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On feedback equivalence of nonlinear systems

A. ISIDORI

Istituto di Automatica, Università di Roma, Via Eudossiana, 18, 00184 Roma, Italy

A.J. KRENER

Department of Mathematics, University of California, Davis, CA 95616, USA

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Conditions are derived under which a given system can be modified by means of feedback to obtain a new system whose output is influenced only by a linear and controllable dynamics.

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1. Introduction

In the last years, beginning with the work of Brockett [1], there has been considerable interest in nonlinear feedback theory and some results of earlier work on linear systems have been generalized to nonlinear systems of the form

$$\dot{x} = g_0(x) + g(x)u, \tag{1.1a}$$

$$y = h(x). \tag{1.1b}$$

The growing literature in this field include now results about feedback equivalence to linear systems [1], [2], disturbance decoupling [3]–[7], noninteracting control [3], [8], feedback equivalence to single-input systems [9].

In this paper we give conditions under which there exists a feedback control law of the form $u = \alpha(x) + \beta(x)v$ such that in suitable local coordinates (1.1) becomes

$$\dot{x}_1 = g_0^1(x_1, x_2) + g^1(x_1, x_2)v, \tag{1.2a}$$

$$\dot{x}_2 = A_2 x_2 + B_2 v, \tag{1.2b}$$

$$y = h(x_2), \tag{1.2c}$$

with (A_2, B_2) a controllable pair. In other words, we give conditions under which (1.1) can be modified by means of feedback to obtain a system whose output is influenced only by a linear and controllable dynamics. In doing this we essentially combine the results of Brockett [1] and Jakubczyk and Respondek [2] on feedback equivalence of (1.1a) to a controllable linear dynamics with previous results of ours [3] about observability of systems under feedback.

2. Main result

We consider control systems of the form (1.1), where

$$g(x)u = \sum_{i=1}^{m} g_i(x)u_i$$

and where $x \in \mathbb{R}^n$, $u_i \in \mathbb{R}$, $y \in \mathbb{R}^p$. The vector fields g_i are complete smooth vector fields on $\mathbb{R}^n$ (smooth means either C^∞ or analytic) and $f(0) = 0$; h is a smooth function.

Following Jakubcyk and Respondek [2], we consider the group of transformations generated by

(i) change of coordinates in the state space (i.e. a diffeomorphism φ: $\mathbb{R}^n \to \mathbb{R}^n$ subject to the constraint $\varphi(0) = 0$;

(ii) feedback of the form $u = \alpha(x) + \beta(x)v$, i.e.

$$u_i = \alpha_i(x) + \sum_{j=1}^{m} \beta_{ij}(x)v_j$$

where α_i and β_{ij} are smooth functions, $\alpha(0) = 0$ and β is nonsingular at 0.

A system obtained from (1.1) by means of a transformation belonging to this group is called *feedback equivalent* to (1.1). For further details, the reader is referred to [2], where the authors give necessary and sufficient conditions under which (1.1a) is feedback equivalent to a linear system

$$\dot{x} = Ax + Bv$$

with (A, B) a controllable pair. Their results will be used in the sequel.

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In order to solve the synthesis problem outlined in the introduction, it is needed to make the system as much 'unobservable' as possible. To this end, it is convenient to look at the so-called (f, g) invariant distributions of (1.1a), in particular to the family of those annihilating the output map. A distribution Δ on $\mathbb{R}^n$ is (f, g) invariant if there exists a nonlinear feedback $u = \alpha(x) + \beta(x)v$ such that the modified dynamics

$$\begin{aligned}\dot{x} &= g_0(x) + g(x)\alpha(x) + g(x)\beta(x)v \\ &= \tilde{g}_0(x) + \tilde{g}(x)v \qquad (2.1)\end{aligned}$$

leaves Δ invariant, i.e.

$$[g_0 + g\alpha, \Delta](x) \subseteq \Delta(x), \qquad (2.2a)$$

$$[g\beta, \Delta] \subseteq \Delta(x). \qquad (2.2b)$$

A distribution Δ on $\mathbb{R}^n$ is *locally* (f, g) invariant if

$$[g^0, \Delta](x) \subseteq \Delta(x) + \mathcal{R}(g(x)), \qquad (2.3a)$$

$$[g, \Delta](x) \subseteq \Delta(x) + \mathcal{R}(g(x)), \qquad (2.3b)$$

where

$$\mathcal{R}(g(x)) = \operatorname{span}\{g_1(x),\dots,g_m(x)\}.$$

It easy to see that if β is invertible (2.2) implies (2.3). The converse in true under some mild assumptions on Δ. We say that a distribution Δ *separates the controls* if there exists a feedback pair (α, β) with invertible β and a partition of $\beta = (\beta_1 \cup \beta_2)$ such that

$$\Delta(x) \cap \mathcal{R}(g(x)) = \mathcal{R}(g(x)\beta_1(x)),$$

$$\Delta(x) \cap \mathcal{R}(g(x)\beta_2(x)) = \{0\}.$$

Such feedback is said to be separating. A distribution is *nonsingular* on some open subset U of $\mathbb{R}^n$ if the dimension of $\Delta(x)$ is constant over U. The following lemma provides the bridge between (2.3) and (2.2) [4], [5], [6].

