frac{\partial^2 f_{1I}(\xi_2)}{\partial \xi_2^2}\xi_3^2 + \frac{\partial f_{1I}(\xi_2)}{\partial \xi_2}v \end{aligned} \qquad (2.8)$$

Under the new coordinates we have

$$\begin{aligned} \dot{x}_1 &= x_2 + f_{1III}(\xi_1, \xi_2, \xi_3)\xi_3^2 \\ \dot{x}_2 &= x_3, \ \ \dot{x}_3 = \nu \end{aligned} \qquad (2.9)$$

From (2.8) we have $\xi_3 = x_3 + \psi(x)x_3$ for some function $\psi(x)$. Substitute this into (2.9), we obtain

$$\begin{aligned} \dot{x}_1 &= x_2 + f_{1III}(\xi(x))(1 + \psi(x))^2 x_3^2 \\ \dot{x}_2 &= x_3, \ \ \dot{x}_3 = u \end{aligned}$$

This is a system in normal form and the transformations used are analytic. □

2.2 Systems with a single uncontrollable mode

This is the case in which the problem of convergence is completely solved. The proof is based on the Cauchy-Kowalevsakaya theorem of partial differential equations.

Theorem 2.2. Consider an analytic control system defined by (2.1). Assume $n_0 = 1$. Then (2.1) is analytically equivalent to its normal form

$$\dot{x} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} p^{[2+]}(x_0) \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} \gamma(x_0)x_{11} + \delta_1(x_0, x_{11})x_{11}^2 + \delta_2(x_0, x_1)x_{12}^2 \\ 0 \\ 0 \end{bmatrix}$$

where $p(x_0)$ is the Poincaré normal form, i.e.

$$p^{[2+]}(x_0) = \begin{cases} 0, & \text{if } \lambda \neq 0 \\ \text{analytic function,} & \text{if } \lambda = 0 \end{cases} \qquad (2.10)$$

Proof. Because $g(0) \neq 0$, there exists a change of coordinates so that, under the new coordinate, $g = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. So, we can assume $g(\xi)$ to be a constant vector. In this case, u is not contained in any nonlinear term. Because $n_0 = 1$, there exists a linear change of coordinates $x = T\xi$ and a linear feedback $v = K\xi + Pu$ such that (2.1) is transformed into a system in the following form

$$\dot{x} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} f_0^{[2+]}(x) \\ f_{11}^{[2+]}(x) \\ f_{12}^{[2+]}(x) \end{bmatrix} \qquad (2.11)$$

where $x = \begin{bmatrix} x_0 & x_{11} & x_{12} \end{bmatrix}^T$, and $f_0^{[2+]}(x)$, $f_{11}^{[2+]}(x)$ $f_{12}^{[2+]}(x)$ are analytic functions contain only the quadratic and higher order terms of x in their Taylor expansions around the origin.

To transform the system into its normal form, we first cancel $f_{11}^{[2+]}$ and $f_{12}^{[2+]}$ by change of coordinates and state feedback. For this purpose, let $\bar{x}_{11} = x_{11}$, $\bar{x}_{12} = x_{12} + f_{11}^{[2+]}(x)$, then

$$\begin{aligned} \dot{\bar{x}}_{11} &= \bar{x}_{12}, \\ \dot{\bar{x}}_{12} &= u + f_{12}^{[2+]}(x) + \frac{\partial f_{11}^{[2+]}}{\partial x_0}(\lambda x_0 + f_0^{[2+]}(x)) \\ &\quad + \frac{\partial f_{11}^{[2+]}}{\partial x_{11}}(x_{12} + f_{11}^{[2+]}(x)) + \frac{\partial f_{11}^{[2+]}}{\partial x_{12}}(u + f_{12}^{[2+]}(x)) \end{aligned}$$

By defining $v = \dot{\bar{x}}_{12}$, we obtain

$$\begin{aligned}\dot{x}_0 &= \lambda x_0 + f_0^{[2+]}(x),\\ \dot{\bar{x}}_{11} &= \bar{x}_{12}, \ \ \dot{\bar{x}}_{12} = v\end{aligned}$$

Now, we can consider (2.11) in which

$$f_{11}^{[2+]}(x) = 0, \ \ f_{12}^{[2+]}(x) = 0 \tag{2.12}$$

Now, we focus on the uncontrollable part.

