(f,g) INVARIANT DISTRIBUTIONS, CONNECTIONS AND PONTRYAGIN CLASSES*

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Summary

The concept of (A,B) invariance in linear systems theory generalizes to two concepts for nonlinear systems, (f,g) invariance and local (f,g)invariance. The former implies the latter and the latter implies the former only locally. The topological obstruction to global (f,g) invariance are discussed.

(A,B) Invariance

The fundamental geometric concept in the study of decoupling of a linear system by feedback is that of an (A,B) invariant subspace [1]. Recall for the linear system

$$\dot{x} = Ax + Bu \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$$

a linear subspace $\mathcal{V}$ is an (A,B) invariant if there exists mxn matrix F defining a feedback law $u = Fx + v$ such that the modified dynamics

$$\dot{x} = (A+BF)x + Bv$$

leaves $\mathcal{V}$ invariant, i.e., $(A+BF)\,\mathcal{V} \subseteq \mathcal{V}$. (1)

It is well known and easy to see that this is equivalent to

$$A\mathcal{V} \subseteq \mathcal{V} + \mathcal{R}(B) \tag{2}$$

where $\mathcal{R}(B)$ denotes the subspace spanned by the columns of B. For reasons that will soon be apparent we refer to (1) and (2) as the global and local characterizations of (A,B) invariance. The former is useful for it directly relates to the system dynamics but since (1) is nonlinear in the two unknowns F and $\mathcal{V}$ it is somewhat difficult to work with. On the other hand (2) is linear in only one unknown so is much easier to use. In particular it shows that the class of (A,B) invariant subspaces is a semilattice under inclusion and vector space addition and this is extremely useful.

(f,g) Invariance

Similar concepts arise in the study of decoupling of nonlinear systems of the form

$$\dot{x} = f(x,u) = g^0(x) + g(x)u \tag{3}$$

$$= g^0(x) + \sum_{j=1}^{m} g^j(x)u_j$$

where x are local coordinates of a smooth (either) C^∞ or C^ω) n dimensional manifold M, $u \in \mathbb{R}^m$, g^0 and the m columns of g are the local coordinate descriptions of smooth vector fields on M. Of course (3) is to be interperted locally, that is, differing descriptions of this type are needed in different coordinate charts. We do assume though that the vector fields represented locally by g^0 and $g^1,..g^m$ are globally defined on all of M.

The nonlinear generalization of a linear subspace is twofold, if we view the subspace as a space of velocities then the nonlinear generalization is a distribution. The space $\mathcal{X}(M)$ of all smooth vector fields on M is a vector space over $\mathbb{R}$ and a module over the smooth functions $\mathcal{F}(M)$. Under the lie bracket, $\mathcal{X}(M)$ is a Lie algebra over $\mathbb{R}$. A distribution Δ is a submodule and hence a subspace of $\mathcal{X}(M)$. We denote by $\Delta(x)$ the linear subspace of the target space at x obtained by evaluating the vector fields of Δ at x. If the dimension of $\Delta(x)$ is constant, say d, Δ is said to be nonsingular and of dimension d. A linear subspace $\mathcal{V} \subseteq \mathbb{R}$ can be viewed as a nonsingular distribution Δ on $\mathbb{R}^n$ where each $\Delta(x) = \mathcal{V}$ canonically inbedded in $T_x\mathbb{R}^n$.

Suppose $\mathcal{V}$ is a d dimensional subspace of $\mathbb{R}^n$, then $\mathcal{V}$ induces a canonical projection $\pi: \mathbb{R}^n \to \mathbb{R}^n/\mathcal{V}$. The level sets of this projection, the cosets $x+\mathcal{V}$, define a partition of $\mathbb{R}^n$ into affine d dimensional planes. The nonlinear generalization of this is a foliation (We consider only nonsingular foliations).