Lemma 2.1. *If Δ is nonsingular on U, involutive and separates the controls, then the following are equivalent:*

(i) *Δ is locally (f, g) invariant,*

(ii) *locally around each $x \in U$ there exists a separating feedback pair (α, β) such that (2.2) are satisfied.*

A distribution Δ annihilates the output map h if all tangent vectors $v \in \Delta(x)$ annihilate $\mathrm{d}h(x)$, i.e. are such that $\mathrm{d}h(x)v = 0$ (where $\mathrm{d}h(x)$ denotes the differential of h at x). In view of (2.3) it is seen that the family of all locally (f, g) invariant distributions annihilating the output map is closed under addition and thus has a unique maximal element, denoted Δ_h^*. It is easy to see that Δ_h^* is involutive [3], thus if it is also nonsingular on some subset U of $\mathbb{R}^n$ and separates the controls, then Δ_h^* becomes invariant locally around each $x \in U$ under a suitable feedback.

The relevance of Δ_h^* to our present synthesis problem is due to the following property, shown in [3]. Let (α, β) be a feedback pair such that (2.2) hold (on some open subset U), i.e. such that the modified dynamics (2.1) satisfy

$$[\tilde{g}_0, \Delta_h^*](x) \subseteq \Delta_h^*(x), \qquad (2.4a)$$

$$[\tilde{g}, \Delta_h^*](x) \subseteq \Delta_h^*(x), \qquad (2.4b)$$

for all $x \in U$. Let Δ_h^* be nonsingular on U. Then locally around each point in U it is possible to choose coordinates $x = (x_1, x_2)$ in such a way that (2.1) becomes

$$\dot{x}_1 = \tilde{g}_0^1(x_1, x_2) + \tilde{g}^1(x_1, x_2)v, \qquad (2.5a)$$

$$\dot{x}_2 = \tilde{g}_0^2(x_2) + \tilde{g}^2(x_2)v, \qquad (2.5b)$$

$$y = h(x_2). \qquad (2.5c)$$

Since Δ_h^* is maximal, then the dimension of x_1 is maximal over all such decompositions possible. In this way the system is made as much 'unobservable' as possible by feedback.

On the basis of those concepts, it is possible to state the main result of this section.

Theorem 2.1. *Let Δ_h^* be the maximal locally (f, g) invariant distribution annihilating the output map. Assume that Δ_h^* separates the controls and is nonsingular on some neighbourhood U of 0. Let μ denote the codimension of Δ_h^* on U. Assume that the following conditions are satisfied on U:*

(a) *For all $0 \leqslant k \leqslant h \leqslant \mu - 1$ and $1 \leqslant i, j \leqslant m$, there exist smooth functions c_{rs} such that*

$$\left[\mathrm{ad}_{g_0}^h g_i, \mathrm{ad}_{g_0}^k g_j\right] = \sum_{\substack{1 \leqslant s \leqslant m \\ 0 \leqslant r \leqslant h}} c_{rs}\, \mathrm{ad}_{g_0}^r g_s \mod \Delta_h^*.$$

(b) $\dim\left[\operatorname{span}\{\mathrm{ad}_{g_0}^r g(x) | 0 \leqslant r | k\} + \Delta_h^*(x)\right] = r_k(x) = \text{const}.$

(c) $\dim\left[\operatorname{span}\{\mathrm{ad}_{g_0}^r g(x) | 0 \leqslant r \leqslant \mu - 1\} + \Delta_h^*(x)\right] = n.$

Then the system (1.1) *is locally feedback equivalent to a system of the form* (1.2), *with* (A_2, B_2) *a controllable pair.*

Remark 2.1. The above conditions are exactly the ones found by Jakubczyk and Respondek, but this time they are required to hold 'mod Δ_h^*'.