$$\begin{aligned}f_0^{[2+]}(x) &= (f_0^{[2+]}(x_0, x_{11}, x_{12}) - f_0^{[2+]}(x_0, 0, 0))\\ &\quad + f_0^{[2+]}(x_0, 0, 0)\end{aligned}$$

If $\lambda \neq 0$, based on the Poincaré normal form theory (Arnold, 1983) we know there exists a change of coordinates $\bar{x}_0 = x_0 + \varphi^{[2+]}(x_0)$ such that $\dot{x}_0 = \lambda x_0 + f_0^{[2+]}(x_0, 0, 0)$ is transformed into $\dot{\bar{x}}_0 = \lambda \bar{x}_0$. When $\lambda = 0$, then all terms in the Taylor expansion of $f_0^{[2+]}(x_0, 0, 0)$ are resonant, thus they are already in Poincaré normal form. So we can assume (2.11) satisfies

$$f_0^{[2+]}(x) = \bar{f}^{[2+]}(x_0, x_{11}, x_{12}) + p^{[2+]}(x_0)$$

in which $p^{[2+]}(x_0)$ satisfies (2.10), and

$$\bar{f}^{[2+]}(x_0, 0, 0) = 0$$

This function can be expressed as

$$\begin{aligned}&\bar{f}^{[2+]}(x_0, x_{11}, x_{12}) = \gamma(x_0)x_{11} + q(x_0, x_{11})x_{12} +\\ &\delta_1(x_0, x_{11})x_{11}^2 + \delta_2(x_0, x_1)x_{12}^2\end{aligned}$$

where $\gamma(x_0)$, $q(x_0, x_{11})$, $\delta_1(x_0, x_{11})$, and $\delta_2(x_0, x_1)$ are analytic functions and $\gamma(0) = 0$, $q(0, 0) = 0$. Now, let us find a change of coordinates that permits us to cancel $q(x_0, x_{11})x_{12}$ in $\bar{f}(x_0, x_{11}, x_{12})$. Consider the change of coordinates

$$\bar{x}_0 = x_0 - \phi^{[2+]}(x_0, x_{11}).$$

Then

$$\begin{aligned}\dot{\bar{x}}_0 &= \dot{x}_0 - \frac{\partial \phi^{[2+]}(x_0, x_{11})}{\partial x_0}\dot{x}_0 - \frac{\partial \phi^{[2+]}(x_0, x_{11})}{\partial x_{11}}\dot{x}_{11},\\ &= \left(1 - \frac{\partial \phi^{[2+]}(x_0, x_{11})}{\partial x_0}\right)\Big(\lambda x_0 + p^{[2+]}(x_0)\\ &\quad + \gamma(x_0)x_{11} + q(x_0, x_{11})x_{12} + \delta_1(x_0, x_{11})x_{11}^2\\ &\quad + \delta_2(x_0, x_1)x_{12}^2\Big) - \frac{\partial \phi^{[2+]}(x_0, x_{11})}{\partial x_{11}}x_{12}\end{aligned}$$

We want to prove that the following PDE

$$\left(1 - \frac{\partial \phi^{[2+]}(x_0, x_{11})}{\partial x_0}\right)q(x_0, x_{11}) - \frac{\partial \phi^{[2+]}(x_0, x_{11})}{\partial x_{11}} = 0$$

admits an analytic solution $\phi^{[2+]}(x_0, x_{11})$ around the origin. Indeed, let $y_1 = \phi^{[2+]}(x_0, x_{11})$ and $y_2 = x_{11}$, then the equation becomes

$$\begin{aligned}\frac{\partial y_1}{\partial x_{11}} &= -\frac{\partial y_1}{\partial x_0}q(x_0, y_2) + q(x_0, y_2),\\ \frac{\partial y_2}{\partial x_{11}} &= 1\end{aligned} \tag{2.13}$$

with boundary condition $y_1 = y_2 = 0$ at $x_{11} = 0$. The Cauchy-Kowalevski Theorem (John, 1982) guarantees that (2.13) has an analytic solution around the origin. Moreover, every term in y_1 contains x_{11} because of the boundary condition. So, if y_1 has a linear term, it must be x_{11}. However, the right hand side of the first equation in (2.13) is at least linear. Therefore, the lowest nonzero term in the Taylor expansion of $y_1 = \phi^{[2+]}(x_0, x_{11})$ must be at least quadratic and

