The Controlled Center Systems

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Abstract: In this paper we propose a methodology to stabilize systems with control bifurcations by introducing “The Controlled Center System”. This system is a reduced-order controlled dynamics consisting of the linearly uncontrollable dynamics with the first variable of the linearly controllable dynamics as input. The controller of the full order system is then constructed. We apply this methodology to systems with a transcontrollable, a Hopf, and a double-zero, control bifurcation.

I. Introduction

Center manifold theory plays an important role in the study of the stability of nonlinear systems when the equilibrium point is not hyperbolic. The center manifold is an invariant manifold of the differential (difference) equation which is tangent at the equilibrium point to the eigenspace of the neutrally stable eigenvalues. After determining the reduced dynamics on the center manifold, we study its stability and then conclude about the stability of the full order system [6]. This theory can be viewed as a model reduction technique for nonlinear dynamical systems with non-hyberbolic equilibrium points. Indeed, the stability properties of a dynamical system around an equilibrium where one or more eigenvalues of its linear part are on the imaginary axis are characterized by the local asymptotic stability of the dynamics on the center manifold. Thus, this leads to a reduction of the dimension of the dynamics that needs to be analyzed to determine local asymptotic stability of the equilibrium.

For a nonlinear control system around an equilibrium, the local asymptotic stability of the linear controllable directions can be easily achieved by linear feedback. Therefore the stabilizability of the whole system should depend on a reduced order model that corresponds to the stabilizability of the linearly uncontrollable directions. The *Controlled Center Dynamics* introduced in [9] formalizes this intuition. By assuming that the stabilizing feedback has a certain structure and is characterized by certain parameters, the controlled center dynamics is a reduced order dynamical system characterized by the parameters of the feedback. By finding the conditions under which this dynamical system is stable, we deduce conditions on the parameters of the feedback, and thus deduce a stabilizing controller for the full order system.

In this paper, we present a slightly different approach. Instead of assuming that the feedback has a certain structure and is characterized by certain parameters, we synthesize a controller on a reduced-order control system called the *Controlled Center System*. This system is a controlled dynamical system consisting of the linearly uncontrollable dynamics with the first variable of the linearly controllable dynamics playing the role of the input. By constructing a stabilizing controller, that satsifies certain conditions, for this reduced order control system, we are able to deduce a stabilizing controller for the full order system.

The paper is organized as follows. In section 2, we review the controlled center dynamics approach. Then, in section 3, we introduce the quadratic controlled center systems and propose a methodology to stabilize systems with control bifurcations. We apply this approach to systems with a trancontrollable and Hopf control bifurcation. Finally, in section 4, we introduce the cubic controlled center systems, and then apply this technique to systems with a double-zero control bifurcation.

II. Review of the Controlled Center Dynamics

Consider the following nonlinear system

$$\dot{\zeta} = f(\zeta, v) \tag{1}$$

the variable $\zeta \in \mathbb{R}^n$ is the state, $v \in \mathbb{R}$ is the input variable. The vectorfield $f(\zeta)$ is assumed to be C^k for some sufficiently large k.

Assume $f(0,0)=0$, and suppose that the linearization of the system at the origin is uncontrollable with the uncontrollable modes being on the imaginary axis. Thus

$$\text{rank}([B\ AB\ A^2B\ \cdots\ A^{n-1}B]) = n - r, \tag{2}$$

with $A = \frac{\partial f}{\partial \zeta}(0,0)$,$B = \frac{\partial f}{\partial v}(0,0)$, and $r > 0$. Let us denote by $\Sigma_{\mathcal{U}}$ the system (1) under the above assumptions.

The system $\Sigma_{\mathcal{U}}$ is not linearly controllable at the origin, and a change of some control properties may occur around this equilibrium point, this is called a control bifurcation if it is linearly controllable at other equilibria [16].