The proof of this theorem is essentially based on an invariance property of (a), (b), (c) under feedback. We recall first that if Δ is an involutive and nonsingular distribution of dimension d defined on some open subset U then, by Frobenius' theorem, locally around each point in U it is possible choose local coordinates $x = (x_1, x_2)$ such that

$$\Delta = \operatorname{span}\{\partial/\partial x_{11}, \ldots, \partial/\partial x_{1d}\},$$

where $x_{11}, \ldots, x_{1d}$ denotes components of x_1. The $(n-d)$-dimensional submanifold N passing through (x_1^0, x_2^0) and given by $x_1 = x_1^0$ is a submanifold which is said *complementary* to Δ. If f is a vector field on U which leaves Δ invariant, i.e. such that

$$[f, \Delta] \subseteq \Delta,$$

then locally f can be projected onto N, i.e. there exists a vector field $\hat{f}$ defined on N such that

$$(\mathrm{d}\pi) f = \hat{f} \circ \pi$$

where π denotes the projection $(x_1, x_2) \to (x_1^0, x_2)$. Based on these concepts, we can state the following lemma.

Lemma 2.2. *Assume* g_0, g *and* Δ_h^* *satisfy the hypotheses of the above theorem and let* $\tilde{g}_0$ *and* $\tilde{g}$ *be such that* (2.4) *is satisfied. Let* N *be a complementary submanifold of* Δ_h^* *passing through* $x_0 \in V$ *and let* $\hat{g}_0$ *and* $\hat{g}$ *denote local projections of* $\tilde{g}_0$ *and* $\tilde{g}$ *onto* N.

Then the following conditions are satisfied:

(a′) *For all* $0 \leqslant k \leqslant h \leqslant \mu^{-1}$ *and* $1 \leqslant i, j \leqslant m$ *there exist smooth functions* $\hat{c}_{rs}$ *defined on* N *such that*

$$\left[\mathrm{ad}^h_{\hat{g}_0}\hat{g}_i, \mathrm{ad}^k_{\hat{g}_0}\hat{g}_j\right] = \sum_{\substack{1 \leqslant s \leqslant m \\ 0 \leqslant r \leqslant h}} \hat{c}_{rs}\mathrm{ad}^r_{\hat{g}_0}\hat{g}_s.$$

(b′) $\dim\left[\operatorname{span}\left\{\mathrm{ad}^r_{\hat{g}_0}\hat{g}(x) | 0 \leqslant r \leqslant k\right\}\right] = r_k(x) = \text{const.}$

(c′) $\dim\left[\operatorname{span}\left\{\mathrm{ad}^r_{\hat{g}_0}\hat{g}(x) | 0 \leqslant r \leqslant \mu - 1\right\}\right] = \mu.$

Proof of the lemma. It is shown firstly that the iterated use of the hypothesis (a′) makes it possible to write

$$\mathrm{ad}^k_{\tilde{g}_0} = \left(\mathrm{ad}^k_{g_0}g\right)\beta + \sum_{i=0}^{k-1}\left(\mathrm{ad}^i_{g_0}g\right)\gamma_i^k \mod \Delta_h^* \quad (2.6)$$

for all $0 \leqslant k \leqslant \mu - 1$, where γ_i^k are suitable $m \times m$ matrices of smooth functions. This is obviously true for $k = 0$. By induction, assuming (2.6) true for a given k, we have

$$\mathrm{ad}^{k+1}_{\tilde{g}_0}\tilde{g} = \left[\tilde{g}_0, \left(\mathrm{ad}^k_{g_0}g\right)\beta + \sum_{i=0}^{k-1}\left(\mathrm{ad}^i_{g_0}g\right)\gamma_i^k + v\right],$$

where v is a smooth vector field in Δ_h^*,

$$= \left[g_0 + g\alpha, \left(\mathrm{ad}^k_{g_0}g\right)\beta + \sum_{i=0}^{k-1}\left(\mathrm{ad}^i_{g_0}g\right)\gamma_i^k\right] \mod \Delta_h^*,$$

because $[\tilde{g}_0, v] \in \Delta_h^*$,

$$= \left(\mathrm{ad}^{k+1}_{g_0}g\right)\beta + \sum_{i=0}^{k}\left(\mathrm{ad}^i_{g_0}g\right)\gamma_i^{k+1} \mod \Delta_h^*,$$

due to the hypotesis (a′), for suitable γ_i^{k+1}.

Since β is nonsingular, (2.6) can be inverted, thus obtaining

$$\mathrm{ad}^k_{g_0}g = \left(\mathrm{ad}^k_{\tilde{g}_0}\tilde{g}\right)\beta^{-1} + \sum_{i=0}^{k-1}\left(\mathrm{ad}^i_{\tilde{g}_0}\tilde{g}\right)\delta_i^k \mod \Delta_h^* \quad (2.7)$$

for all $0 \leqslant k \leqslant \mu - 1$, where δ_i^k are suitable $m \times m$ matrices of smooth functions.