$$\phi^{[2+]}(x_0, 0) = 0 \tag{2.14}$$

Under this transformation, $\dot{\bar{x}}_0$ has the following form

$$\begin{aligned}\dot{\bar{x}}_0 &= \left(1 - \frac{\partial \phi^{[2+]}(x_0, x_{11})}{\partial x_0}\right)\Big(\lambda x_0 + p^{[2+]}(x_0)\\ &\quad + \gamma(x_0)x_{11} + \delta_1(x_0, x_{11})x_{11}^2 + \delta_2(x_0, x_1)x_{12}^2\Big)\\ &= \lambda x_0 + \tilde{p}^{[2+]}(x_0) + \tilde{\gamma}(x_0)x_{11}\\ &\quad + \tilde{\delta}_1(x_0, x_{11})x_{11}^2 + \tilde{\delta}_2(x_0, x_1)x_{12}^2\end{aligned} \tag{2.15}$$

for some analytic functions $\tilde{p}^{[2+]}(x_0)$, $\tilde{\gamma}(x_0)x_{11}$, $\tilde{\delta}_1(x_0, x_{11})x_{11}^2$, and $\tilde{\delta}_2(x_0, x_1)$. In the inverse transformation x_0 is a function of $\bar{x}_0$ and x_{11}, i.e.

$$x_0 = \bar{x}_0 + \psi^{[2+]}(\bar{x}_0, x_{11})$$

for some analytic function $\psi^{[2+]}(\bar{x}_0, x_{11})$. Because of (2.14) we have $\psi(\bar{x}_0, 0) = 0$. Therefore, under the new coordinate, (2.15) is equivalent to a system in the following form

$$\begin{aligned}\dot{\bar{x}}_0 &= \lambda \bar{x}_0 + \bar{p}^{[2+]}(\bar{x}_0) + \bar{\gamma}(\bar{x}_0)x_{11} + \bar{\delta}_1(\bar{x}_0, x_{11})x_{11}^2\\ &\quad + \bar{\delta}_2(\bar{x}_0, x_1)x_{12}^2\end{aligned}$$

This is in normal form. □

2.3 Systems with two uncontrollable modes

If $n_0 = 2$, the linearization of the system has two uncontrollable modes, and one controllable mode. Due to the small dimension of the controllable part, the convergence of the normal form is fully determined by the uncotrollable variables. As a result, the convergence of the normal form depends on the convergence of the Poincaré normal form in the uncontrollable dynamics. Consider a system defined by (2.1) satisfying $n_0 = 2$. By an analytic change of coordinates, we can assume $g(\xi) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$, the system has the following form

$$\begin{aligned}\begin{bmatrix}\dot{\xi}_{01}\\ \dot{\xi}_{02}\end{bmatrix} &= A_0\xi_0 + A_1\xi_1 + f_0^{[2+]}(\xi_0, \xi_1)\\ \dot{\xi}_1 &= v\end{aligned}$$

The controllability matrix

$$\begin{bmatrix} \begin{bmatrix}0\\0\end{bmatrix} & A_1 & A_0A_1\\ 1 & 0 & 0\end{bmatrix}$$

has rank 1, we know $A_1 = 0$. Therefore, the system is analytically equivalent to

$$\begin{aligned} \begin{bmatrix} \dot{\xi}_{01} \\ \dot{\xi}_{02} \end{bmatrix} &= A_0\xi_0 + f_0^{[2+]}(\xi_0, \xi_1) \\ \dot{\xi}_1 &= v \end{aligned} \tag{2.16}$$

where A_0 is a 2×2 matrix in Jordan canonical form, and $\xi_0 = \begin{bmatrix} \xi_{01} & \xi_{02} \end{bmatrix}^T$. The higher order vector field $f_0^{[2+]}(x_0, x_1)$ can be expressed in the following form

$$\begin{aligned} f_{01}^{[2+]}(\xi_0, \xi_1) = p_1^{[2+]}(\xi_0) + \gamma_1(\xi_0)\xi_1 + \delta_1(\xi_0, \xi_1)\xi_1^2 \\ f_{02}^{[2+]}(\xi_0, \xi_1) = p_2^{[2+]}(\xi_0) + \gamma_2(\xi_0)\xi_1 + \delta_2(\xi_0, \xi_1)\xi_1^2 \end{aligned}$$