From linear control theory [11], we know that there exist a linear change of coordinates and a linear feedback transforming the system $\Sigma_{\mathcal{U}}$ to

$$\begin{aligned} \dot{x}_1 &= A_1x_1 + \bar{f}_1(x_1, x_2, u), \\ \dot{x}_2 &= A_2x_2 + B_2u + \bar{f}_2(x_1, x_2, u), \end{aligned} \tag{3}$$

with $x_1 \in \mathbb{R}^r$, $x_2 \in \mathbb{R}^{n-r}$, $u \in \mathbb{R}$, $A_1 \in \mathbb{R}^{r\times r}$ is in the real Jordan form and its eigenvalues are on the imaginary axis,

$A_2 \in \mathbb{R}^{(n-r)\times(n-r)}$, $B_2 \in \mathbb{R}^{(n-r)\times 1}$ are in the Brunovskỳ form, i.e.

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

and $\bar{f}_k(x_1, x_2, u) = O(x_1, x_2, u)^2$, for $k = 1, 2$.
Now, consider the feedback given by

$$u(x_1, x_2) = \kappa(x_1) + K_2 x_2, \quad (4)$$

with κ is a piecewize smooth function and $K_2 = \left[\ k_{2,1} \ \ \cdots \ \ k_{2,n-r} \ \right]$.
Because (A_2, B_2) is controllable, the eigenvalues in the closed-loop system associated with the equation of x_2 can be placed at arbitrary given points in the complex plane by selecting values for K_2. If one of these controllable eigenvalues is placed in the right-half plane, the closed-loop system is unstable around the origin. Therefore, we assume that K_2 has the following property.

Property $\mathcal{P}$: The matrix $\bar{A}_2 = A_2 + B_2 K_2$ is Hurwitz.

Let us denote by $\mathcal{F}$ the feedback (4) with the property $\mathcal{P}$.
Now consider the closed loop system (3)-(4), given by

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + \bar{f}_1(x_1, x_2, \kappa(x_1) + K_2 x_2), \\ \dot{x}_2 &= A_2 x_2 + B_2(\kappa(x_1) + K_2 x_2) + \bar{f}_2(x_1, x_2, \kappa(x_1) + K_2 x_2). \end{aligned} \quad (5)$$

This system possesses r eigenvalues on the imaginary axis, and $n - r$ eigenvalues in the open left half plane. Thus, a center manifold exists [6]. It is represented locally around the origin as

$$W^c = \{(x_1, x_2) \in \mathbb{R}^r \times \mathbb{R}^{n-r} | x_2 = \Pi(x_1), |x_1| < \delta, \Pi(0) = 0\} \quad (6)$$

for δ sufficiently small.
For any point (x_1, x_2) in W^c we have

$$x_2 = \Pi(x_1),$$

hence

$$\dot{x}_2 = \frac{\partial \, \Pi(x_1)}{\partial \, x_1} \dot{x}_1. \quad (7)$$

Since the points in W^c obeys the dynamics generated by the closed-loop system (5), and since in W^c the feedback law (4) is

$$u(x_1, x_2)|_{x_2 = \Pi(x_1)} = \kappa(x_1) + K_2 \Pi(x_1).$$

Then, substituting

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + \bar{f}_1(x_1, \Pi(x_1), \kappa(x_1) + K_2 \Pi(x_1)), \\ \dot{x}_2 &= A_2 \Pi(x_1) + B_2(\kappa(x_1) + K_2 \Pi(x_1)) \\ &\quad + \bar{f}_2(x_1, \Pi(x_1), \kappa(x_1) + K_2 \Pi(x_1)), \end{aligned}$$

into (7) gives the PDE satisfied by Π and κ

$$\begin{aligned} &\bar{A}_2 \Pi(x_1) + B_2 \kappa(x_1) + \bar{f}_2(x_1, \Pi(x_1), \kappa(x_1) + K_2 \Pi(x_1)) = \\ &\frac{\partial \Pi}{\partial x_1}(x_1) \left(A_1 x_1 + \bar{f}_1(x_1, \Pi(x_1), \kappa(x_1) + K_2 \Pi(x_1)) \right). \end{aligned} \quad (8)$$