Now, it is shown that, as consequence of (a) and (2.7), for all $0 \leqslant k \leqslant h \leqslant \mu - 1$ and $1 \leqslant i, j \leqslant m$ there exist smooth functions $\tilde{c}_{rs}$ such that

$$\left[\mathrm{ad}^h_{\tilde{g}_0}\tilde{g}_i, \mathrm{ad}^k_{\tilde{g}_0}\tilde{g}_j\right] = \sum_{\substack{1 \leqslant s \leqslant m \\ 0 \leqslant r \leqslant h}} \tilde{c}_{rs}\mathrm{ad}^r_{\tilde{g}_0}\tilde{g}_s \mod \Delta_h^*. \quad (2.8)$$

To this end, one proceeds again by induction. Replacing the right-hand-side of (2.7) on both sides of the equality written under (a), one typically gets equalities written (with an abuse of notation) in the form

$$\left[\left(\mathrm{ad}^k_{\tilde{g}_0}\tilde{g}\right)\beta^{-1} + \sum_{i=0}^{k-1}\left(\mathrm{ad}^i_{\tilde{g}_0}\tilde{g}\right)\delta_i^k + v,\right.$$

$$\left(\mathrm{ad}^h_{\tilde g_0}\tilde g\right)\beta^{-1}+\sum_{i=0}^{h-1}\left(\mathrm{ad}^i_{\tilde g_0}\tilde g\right)\delta_i^h+w\Bigg]$$
$$=\text{combination of elements of the form } \mathrm{ad}^i_{\tilde g_0}\tilde g, \text{ with } i\leq\max(k,h),$$

which hold mod Δ_h^*. The vector fields v and w are in Δ_h^*. Since Δ_h^* is involutive and invariant also under $\mathrm{ad}^i_{\tilde g_0}\tilde g$ (use the Jacobi identity) they can be omitted. From the resulting equality, an induction argument yields the desired result.

Thus, it has been proved that the equality written under (a) remains valid when g_0 and g are replaced with $\tilde g_0$ and $\tilde g$. The same can be said about (b) and (c) because, thanks to (2.6),

$$\mathrm{span}\left\{\mathrm{ad}^r_{g_0}g(x)\,|\,0\leq r\leq k\right\}+\Delta_h^*(x)$$
$$=\mathrm{span}\left\{\mathrm{ad}^r_{\tilde g_0}\tilde g(x)\,|\,0\leq r\leq k\right\}+\Delta_h^*(x).$$

At this point it is sufficient to project onto N all the relations obtained hitherto (i.e. (a), (b), (c) with $\tilde g_0$ and $\tilde g$ in the place of g_0 and g) in order to get (a′), (b′), (c′). □

The proof of Theorem 2.1 is now completed in this way. Since Δ_h^* is locally (f,g) invariant, by Lemma 2.1 one can find locally around 0 a feedback pair (α,β) such that (2.2) are satisfied. By proper choice of local coordinates, (2.1) becomes (2.5), where $\tilde g_0^2$ and $\tilde g^2$ represent local projections of $\tilde g_0$ and $\tilde g$ onto the complementary submanifold of Δ_h^* given by $x_1=0$. Moreover, the feedback pair (α,β) can always be chosen in such a way that $\alpha(0)=0$ and, thus, $\tilde g_0^2(0)=0$. Thanks to Lemma 2.2, the vector fields $\tilde g_0^2$ and $\tilde g^2$ satisfy all the conditions, given in [3], under which (2.5b) is feedback equivalent to (1.2b). Thus, the theorem is proved. □

References

[1] R.W. Brockett, Feedback invariants for nonlinear systems, *Proc. VII IFAC Congress, Helsinki* (1978) pp. 1115–1120.

[2] B. Jakubczyk and W. Respondek, On linearization of control systems, *Bull. Acad. Polonaise Sci. Sér. Sci. Math.* 28 (1980) 517–522.

[3] A. Isidori, A.J. Krener, C. Gori-Giorgi and S. Monaco, Nonlinear decoupling via feedback: a differential geometric approach, *IEEE Trans. Aut. Control* 26 (1981) 331–345.

[4] R.M. Hirschorn, (A,B) invariant distributions and the disturbance decoupling of nonlinear systems, *SIAM J. Control Optim.* 17 (1981) 1–19.

[5] H. Nijmeijer, Controlled invariance for affine control systems, *Int. J. Control* 34 (1981) 825–833.

[6] A. Isidori, A.J. Krener, C. Gori-Giorgi and S. Monaco, Locally (f,g) invariant distributions, *Systems Control Lett.* 1 (1981) 12–15.

[7] H. Nijmeijer and A.J. van der Schaft, Controlled invariance for nonlinear control systems, to appear in *IEEE Trans. Aut. Control.*

[8] D. Claude, Decoupling of nonlinear systems, *Systems Control Lett.* 1 (1982) 242–248.

[9] J. Tsinias and N. Kalouptsidis, Transforming a controllable multiinput nonlinear system to a single input controllable system by feedback, *Systems Control Lett.* 1 (1981) 173–178.