Except for the vector field

$$p^{[2+]}(\xi_0) = \begin{bmatrix} p_1^{[2+]}(\xi_0) \\ p_2^{[2+]}(\xi_0) \end{bmatrix}$$

the system is already in normal form. Therefore, the convergence of normal form boils down to the normal form of the following system without control

$$\dot{\xi}_0 = A_0\xi_0 + p^{[2+]}(\xi_0) \tag{2.17}$$

This is the classical Poicaré normal form.

Proposition 2.1. Consider three dimensional systems satisfying $n_0 = 2$. Then the system (2.16) is analytically equivalent to its normal form if (2.17) admits an analytic Poincaré normal form under analytic change of coordinates.

The condition in this result is sufficient, but not necessary. For example, consider

$$\begin{aligned} \begin{bmatrix} \dot{\xi}_{01} \\ \dot{\xi}_{02} \end{bmatrix} &= A_0\xi_0 + q_1^{[2+]}(\xi_0)q_2^{[2+]}(\xi_0) + q_1^{[2+]}(\xi_0)\xi_1 \\ \dot{\xi}_1 &= v \end{aligned}$$

Then the change of coordinates

$$\xi_0 = x_0,\ \xi_1 = x_1 - q_2^{[2+]}(x_0)$$

plus a suitable state feedback cancels $q_1^{[2+]}(\xi_0)q_2^{[2+]}(\xi_0)$, no matter it has a convergent Poincaré normal form or not.

3. STABILITY AND BEHAVIOR OF A THREE DIMENSIONAL NORMAL FORM

Normal forms are representatives of general nonlinear control systems. Therefore, the dynamic behavior of normal forms qualitatively characterizes the fundamental properties of nonlinear control systems. In this section, we develop a Lyapunov function and analyze its behavior for the normal forms defined by (3.1), which is a subset of the normal forms in Theorem 2.2. In addition, it is proved that the controlled Lorenz equation is globally and analytically equivalent to a special case of (3.1). As a result, the normal form could have chaotic behavior. Consider the following normal form with one uncontrollable mode

$$\begin{aligned} \dot{x}_0 &= -\lambda x_0 + \delta_1 x_1^2 + \delta_2 x_2^2 \\ \dot{x}_1 &= x_2 \\ \dot{x}_3 &= u \end{aligned} \tag{3.1}$$

In this section, we assume $\lambda > 0$. The system has interesting behavior when its trajectories are attracted to a bounded set. For this purpose, we need to generate a Lyapunov function. let $x_1 = z_1$, $x_2 = -dz_1 + ez_2$, and $z_0 = x_0 + px_1^2$. Define the Lyapunov function by

$$V = \frac{1}{2}(z_1^2 + z_2^2 + (z_0 - q)^2) \tag{3.2}$$

Consider the feedback

$$\begin{aligned} u = &-k_1x_1 - k_2x_2 + k_{01}x_0x_1 + k_{02}x_0x_2 \\ &+k_{111}x_1^2 + k_{112}x_1^2x_2 \end{aligned} \tag{3.3}$$

We consider three family of feedbacks. If in $V(z)$ we select $q = 0$, then it can be proved that the following feedback globally stabilizes the system.

Feedback I

$$\begin{cases} k_2 > 0 \\ 0 < d < k_2 \text{ and } d \neq \lambda/2, \ T = \dfrac{\delta_1 + \delta_2 d^2}{\lambda - 2d} \\ k_1 > (k_2 - d)d, \qquad e^2 = k_1 - (k_2 - d)d \\ k_{01} = e^2(\delta_2 d + 2T) \\ k_{02} = -\delta_2 e^2 \\ k_{111} = -e^2T(\delta_2 d + 2T) \\ k_{112} = e^2\delta_2 T \end{cases} \tag{3.4}$$

If we select $q \neq 0$, then we can prove that the trajectory approaches a bounded attractor under the following feedbacks. The notation $I_{(a,b)}$ represents the open interval between a and $sign(b)\infty$.