The center manifold theorem ensures that this equation has a local solution for any smooth $\kappa(x_1)$. The reduced dynamics of the closed loop system (5) on the center manifold is given by

$$\dot{x}_1 = f_1(x_1; \kappa) \quad (9)$$

where

$$f_1(x_1; \kappa) = A_1 x_1 + \bar{f}_1(x_1, \Pi(x_1), \kappa(x_1) + K_2 \Pi(x_1))$$

According to the center manifold theorem, we know that if the dynamics (9) is locally asymptotically stable then the closed loop system (3)-(4) is locally asymptotically stable (see [6], for example).
The part of the feedback $\mathcal{F}$ given by $\kappa(x_1)$ determines the controlled center manifold $x_2 = \Pi(x_1)$ which in turn determines the dynamics (9). Hence the problem of stabilization of the system (3) reduces the problem to stabilizing the system (9) after solving the PDE (8), i.e. finding $\kappa(x_1)$ such that the origin of the dynamics (9) is asymptotically stable. Thus we can view $\kappa(x_1)$ as a pseudo control.
Let $\hat{f}_1(x_1) = A_1 x_1 + \bar{f}_1(x_1, \Pi(x_1), \kappa(x_1) + K_2 \Pi(x_1))$, and $\hat{f}_{2,i}(x_1) = \bar{f}_{2,i}(x_1, \Pi(x_1), \kappa(x_1) + K_2 \Pi(x_1))$, for $i = 1, \cdots, n - r$. By expliciting (8), we obtain

$$\Pi_{i+1}(x_1) + \hat{f}_{2,i}(x_1) = \frac{\partial \Pi_i}{\partial x_1}(x_1) \hat{f}_1(x_1),$$

for $i = 1, \cdots, n - r - 1$, and

$$\sum_{i=1}^{n-r} k_{2,i} \Pi_i(x_1) + \kappa(x_1) + \hat{f}_{2,n-r}(x_1) = \frac{\partial \Pi_{n-r}}{\partial x_1}(x_1) \hat{f}_1(x_1). \quad (10)$$

Instead of viewing the feedback $\kappa(x_1)$ as determining the center manifold $\Pi(x_1)$, we can view the first coordinate function of the center manifold $\Pi_1(x_1)$ as determining the other coordinate functions $\Pi_2(x_1), \cdots \Pi_{n-r}(x_1)$ and the feedback $\kappa(x_1)$. Thus we can view Π_1 as a pseudo control and write the dynamics as

$$\dot{x}_1 = A_1 x_1 + \bar{f}_1(x_1; \Pi_1). \quad (11)$$

We will call this dynamics the *Controlled Center Dynamics*.

III. The Quadratic Controlled Center Systems

We know from [12], [16] that there exist a quadratic change of coordinates and feedback which brings $\Sigma_{\mathcal{U}}$ to its quadratic normal form $\Sigma_{\mathcal{NF}}$ given by

$$\begin{aligned} \dot{z}_1 &= A_1 z_1 + \mathcal{R}^{[2]}(z_1) + \Gamma z_1 z_{2,1} + \sum_{i=1}^{r} \sum_{j=1}^{n-r+1} \delta_i^j \, z_{2,j}^2 e_1^i, \\ \dot{z}_2 &= A_2 z_2 + B_2 u + \sum_{i=1}^{n-r} \sum_{j=i+2}^{n-r+1} \theta_i^j \, z_{2,j}^2 \, e_2^i, \end{aligned} \quad (12)$$

with $\delta_i^j, \theta_i^j \in \mathbb{R}$, $\Gamma \in \mathbb{R}^{r \times r}$, $z_{2,n-r+1} = u$, and e_1^i (resp. e_2^i) is the $i^{th}-$ unit vector in the z_1-space (z_2-space); $\mathcal{R}^{[2]}(z_1)$ are the quadratic resonant terms.

