(Ad f, G) INVARIANT AND CONTROLLABILITY DISTRIBUTIONS

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1. INTRODUCTION

In the last few years there has been an increasing interest in nonlinear feedback systems, and a systematic work of generalization of Wonham's geometric approach to linear feedback systems is being set up (see [1-6]).

Key tools are those of f invariance and (f,g) invariance for distributions, introduced in [7], [1] and [2]. In this paper, we compare previous definitions of "f invariance" and introduce a new notion, based on Sussmann's results about the integrability of C^∞ distributions [7], which we term (Ad f, G) invariance. Then we also introduce the concept of (Ad f, G) controllability subdistribution (a generalization of the notion of an (A,B) controllability subspace).

2. MATHEMATICAL PRELIMINARIES

Throughout this paper we consider nonlinear systems described by differential equations of the form

(2.1 a) $$\dot{x} = f(x,u) = g_0(x) + \sum_{i=1}^{m} g_i(x)u_i$$

(2.1 b) $$y = h(x)$$

The state x belongs to an n-dimensional C^∞ manifold M, $u_i \in R$, the vector fields $g_0(x)$, $g_1(x), \ldots, g_m(x)$ are complete C^∞ vector fields on M and $h : M \to \mathbb{R}^p$ is a C^∞ function. Occasionally, we shall make an explicit assumption of analyticity.

The following notions are standard. A C^∞ distribution Δ is a mapping assigning to each $x \in M$ a linear subspace $\Delta(x)$ of T_xM, with the property that for all $x \in M$ there exists a neghbourhood U of x and a set of C^∞ vector fields $\{X_i\}_{i\in I}$ defined

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on M such that $\Delta(x)$ is spanned by the set of vectors $\{X_i(x)\}_{i \in I}$. A vector field X belongs to a distributions Δ if $X(x) \in \Delta(x)$ for all $x \in M$. A distribution Δ_1 contains a distribution Δ_2 if $\Delta_1(x) \supseteq \Delta_2(x)$ for all $x \in M$. A distribution Δ is <u>involutive</u> if $X \in \Delta$, $Y \in \Delta$ implies $[X,Y] \in \Delta$. A distribution Δ is <u>nonsingular</u> if the dimension of $\Delta(x)$ is constant over M. An <u>integral submanifold</u> N of Δ is a connected, immersed submanifold $N \subseteq M$ such that, for each $x \in N$, $T_xN = \Delta(x)$. An integral submanifold N of Δ is <u>maximal</u> if every integral submanifold N' of Δ with the property that $N' \supseteq N$ coincides with N. A distribution Δ is <u>integrable</u> if its maximal integral submanifolds define a partition of M.

Let X be a complete vector field on M and let $\phi_t^X(x)$ denote the corresponding <u>flow</u>, i.e. the C^∞ mapping $\mathbb{R} \times M \to M$ with the property that

$$\frac{d}{dt}\phi_t^X(x) = X(\phi_t^X(x))$$

$$\phi_0^X(x) = x$$

For each t, ϕ_t^X defines a diffeomorphism $x \to \phi_t^X(x)$. Let Y be another vector field on M. For all $t \in \mathbb{R}$, there exists a unique vector field, denoted $Ad^tX(Y)$, which is <u>ϕ_t^X-related</u> to Y, i.e. that satisfies the condition

$$(\phi_t^X)_* Ad^tX(Y) = Y \circ \phi_t(x)$$

for all $x \in M$.

The following two Defintions clarify the concepts of "X invariance" for a distribution Δ.

<u>Definition 2.1.</u> A distribution Δ is Ad X invariant (X-invariant in [7]) if for all $Y \in \Delta$ and for all $t \in \mathbb{R}$

$$Ad^tX(Y) \in \Delta$$

A distribution Δ is ad X invariant (X-invariant in [1]) if for all $Y \in \Delta$

$$[X,Y] \in \Delta$$

<u>Remark 2.1.</u> Clearly, a distribution is Ad X-invariant iff for all $t \in \mathbb{R}$ $(\phi_t^X)_*$ maps $\Delta(x)$ into $\Delta(\phi_t^X(x))$, for all $x \in M$ (see [7]).

<u>Remark 2.2.</u> The vector field $Ad^tX(Y)$ can be given a Taylor series expansion via the Campbell-Backer-Hausdorff formula

$$Ad^tX(Y)(x) = \sum_{k=0}^{\infty} \frac{t^k}{k!} ad^kX(Y)(x)$$

where

$$ad^oX(Y) = Y \quad \text{and} \quad ad^kX(Y) = [X, ad^{k-1}X(Y)]$$

Thus, by differentiation, we see that Ad X invariance implies ad X invariance. The converse is clearly true in C^{ω}. In C^{∞}, the two notions are equivalent only under some suitable extra assumption like, e.g., the nonsingularity of Δ.