Feedback II , $\delta_2 \neq 0$

$$\begin{cases} k_2 \in \mathbb{R} \\ k_{02} \in I_{(0,-\delta_2)}, \ \ e^2 = \left|\dfrac{k_{02}}{\delta_2}\right| \\ d > 0 \text{ and } d \neq \lambda/2, \ \ T = \dfrac{\delta_1 + \delta_2 d^2}{\lambda - 2d} \\ k_1 \in I_{(e^2 + (\delta_2 d + 2T)(d - k_2)/\delta_2, (\delta_2 d + T)/\delta_2)} \\ \\ k_{01} = e^2(\delta_2 d + 2T) \\ k_{111} = -e^2T(\delta_2 d + 2T) \\ k_{112} = e^2\delta_2 T \end{cases} \tag{3.5}$$

Feedback III , $\delta_2 = 0,\ \delta_1 \neq 0$

$$\begin{cases} k_2 > 0 \\ k_1 \in \mathbb{R} \\ 0 < d < k_2 \text{ and } d \neq \lambda/2,\ T = \dfrac{\delta_1}{\lambda - 2d} \\ k_{111} < 0, \qquad\qquad e^2 = -\dfrac{k_{111}}{2T^2} \\ k_{01} = 2e^2 T \\ k_{02} = 0,\ k_{112} = 0 \end{cases} \tag{3.6}$$

Theorem 3.1. Consider the normal form (3.1).

(i) Given any feedback in the form of I, or II, or III. Then there exists a bounded set D so that all trajectories of the closed-loop system asymptotically approach D.

(ii) Under Feedback I the system is globally asymptotically stable at the origin.

While the system is asymptotically stable under Feedback I, its behavior under Feedbacks II and III becomes more complicated. All we know is that the system has a bounded attractor. However, the behavior inside the bounded attractor is unknown. In fact, depending on the feedback, bifurcations and chaos could occur. As an example, in the following we prove that the Lorenz system is equivalent to a normal form (3.1) with a Feedback III. Consider a controlled Lorenz system

$$\begin{aligned} \dot{x} &= \alpha(y - x) \\ \dot{y} &= \beta x - xz - y + v \\ \dot{z} &= xy - \lambda z \end{aligned} \tag{3.7}$$

where $\alpha > 0$, $\beta > 0$, and $\lambda > 0$ are constant parameters, u is a control input. Define a globally invertible transformation

$$\begin{aligned} x_1 &= x,\ \ x_2 = \alpha(y - x), \\ x_0 &= z - \frac{1}{2\alpha}x^2, \\ u &= \alpha(\beta + \alpha)x - \alpha(1 + \alpha)y - \alpha xz + \alpha v, \end{aligned} \tag{3.8}$$

Then the Lorenz system is transformed into the following normal form

$$\begin{aligned} \dot{x}_0 &= -\lambda x_0 + (1 - \frac{\lambda}{\alpha})x_1^2 \\ \dot{x}_1 &= x_2,\ \ \dot{x}_2 = u \end{aligned} \tag{3.9}$$

It is a special case of the normal form (3.1) with $\delta_1 = 1 - \frac{\lambda}{\alpha}$, $\delta_2 = 0$, for some $\alpha > 0$. The classical Lorenz system is (3.7) with $v = 0$. It is equivalent to (3.9) in which u is defined by (3.3) and the coefficients $k_1 = \alpha - \alpha\beta$, $k_2 = \alpha + 1$, $k_{01} = -\alpha$, $k_{02} = 0$, $k_{111} = -\dfrac{1}{2}$, $k_{112} = 0$. It is a special case of Feedback III with $d = \alpha$. It is well known that the system has bifurcations and chaos, depending on the value of α, β, and λ.

Proposition 3.1. Under the change of coordinates and state feedback (3.8), the controlled Lorenz system (3.7) is globally and analytically equivalent to its normal form (3.9). The classical Lorenz system with $v = 0$ is equivalent to (3.9) under a Feedback III.

4. CONCLUSION

The convergence of control system normal forms is a long time open problem. In this paper, the problem is solved for three dimensional control systems with a single uncontrollable mode and a single input. The problem is partially solved for linearly controllable systems and also for systems with two controllable modes. A controlled Lorenz system is proved to be analytically equivalent to a normal form. In addition, a family of feedbacks is derived under which the normal form has a bounded attractor which include the well known Lorenz attractor as a special case.

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