Definition 3.1: Consider a nonlinear system with a control bifurcation in its normal form $\Sigma_{\mathcal{NF}}$. We define the *quadratic controlled center system* as

$$\dot{z}_1 = A_1 z_1 + \mathcal{R}^{[2]}(z_1) + \Gamma z_1 z_{2,1} + \Delta z_{2,1}^2, \tag{13}$$

with $\Delta = \sum_{i=1}^{r} \delta_1^i \, e_1^i \in \mathbb{R}^{n\times 1}$.

This system can be viewed as a control system where $z_{2,1}$ plays the role of the input.

Our goal in this section is to find a mapping $\pi_1 : z_1 \to z_{2,1}$ which locally asymptotically stabilizes the controlled center system and which allows at the same time to find a controller $u(z_1, z_2) = \kappa(z_1) + K_2 z_2$ that locally asymptotically stabilizes the full order system (12).

Let V be a continuously differentiable, positive definite, function; then the derivative of V along the trajectories of (13) is given by

$$\dot{V} = \frac{\partial V}{\partial z_1}\dot{z}_1 = P_1(z_1) + P_2(z_1)z_{2,1} + P_3(z_1)z_{2,1}^2,$$

with $P_1(z_1) = \frac{\partial V}{\partial z_1}\cdot(A_1 z_1 + \mathcal{R}^{[2]}(z_1))$, $P_2(z_1) = \frac{\partial V}{\partial z_1}\cdot \Gamma z_1$, and $P_3(z_1) = \frac{\partial V}{\partial z_1}\cdot\Delta$. If we find a mapping $\pi_1 : z_1 \to z_{2,1}$ such that $\dot{V}$ is negative definite in some neighborhood of the origin $z_1 = 0$, then the origin, $z_1 = 0$, for the controlled center system is locally asymtptotically stable.

When z_1 is such that $P_3(z_1) < 0$, it is sufficient to choose $z_{2,1} = \pi_1(z_1)$ sufficiently large in order to have $\dot{V} < 0$. But when z_1 is such that $P_3(z_1) \geq 0$, then then we have to find π_1 such that $\dot{V} < 0$. In this case, by viewing $\dot{V}$ as a polynomial of degree two in $z_{2,1}$, it is necessary that the discriminant of $\dot{V}$ satisfies

$$P_2^2(z_1) - 4P_1(z_1)P_3(z_1) > 0, \quad \text{for every } z_1 \in \mathbb{R}^r. \tag{14}$$

This will allow $\dot{V}$ to change its sign when viewed as a function of z_{21}.

Let $\overline{\mathcal{V}}$ be defined as

$$\overline{\mathcal{V}} = V(z_1) + z_2^T P z_2, \tag{15}$$

with $\bar{A}_2^T P + P\bar{A}_2 = -Q$, with $Q > 0$. Then the derivative of $\overline{\mathcal{V}}$ along the trajectories of (12) is

$$\begin{aligned}
\dot{\overline{\mathcal{V}}} &= \frac{\partial V}{\partial z_1}\dot{z}_1 + \dot{z}_2^T P z_2 + z_2^T P \dot{z}_2, \\
&= \frac{\partial V}{\partial z_1}(A_1 z_1 + \mathcal{R}^{[2]}(z_1) + \Gamma z_1 z_{2,1} + \sum_{i=1}^{r}\sum_{j=1}^{n-r+1} \delta_i^j \, z_{2,j}^2 e_1^i) \\
&+ z_2^T(\bar{A}_2^T P + P\bar{A}_2)z_2 + (B_2^T P z_2 + z_2^T P B_2)\kappa(z_1) \\
&= -z_2^T Q z_2 + (B_2^T P z_2 + z_2^T P B_2)\kappa(z_1) \\
&+\frac{\partial V}{\partial z_1}(A_1 z_1 + \mathcal{R}^{[2]}(z_1) + \Gamma z_1 z_{2,1} + \sum_{i=1}^{r}\sum_{j=1}^{n-r+1} \delta_i^j \, z_{2,j}^2 e_1^i).
\end{aligned}$$