Remark 2.3. A distribution is involutive iff it is ad X invariant for all $X \in \Delta$.

The basic integrability results are the following

Theorem (Sussmann [7]). A distribution Δ is integrable iff it is Ad X invariant for every $X \in \Delta$.

Corollary (Frobenius) A distribution Δ is integrable only if it is involutive.

Corollary (Frobenius) A nonsingular distribution Δ is integrable iff it is involutive.

Corollary (Hermann-Nagano) A C^{ω} distribution Δ is integrable iff it is involutive.

When referred to a collection of vector fields, like the ones appearing on the right-hand-side of (2.1a), Definition 2.1. is extended as follows (again, see [7] and [1], where the same notions are used, with different notation).

Definition 2.2. A distribution Δ is Ad f invariant (resp. ad f invariant) if for every $u \in \mathbb{R}^m$, Δ is Ad $f(\cdot,u)$ invariant (resp. ad $f(\cdot,u)$ invariant).

If Δ is a given distribution, there is a smallest C^{∞} distribution which contains Δ and is Ad f invariant. This distribution will be denoted with the symbol

$$\langle \mathrm{Ad}\ f | \Delta \rangle$$

It is easy to see [7] that the subspace $\langle \mathrm{Ad}\ f|\Delta \rangle(x)$ of T_xM is the linear hull of all the vectors of the form $(g^{-1})_* X \circ g(x)$, where X is a vector field in Δ and g is a diffeomophism of the form $\Phi_{t_1}^{f(\cdot,u_1)} \circ \Phi^{f(\cdot,u_2)} \circ \ldots \circ \Phi_{t_n}^{f(\cdot,u_n)}$, with $n \in \mathbb{Z}$, $t_i \in \mathbb{R}$, $u_i \in \mathbb{R}^m$.

The symbol

$$\langle \mathrm{ad}\ f | \Delta \rangle$$

shall denote the smallest distribution which contains Δ and is ad f invariant. The subspace $\langle \mathrm{ad}\ f|\Delta \rangle(x)$ of T_xM is the linear hull of all vectors of the form $[f(\cdot,u_1), [f(\cdot,u_2), \ldots, f(\cdot,u_n), X] \ldots]](x)$, where $X \in \Delta$, $n \in \mathbb{Z}$, $u_i \in \mathbb{R}^m$.

Let $R(F)$ denote the distribution spanned by the set of vector fields $\{g_i\}_{i=0,1,\ldots,n}$ and $R(G)$ the distribution spanned by the set of vector fields $\{g_i\}_{i=1,\ldots,n}$. The following distributions are of paramount importance in the study of accessibility properties of control systems =

(2.2) $\langle \mathrm{Ad}\ f|R(F) \rangle$ (resp. $\langle \mathrm{ad}\ f|R(F) \rangle$)

(2.3) $\langle \mathrm{Ad}\ f|R(G) \rangle$ (resp. $\langle \mathrm{ad}\ f|R(G) \rangle$)

The distribution $\langle \text{Ad } f|R(F)\rangle$ is integrable (in C^ω, it may properly contain $\langle \text{ad } f|R(F)\rangle$) and related to the partition of M into equivalence classes with respect to the relation of weak accessibility [7]. In C^ω, the distributions $\langle \text{Ad } f|R(G)\rangle$ and $\langle \text{ad } f|R(G)\rangle$ coincide and are related to the partition of M into equivalence classses with respect to the relation of weak accessibility "in zero units of time" [8].

3. (Ad f,G) INVARIANCE

In this and the following section we assume the dynamics (2.1a) be modified by feedback, i.e. that there exists a pair $\alpha(x),\beta(x)$ of $m\times 1$ and $m\times m$ matrix valued C^∞ functions of x such that

$$u_i = \alpha_i(x) + \sum_{j=1}^{m} \beta_{ij}(x)v_i$$

with $v_i \in \mathbb{R}$. The new dynamics shall be written as

$$\dot{x} = \tilde{f}(x,v) = \tilde{g}_o(x) + \sum_{i=1}^{m} \tilde{g}_i(x)v_i \qquad (3.1)$$

For the sake of compactness, we shall introduce the notations

$$G := \text{row}(g_1,\ldots,g_m)$$
$$F := \text{row}(g_o,g_1,\ldots,g_m)$$
$$\gamma := \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix}$$

and write

$$\tilde{g}_o(x) = g_o(x) + G(x)\alpha(x)$$
$$G(x) = G(x)\beta(x)$$
$$\tilde{F}(x) = F(x)\gamma(x)$$

We say that a distribution Δ separates the controls if there exists a feedback γ with invertible β and a partition of $\beta = (\beta_1\ \beta_2)$ such that

$$\Delta \cap R(G) = R(\tilde{G}_1)$$
$$\Delta \cap R(\tilde{G}_2) = \{0\}$$

where $\tilde{G}_i(x) = G(x)\beta_i(x)$, $i = 1,2$. Such feedback γ is said to be separating.