A d dimensional foliation Φ of M is a collection $\Phi = \{\mathcal{U}^\sigma, \pi^\sigma : \sigma \text{ in some index set}\}$ The $\{\mathcal{U}^\sigma\}$ is an open cover of M and the π^σ are submersions

$$\pi^\sigma : \mathcal{U}^\sigma \to \mathbb{R}^{n-d}$$

such that the level sets of π^σ and π^τ coincide when both submersions are restricted $\mathcal{U}^\sigma \cap \mathcal{U}^\tau$. A leaf of the foliation is a maximal immersed submanifold N with the property that each π^σ is constant on $N \cap \mathcal{U}^\sigma$. We frequently identify the foliation with its collection of leaves $\Phi = \{N^\rho : \rho \text{ in some index set}\}$. The leaves partition M.

Every d dimensional foliation Φ defines a d dimensional distribution Δ by letting

$$\Delta(x) = T_x N^\rho \text{ where } x \in N^\rho$$

and Δ be the space of vector fields X such that

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$X(x) \in \Delta(x)$ for all x. Not every distribution comes from a foliation, those that do are said to be integrable. A classical result is

Frobenius Theorem A nonsingular distribution Δ is integrable iff it is involutive. (A distribution is said to be involutive if it closed under Lie bracket or in other words if it is Lie subalgebra of $\mathcal{X}(M)$).

A vector field X leaves a distribution Δ invariant if $Y \in \Delta$ implies $[X,Y] \in \Delta$. Therefore Δ is involutive iff it is left invariant by each of its vector fields. The smallest involutive distribution containing Δ is called its involutive closure and denoted by $\bar{\Delta}$.

We can now define the nonlinear generalization of an (A,B) invariant subspace [2]. Actually there are two generalizations since the local and global characterizations differ in the presence of nonlinearity. A nonlinear feedback γ is $(m+1) \times (m+1)$ matrix valued function of the form

$$\gamma = \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix}$$

where α is $m \times 1$ and β is $m \times m$. If $u = \alpha(x) + \beta(x)v$ we have the modified dynamics

$$\dot{x} = \tilde{f}(x,v) = \tilde{g}^0(x) + g(x)v \tag{4}$$

where

$$\tilde{f}(x,v) = f(x,u(x,v)).$$

and

$$\tilde{g}^0(x) = g^0(x) + g(x)\alpha(x) \quad \tilde{g}(x) = g(x)\beta(x).$$

A distribution Δ is (f,g) invariant if there exists a feedback γ such that Δ is $\tilde{f}(\cdot,v)$ invariant for each constant control v. This is equivalent to Δ being $\tilde{g}^j$ invariant for $j = 0,..,m$. We use the notation

$$[\tilde{f}, \Delta] \subseteq \Delta \tag{5}$$

to mean the bracket of the vector field $\tilde{f}(\cdot,v)$ for any constant v with a vector field of Δ is again a vector field of Δ.

Let $\mathcal{R}(g)$ denote the distribution spanned by $g^1(x),..,g^m(x)$. A distribution Δ is locally (f,g) invariant if the bracket of any vector field $f(\cdot,u)$, u constant, with a vector field of Δ is back in $\Delta + \mathcal{R}(g)$. We denote this by

$$[f,\Delta] \subseteq \Delta + \mathcal{R}(g). \tag{6}$$

Clearly (5) and (6) are the nonlinear generalizations of (1) and (2) however they are not equivalent. It is easy to see the (f,g) invariance implies local (f,g) invariance and in [3] the following was proved.

Lemma Suppose Δ is an involutive, nonsingular locally (f,g) invariant distribution and $\mathcal{R}(g)$, $\mathcal{R}(g) \cap \Delta$ are nonsingular. Then locally there exists an invertible $\gamma(x)$ such that (5) is locally satisfied.

Topological Obstructions

From the above lemma it is clear why local (f,g) invariance is so called. Now we turn to a discussion of the topological obstructions to global (f,g) invariance

From the hypothesis of local (f,g) invariance one can arrive at a partial differential equation which a feedback γ must satisfy so that the modified dynamics leaves Δ invariant. In effect the partial derivatives of γ in the directions Δ are specified by (6).

The integrability (or mixed partial) condition for the solvability of this PDE follows immediately from the Jacobi identity for Lie brackets and the PDE is locally solvable once initial conditions are set.