Let us denote by $z_1^{\nu} = z_{1,1}^{\nu_1}\cdots z_{1,r}^{\nu_r}$ such that $\nu_1 + \cdots + \nu_r = \nu$, and assume that $\pi_1(z_1)$ is chosen such that

$$\frac{\partial V}{\partial z_1}[A_1 z_1 + \mathcal{R}^{[2]}(z_1) + \Gamma z_1 \pi_1(z_1) + \Delta(\pi_1(z_1))^2] = O(z_1^d) \tag{16}$$

is negative definite and

$$\frac{\partial V}{\partial z_1}\sum_{i=1}^{r}\sum_{j=2}^{n-r+1} \delta_i^j \,(\pi_j(z_1))^2 e_1^i + 2\pi_{n-r}(z_1)\kappa(z_1) = O(z_1^{d'}), \tag{17}$$

with $d < d'$. Then,

$$\begin{aligned}
\dot{\overline{\mathcal{V}}} &< -z_2^T Q z_2 \\
&+\frac{\partial V}{\partial z_1}[A_1 z_1 + \mathcal{R}^{[2]}(z_1) + \Gamma z_1 \pi_1(z_1) + \Delta(\pi_1(z_1))^2] + O(z_1^{d'}),
\end{aligned}$$

which is negative definite around the origin. Thus locally asymptotically stabilizing the controlled center system with a "feedback" $\pi_1(z_1)$ satisfying conditions (16) and (17) allows finding a feedback $u(z_1, z_2) = \kappa(z_1) + K_2 z_2$ that locally asymptotically stabilizes the full order system (12), since $\kappa(z_1)$ and $\pi_1(z_1)$ are directly related through (10).

Now, let us apply this approach to systems with a transcontrollable bifurcation or a Hopf control bifurcations. For systems with a transcontrollable bifurcation, we have $A_1 = 0 \in \mathbb{R}$. In this csae, the system has the normal form

$$\begin{aligned}
\dot{z}_1 &= \beta z_1^2 + \gamma z_1 z_{2,1} + \sum_{j=1}^{n-1} \delta_j^1 \, z_{2,j}^2, \\
\dot{z}_2 &= A_2 z_2 + B_2 u + \sum_{i=1}^{n-2}\sum_{j=i+2}^{n-1} \theta_i^j \, z_{2,j}^2 \, e_2^i.
\end{aligned} \tag{18}$$

This system exhibits a transcontrollable bifurcation if $\gamma^2 - 4\beta\delta_1 > 0$ (see [12] and [16]). The controlled center system is

$$\dot{z}_1 = \beta z_1^2 + \gamma z_1 z_{2,1} + \delta_1^1 \, z_{2,1}^2.$$

Consider $V(z_1) = \frac{1}{2}z_1^2$, then

$$\dot{V} = \beta z_1^3 + \gamma z_1^2 z_{21} + \delta_1 z_1 z_{21}^2.$$

If we consider a mapping of the form $\pi_1(z_1) = \alpha z_1$, then

$$\dot{V} = (\beta + \gamma\alpha + \delta_1\alpha^2)z_1^3 = (\beta + \gamma\alpha + \delta_1\alpha^2)\mathrm{sgn}(z_1)|z_1|z_1^2.$$

Thus we have to choose α such that $(\beta + \gamma\alpha + \delta_1\alpha^2)\mathrm{sgn}(z_1) < 0$, i.e. when $z_1 \geq 0$, we choose $\alpha = \alpha_1$ with $\beta + \gamma\alpha_1 + \delta_1\alpha_1^2 < 0$; and when $z_1 < 0$, we choose $\alpha = \alpha_2$ with $\beta + \gamma\alpha_2 + \delta_1\alpha_2^2 > 0$. This choice is always possible since the function $\beta + \gamma X + \delta_1 X^2$ changes its sign because $\gamma^2 - 4\beta\delta_1 > 0$. From [9], we know that when a feedback of the form $u(z_1, z_2) = K_1 z_1 + K_2 z_2$, with $K_2 = \left[\ K_{2,1}\cdots K_{2,n-1}\ \right]$, is used to stabilize systems with a transcontrollable bifurcation, and $\pi(z_1) = \pi^{[1]} z_1$, then $\pi_1^{[1]} = -\frac{K_1}{K_{2,1}}$ and $\pi_i^{[1]} = 0$, for $i = 2, \cdots, n-1$. Thus, an asymptotically stabilizing feedback for (18) is