The following definitions provide nonlinear generalizations of the notion of an (A,B) invariant subspace.

Definition 3.1. A distribution Δ is (Ad f,G) invariant (resp. (ad f,G) invariant) if there exists a feedback γ such that Δ is Ad $\tilde{f}$ invariant (resp. ad $\tilde{f}$ invariant).

Definition 3.2. A distribution Δ is locally (ad f,G) invariant if for every constant $u \in \mathbb{R}^m$

$$X \in \Delta \quad \Rightarrow \quad [f(\cdot,u),X] \in \Delta + R(G)$$

The link between the two Defintions is given by the following Lemma.

Lemma 3.1. If Δ is nonsingular, involutive and separates the controls then the following are equivalent

(a) Δ is locally (ad f,G) invariant,

(b) there exists an open cover $\{U_j\}$ of M and separating feedbacks γ_j defined on U_j such that Δ is (ad $\tilde{f},G$) invariant on U_j under γ_j.

(c) there exists an open cover $\{U_j\}$ and separating feedbacks γ_j on U_j such that Δ is (Ad $\tilde{f},G$) invariant on U_j under γ_j.

Proof. The equivalence of (b) and (c) follows from the nonsingularity of Δ. It is trivial to verify that (b) implies (a). In [4] it is shown that (a) implies (b) using the stronger hypothesis that $\Delta \cap R(G)$ and $R(G)$ are nonsingular. But the proof only uses this to show that Δ separates the controls. Moreover the feedback so constructed is easily seen to be separating. Similar results are found in [5]. ◀

4. CONTROLLABILITY DISTRIBUTIONS

In this section we introduce various nonlinear generalizations of the notion of an (A,B) controllability subspace (see also [6]).

Definition 4.1. A distribution Δ is (Ad f,G) controllable (resp. (ad f,G) controllable) if separates the controls and, for some separating feedback γ,

$$\Delta = \langle \mathrm{Ad}\ \tilde{f} \,|\, R(\tilde{G}_1) \rangle$$

$$(\text{resp.} \quad \Delta = \langle \mathrm{ad}\ \tilde{f} \,|\, R(\tilde{G}_1) \rangle\ )$$

The local version of this definition is based on a generalization of the controllability subspace algorithm, introduced by Wonham [9].

Controllability subdistribution algorithm. Let Δ be a given distribution. It is possible to prove that the class of all distributions $\hat{\Delta}$ satisfying the condition

$$\hat{\Delta} = \Delta \cap ([f,\hat{\Delta}] + R(G)) \tag{4.1}$$

has a unique minimal element, denoted $\Delta^c(\Delta)$. To this end, define a non-decreasing sequence of distributions Δ_k, by $\Delta_o = \{0\}$ and

$$\Delta_k = \Delta \cap ([f,\Delta_{k-1}] + R(G)) \tag{4.2}$$

Clearly, $\Delta_o \subseteq \Delta_1$ and, by induction, it follows that $\Delta_{k-1} \subseteq \Delta_k$. For, if $\Delta_{k-2} \subseteq \Delta_{k-1}$, then $\Delta_{k-1} = \Delta \cap ([f,\Delta_{k-2}] + R(G)) \subseteq \Delta \cap ([f,\Delta_{k-1}] + R(G)) = \Delta_k$. Let

(4.3) $$\Delta^c(\Delta) := \bigcup_{k\geq 0} \Delta_k$$

Clearly, this distribution satisfies (4.1) for, if on some open subset U of M $\Delta_k = \Delta_{k-1}$, then $\Delta_{k+i} = \Delta_k$ on U for all $i\geq 0$. On the other hand, if $\hat{\Delta}$ is any distribution satisfying (4.1), then trivially $\hat{\Delta} \supseteq \Delta_0$ and, by induction, it follows that $\hat{\Delta} \supseteq \Delta_k$ for all $k>0$. Thus the right-hand-side of (4.3) is the unique minimal element of the class of all distributions $\hat{\Delta}$ satisfying the condition (4.1).