Since Δ is nonsingular and involutive it defines a foliation $\Phi = \{N^\rho\}$ of M. The PDE for γ restricts to each leaf so that specifying initial conditions at one point on a leaf allows one to solve for γ in a neighborhood of that point on the leaf. If the leaf is simply connected then the solution can be continued to the whole leaf. Hence the simplest obstructions to global (f,g) invariance lie in the first fundamental groups of the leaves of Φ. If each of these groups is trivial then there is no problem.

Let's assume that this is so, and turn to the problem of setting initial conditions on each of the leaves. Assume that the foliation Φ is regular, i.e., there exists a smooth manifold structure on Φ in such a way that the canonical projection $\pi : M \to \Phi$ is a submersion. Suppose there exists a smooth section of π, that is, a smooth map $\sigma : \Phi \to M$ such that $\pi \circ \sigma = id : \Phi \to \Phi$. This section can be used to set initial conditions for the PDE by specifying that γ be the $(m+1)\times(m+1)$ identity matrix on the image of σ.

From the above discussions we see that the regularity of Φ and the existence of a section of π are important for the question of global (f,g) invariance. In the next sections we relate this to certain topological invariants of M.

Basic Connections and Bott Vanishing

We follow closely the treatment of Bott [5] and refer the reader there for further details. Suppose Δ is a d dimensional nonsingular distribution on M, the Δ can be viewed as a vector bundle over M. (Loosely speaking, a vector bundle over M is a rule which attaches to each $x \in M$, a linear subspace of some fixed dimension in a smooth fashion. The most important example of this in the tangent bundle TM which attaches to each x, the tangent space of M at x denoted by T_xM. Δ is a subbundle of TM).

The normal bundle Q to Δ is the quotient bundle TM/Δ which attaches to each x, the $q = n-d$ dimensional quotient space $T_xM/\Delta(x)$. A connection on Q is a rule for the directional differentiation of sections of Q in the directions of M. Let $\Gamma(Q)$ denote the space of sections of Q and suppose $Z \in \Gamma(Q)$ and $X \in \Gamma(TM) = \mathcal{X}(M)$. We denote by $\nabla_X(Z)$ the derivative of Z in the direction X as defined by the connection. The mapping $X,Z \to \nabla_X(Z)$ is $\mathbb{R}$ bilinear and satisfies the axioms

(i) $\nabla_X(hZ) = X(h)Z = h\,\nabla_X(Z)$

(ii) $\nabla_{hX}(Z) = h\,\nabla_X Z$

where $h \in \mathfrak{F}(M)$.

Let $\pi : TM \to Q$ denote the family of pointwise projections $\pi_x : T_x M \to T_x M/\Delta(x) = Q(x)$. A connection ∇ on Q is called basic if for every $X \in \Delta$ and $Z \in \Gamma(Q)$,

$$\nabla_X Z = \pi[X, \tilde{Z}] \tag{7}$$

where $\tilde{Z} \in \mathfrak{X}(M)$ such that $\pi(\tilde{Z}) = Z$. It is not hard to see that if Δ is involutive hence integrable then (7) can be used to define the connection ∇ at least for $X \in \Delta$. It always is possible to extend this definition to arbitrary X's hence one has

Lemma [5]. If Δ is integrable then Δ admits a basic connection (generally many basic connections.)

Given a connection ∇ on a vector bundle Q there is an associated curvature tensor κ defined for $X, Y \in \mathfrak{X}(M)$ and $Z \in \Gamma(Q)$

$$\kappa(X,Y)Z = \nabla_X(\nabla_Y(Z)) - \nabla_Y(\nabla_X(Z)) - \nabla_{[X,Y]}(Z) .$$

The curvature measures how much the mapping

$$X \to \nabla_X$$

fails to be a Lie algebra hemomorphism. It is not hard to see that $\kappa(X,Y)Z$ is $\mathfrak{F}(M)$ multilinear. Given a local basis of sections of Q, the curvature can be represented as a $q \times q$ matrix κ relative to this basis. Each entry of κ is a two form.