$$u(z_1, z_2) = -K_{2,1}\alpha|z_1| + K_2 z_2.$$

For systems with a Hopf control bifurcation, i.e. $A_1 = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$, and $\omega \neq 0$, the quadratic normal form of this system is

$$\dot{z}_1 = A_1 z_1 + \Gamma z_1 z_{2,1} + \sum_{i=1}^{2}\sum_{j=1}^{n-1} \delta_i^j\, z_{2,j}^2 e_1^i,$$
$$\dot{z}_2 = A_2 z_2 + B_2 u + \sum_{i=1}^{n-2}\sum_{j=i+2}^{n-1} \theta_i^j\, z_{2,j}^2\, e_2^i,$$

and the controlled center system is given by

$$\dot{z}_1 = A_1 z_1 + \Gamma z_1 z_{2,1} + \Delta z_{2,1}^2.$$

We assume that $\Gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{12} & \gamma_{22} \end{bmatrix}$ is such that $\gamma_{11} \neq 0$ or $\gamma_{22} \neq 0$ or $\gamma_{12} \neq 0$, and we define $\Gamma_s = \Gamma + \Gamma^T$.
Let V be a positive definite function given by $V(z_1) = z_1^T z_1$, then

$$\dot{V} = z_1^T \Gamma_s\, z_1\, z_{2,1} + 2 z_1^T \Delta z_{2,1}^2.$$

The condition (14) reduces to $(z_1^T \Gamma_s\, z_1)^2 > 0$ for every $z_1 \in \mathbb{R}^2$. This condition is satisfied since $\gamma_{11} \neq 0$ or $\gamma_{22} \neq 0$ or $\gamma_{12} \neq 0$.
If we consider the mapping $z_{2,1} = \pi_1(z_1) = \alpha\sqrt{|z_1^T \Gamma_s z_1|} z_1^T \Gamma_s z_1$, with $\alpha > 0$, then

$$\dot{V} = -\alpha |z_1^T \Gamma_s z_1|^{5/2} + O(z_1^7)$$

which is negative in a sufficiently small neighborhood of the origin. Hence, the controlled center system is locally asymptotically stable. Moreover, the mapping π_1 is such that $\pi_i(z_1) = O(z_1^7)$, for $i = 2, \cdots, n-2$. Thus, conditions (16) and (17) are satisfied. Using equation (10), we deduce that the feedback

$$u(z_1, z_2) = -K_{2,1}\alpha\sqrt{|z_1^T \Gamma_s z_1|} z_1^T \Gamma_s z_1 + K_2 z_2$$

locally asymptotically stabilizes the full order system.

IV. The Cubic Controlled Center Systems

There are some cases where it is preferable to use cubic normal forms, and this leads to introducing cubic controlled center systems. For instance, in the case of the double-zero bifurcation it is known that the system is unstable when the quadratic terms in the Poincaré normal form are non-zero (see [4] and [15] and references therein), and that conditions of the stability of these systems are expressed in terms of the cubic and quartic terms. Thus we expect using the cubic normal form in the case of systems with a double-zero control bifurcations.
¿From [16], we know that there exist a cubic change of coordinates and feedback which brings $\Sigma_{\mathcal{U}}$ to its cubic normal form given by