Definition 4.2. A distribution Δ is locally (ad f,G) controllable if Δ is locally (ad f,G) invariant and $\Delta^c(\Delta) = \Delta$.

In order to establish a link between the two defintions, we need the following result , which generalizes a property of (A,B) controllability subspaces.

Lemma 4.1. Suppose Δ is (ad f,G) invariant under invertible feedback γ, then

(4.4) $$\Delta^c(\Delta) = \langle \text{ad } \tilde{f} | \Delta \cap R(G) \rangle$$

Proof. We observe, firstly, that from the equality

$$[\tilde{f}(\cdot,v),X] = [g_0,X] + [G,X](\alpha+\beta v) - G\,X(\alpha+\beta v)$$

we can deduce, because of the nonsingularity of β, that

(4.5) $$[\tilde{f},\Delta] + R(G) = [f,\Delta] + R(G)$$

where Δ is a given distribution.

Now we define a nondecreasing sequence of distributions $\bar{\Delta}_k$, by

$$\bar{\Delta}_1 = \Delta \cap R(G)$$
$$\bar{\Delta}_k = [\tilde{f},\bar{\Delta}_{k-1}] + \Delta_1$$

and we show, by induction, that $\bar{\Delta}_k = \Delta_k$, with Δ_k as defined by (4.2). Clearly, $\bar{\Delta}_1 = \Delta_1$. Assume now that $\Delta_{k-1} = \bar{\Delta}_{k-1}$ and observe that, since Δ is ad $\tilde{f}$ invariant, $[\tilde{f},\Delta_{k-1}] \subseteq \Delta$. By (4.5) we have

$$\Delta_k = \Delta \cap ([f,\Delta_{k-1}] + R(G)) = \Delta \cap ([\tilde{f},\Delta_{k-1}] + R(G)) = [\tilde{f},\Delta_{k-1}] + \Delta \cap R(G) =$$
$$= [\tilde{f},\Delta_{k-1}] + \bar{\Delta}_1 = \bar{\Delta}_k$$

Since

$$\langle \text{ad } f | \Delta \cap R(G) \rangle = \bigcup_{k\geq 1} \bar{\Delta}_k$$

the proof is complete. ◀

At this point it is possible to prove a result analogous to Lemma 3.1.

Lemma 4.2. Suppose Δ is nonsingular, involutive and separates the controls;then the following are equivalent:

(a) Δ is locally (ad f,G) controllable.
(b) there exists an open cover $\{U_j\}$ of M and separating feedbacks γ_j defined on U_j such that Δ is (ad f,G) controllable on U_j under γ_j.
(c) there exists an open cover $\{U_j\}$ of M and separating feedbacks γ_j defined on U_j such that Δ is (Ad f,G) controllable on U_j under γ_j.

Proof. (a) ⇒ (b). If Δ is locally (ad f,G) controllable, then

(i) $\qquad [f,\Delta] \subseteq \Delta + R(G)$

(ii) $\qquad \Delta^c(\Delta) = \Delta$

The first, thanks to Lemma 3.1., implies that there exists an open cover $\{U_j\}$ of M and separating (thus nonsingular) feedbacks γ_j defined on U_j such that Δ is (ad f,G) invariant on U_j under γ_j. Thus, by Lemma 4.1., we have that

$$\Delta^c(\Delta) = \langle \mathrm{ad}\ \tilde{f} | \Delta \cap R(G) \rangle = \langle \mathrm{ad}\ \tilde{f} | R(\tilde{G}_1) \rangle$$

on U_j. From this and (ii), the assertion follows.
(b) ⇒ (a). On U_j, under the separting feedback γ_j, we have

$$\Delta = \langle \mathrm{ad}\ \tilde{f} | R(\tilde{G}_1) \rangle = \langle \mathrm{ad}\ \tilde{f} | \Delta \cap R(G) \rangle$$

Δ is (ad $\tilde{f}$,G) invariant under invertible feedback and, thus, by Lemma 4.1., $\Delta = \Delta^c(\Delta)$. Moreover, by Lemma 3.1., Δ is locally (ad f,G) invariant. Thus, Δ is locally (ad f,G) controllable.
(b) ⇔ (c). It is a consequence of nonsingularity. ◀

It is easy to show that the family of all locally (ad f,G) controllable distributions is a semilattice with respect to inclusion and distribution addition. Thus the family of all locally (ad f,G) controllable distributions contained in a given distribution Δ has a unique maximal element. Like in the case of linear systems (see [9]), this can be computed via the controllability subdistribution algorithm, applied to the unique maxial locally (ad f,G) invariant distribution contained in Δ.

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