On any vector bundle Q over a paracompact manifold M one can always construct a smoothly varying metric on each of the fibers. Let $\langle \, , \rangle$ denote such a metric on Q and let $Z_1, Z_2 \in \Gamma(Q)$. A connection ∇ is compatible with the metric if

$$\nabla_X \langle Z_1, Z_2 \rangle = \langle \nabla_X Z_1, Z_2 \rangle + \langle Z_1, \nabla_X Z_2 \rangle .$$

If the curvature matrix κ for such a connection is written relative to an orthonornal basis for Q the κ is skew-symmetric. It is not hard to show that given a vector bundle Q with a metric there exists a compatible connection. Moreover if $Q = TM/\Delta$ with Δ integrable then there exists a basic connection compatible with this metric.

Consider the 4j form

$$\text{trace} \quad (\kappa^{2j})$$

where κ^{2j} is the 2j exterior power of the matrix of two forms κ. It is not hard to see that trace (κ^{2j}) is a cocycle. The j^{th} Pontryagin class $p_j(Q)$ of Q is the cohomolgy class of trace (κ^{2j}). These classes for $j = 1,..,[q/2]$ along with the constant function, 1, generate the Pontryagin ring $\text{Pont}^*(Q)$ of Q. $\text{Pont}^*(Q)$ is a subring of the de Rham cohomolgy ring $H^*(M)$. We denote by $\text{Pont}^k(Q)$ the elements of degree k.

This construction can be shown to be independent of the choice of connection ∇ and corresponding curvature κ, it only depends on Q up to vector bundle isomorphism.

As an aside we don't consider the odd powers of κ because if κ is compatible with a metric then $\kappa^t = -\kappa$ hence

$$\text{trace } (\kappa^{2j+1}) = 0 \quad \text{for all } j .$$

A bundle Q with connection ∇ is flat if the associated curvature $\kappa = 0$. Suppose $Q = TM/\Delta$ with Δ integrable. Let ∇ be a basic connection on Q then ∇ is defined by (7) in such a way that the mapping $X \to \nabla_X$ is a Lie algebra homomorphism for $X \in \Delta$. In other words if $X, Y \in \Delta$ and $Z \in \Gamma(Q)$ then $\kappa(X,Y)Z = 0$. Therefore we say that Q is flat in the directions of Δ.

This implies that every two form which is an entry of κ is an element of the ideal $\mathcal{J}$ of one forms which annihilate Δ. Let $\mathcal{J}^k$ be the span of products of k elements of $\mathcal{J}$. There are only $q = n-d$ linearly independent one forms annihilating Δ hence $\mathcal{J}^k = 0$ for $k > q$. The generator $p_j(Q)$ of $\text{Pont}^*(Q)$ is of degree 4j and is in $\mathcal{J}^{2j}$. From this we conclude

Bott's Vanishing Theorem [5]. If Q is a vector bundle isomorphic to TM/Δ with a Δ integrable then $\text{Pont}^k(Q) = 0$ for $k > 2q$.

Additional Vanishing

Suppose Δ is nonsingular, involutive and globally (f,g) invariant, so there exists a suitable feedback γ such that $\tilde{f} = f\gamma$ then

$$[\tilde{f}, \Delta] \subseteq \Delta.$$

Now suppose there exists controls $\{v^i : i=1,..,j\}$ such that $\{\tilde{f}^i(x) = \tilde{f}(x,v^i) : i=1,..,j\}$ are everywhere linearly independent modulo Δ. Then it is possible using the same techniques as found in [5] to construct a basic connection on $Q = TM/\Delta$ with the additional property that if $Z \in \Gamma(Q)$ and $\tilde{Z} \in \mathfrak{X}(M)$ with $\pi(\tilde{Z}) = Z$ then

$$\nabla_{\tilde{f}^i}(Z) = \pi[\tilde{f}^i, \tilde{Z}]$$

for $i = 1,..,j$. Therefore in addition to being flat in the directions of Δ, Q is also flat in the directions $\{\tilde{f}^i : i=1,..,j\}$. Using Bott's argument we can show that

$$\text{Pont}^k(Q) = 0$$

for $k > 2q - 2j$. This is a further obstruction to global invariance.

Because of space limitations we have only sketched out the beginnings of an obstruction theory for (f,g) invariance. We do not mean to imply that these are the only possible obstructions. It is not hard to see that there are obstructions which live in the Stiefel-Whitney classes. We will go into this at a later date.

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