$$\begin{aligned}
\dot{z}_1 &= A_1 z_1 + \mathcal{R}^{[3]}(z_1) + \Gamma z_1 z_{2,1} + \sum_{i=1}^{r}\sum_{j=1}^{n-r+1} \delta_i^j\, z_{2,j}^2 e_1^i \\
&+ \sum_{i=1}^{r}\Bigg[\sum_{j=1}^{r}\Bigg(\sum_{k=j}^{r}\gamma_i^{jk} z_{1,k} z_{2,1} + \sum_{k=1}^{n-r+1}\delta_i^{jk} z_{2,k}^2\Bigg) z_{1,j} \\
&+ \sum_{j=1}^{n-r+1}\sum_{k=j}^{n-r+1}\varphi_i^{jk} z_{2,j} z_{2,k}^2\Bigg] e_1^i, \\
\dot{z}_2 &= A_2 z_2 + B_2 u + \sum_{i=1}^{n-r}\sum_{j=i+2}^{n-r+1} \theta_i^j\, z_{2,j}^2\, e_2^i,
\end{aligned} \tag{19}$$

with $\delta_i^j, \theta_i^j, \gamma_i^{jk}, \delta_i^{jk}, \varphi_i^{jk} \in \mathbb{R}$, $\Gamma \in \mathbb{R}^{r\times r}$, $z_{2,n-r+1} = u$, and e_1^i (resp. e_2^i) is the $i^{th}-$ unit vector in the z_1-space (z_2-space); $\mathcal{R}^{[3]}(z_1)$ are the quadratic and cubic resonant terms.
Definition 4.1: Consider a nonlinear system with a control bifurcation in its normal form (19). We define the *cubic controlled center system* as

$$\begin{aligned}
\dot{z}_1 = \Psi(z_1) &= A_1 z_1 + \mathcal{R}^{[3]}(z_1) + \Gamma z_1 z_{2,1} + \Delta z_{2,1}^2 \\
&+ \sum_{i=1}^{r}\Bigg[\sum_{j=1}^{r}\sum_{k=j}^{r}\Big(\gamma_i^{jk} z_{1,k} + \delta_i^{j1} z_{2,1}\Big) z_{1,j} z_{2,1}\Bigg] e_1^i + \Phi z_{2,1}^3,
\end{aligned} \tag{20}$$

with $\Delta = \sum_{i=1}^{r} \delta_1^i\, e_1^i \in \mathbb{R}^{n\times 1}$, $\Phi = \sum_{i=1}^{r} \varphi_i^{11} e_1^i$.
As in the case of the quadratic controlled center system, this system can be viewed as a control system where $z_{2,1}$ plays the role of the input.
Moreover, similarly to the precedent section, the goal is to find a mapping $\pi : z_1 \mapsto z_{2,1}$ which locally asymptotically stabilizes the system (20) and allows to construct a feedback $u(z_1, z_2) = \kappa(z_1) + K_2 z_2$ that locally asymptotically stabilizes (19).
Let V be a continuously differentiable, positive definite, function, and let $\overline{\mathcal{V}}$ be defined as

$$\overline{\mathcal{V}} = V(z_1) + z_2^T P z_2, \tag{21}$$

with $\bar{A}_2^T P + P\bar{A}_2 = -Q$, with $Q > 0$. Following similar steps as before, if we assume that $\pi_1(z_1)$ is chosen such that

$$\frac{\partial V}{\partial z_1}\Psi(z_1) = O(z_1^d) \tag{22}$$

is negative definite and

$$\begin{aligned}
&\frac{\partial V}{\partial z_1}\sum_{i=1}^{r}\Bigg[\sum_{j=2}^{n-r+1}\delta_i^j\,(\pi_j(z_1))^2 + \sum_{j=1}^{r}\sum_{k=2}^{n-r+1}\delta_i^{jk}(\pi_k(z_1))^2 z_{1,j} \\
&+ \sum_{j=2}^{n-r+1}\sum_{k=j}^{n-r+1}\varphi_i^{jk}\pi_j(z_1)\pi_k(z_1)^2\Bigg] e_1^i + 2\pi_{n-r}(z_1)\kappa(z_1) = O(z_1^{d'}),
\end{aligned} \tag{23}$$

with $d < d'$. Then,

$$\dot{\overline{\mathcal{V}}} < -z_2^T Q z_2 + \frac{\partial V}{\partial z_1}\Psi(z_1) + O(z_1^{d'}),$$

which is negative definite around the origin. Thus locally asymptotically stabilizing the controlled center system with a "feedback" $\pi_1(z_1)$ satisfying conditions (22) and (23) allows finding a feedback $u(z_1, z_2) = \kappa(z_1) + K_2 z_2$ that locally asymptotically stabilizes the full order system (12), since $\kappa(z_1)$ and $\pi_1(z_1)$ are directly related through (10).

Let us apply this approach to a system with a double-zero control bifurcation, i.e. when $A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. In this case not all the quadratic and cubic terms are resonant [4]. A possible normal form is given by (19) with

$$\mathcal{R}^{[3]}(z_1) = \begin{bmatrix} z_{11}^2(b + cz_{11}) \\ dz_{11}^3 \end{bmatrix},$$

and b, c, d are real numbers such that $dc \neq 0$. Let us assume that $d < 0$, and $\gamma_{21} \neq 0$ or $\gamma_{22} \neq 0$.

The cubic controlled center system is given by

$$\begin{aligned} \dot{z}_1 &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z_1 + \begin{bmatrix} z_{11}^2(b + cz_{11}) \\ dz_{11}^3 \end{bmatrix} + \Gamma z_1 z_{2,1} + \Delta z_{2,1}^2 \\ &+ \sum_{i=1}^{2} \sum_{j=1}^{2} (\sum_{k=j}^{2} \gamma_i^{jk} z_{1,k} z_{2,1} + \delta_i^{j,1} z_{2,1}^2) z_{1,j} e_1^i + \Phi z_{2,1}^3. \end{aligned}$$

Let V be a positive definite function given by

$$V(z) = -\frac{d}{4} z_{11}^4 + \frac{1}{2} z_{12}^2.$$

Then, along the trajectories of the controlled center system, $\dot{V}$ is given by

$$\begin{aligned} &\dot{V} = -dbz_{11}^5 - cdz_{11}^6 \\ &+\Big[z_{12}(\gamma_{21}z_{11} + \gamma_{22}z_{12} + \gamma_2^{11}z_{11}^2 + \gamma_2^{12}z_{11}z_{12} + \gamma_2^{22}z_{12}^2) \\ &-dz_{11}^3(\gamma_{11}z_{11} + \gamma_{12}z_{12} + \gamma_1^{11}z_{11}^2 + \gamma_1^{12}z_{11}z_{12} + \gamma_1^{22}z_{12}^2)\Big] z_{21} \\ &+\Big[(\delta_2 + \delta_2^{11}z_{11} + \delta_2^{21}z_{12})z_{12} - (\delta_{11} + \delta_1^{11}z_{11} + \delta_1^{21}z_{12})dz_{11}^3\Big] z_{21}^2 \\ &+(-d\varphi_1^{11}z_{11}^3 + \varphi_2^{11}z_{12})z_{21}^3. \end{aligned}$$

Let us consider the mapping π_1 defined by $\pi_1(z_1) = -(\gamma_{21}z_{11}z_{12} + \gamma_{22}z_{12}^2)$, then

$$\dot{V} = -(\gamma_{21}z_{11}z_{12} + \gamma_{22}z_{12}^2)^2 + O(z_{11}, z_{12})^5, \qquad (24)$$

which is negative semidefinite in some neighborhood of the origin since $\gamma_{21} \neq 0$ or $\gamma_{22} \neq 0$. Moreover, since $dc \neq 0$, we can check that the set for which $\dot{V} = 0$ reduces to the origin. Thus according to LaSalle's theorem [14], the origin for the controlled center system is locally asymptotically stable.

The mapping $z_{21} = \pi_1(z_1) = -(\gamma_{21}z_{11}z_{12} + \gamma_{22}z_{12}^2)$ satisfies conditions (16) and (17). Using (10), we deduce that

$$u(z_1, z_2) = K_{21}(\gamma_{21}z_{11}z_{12} + \gamma_{22}z_{12}^2) + K_2 z_2,$$

locally asymptotically stabilizes (19) when $A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

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