Linearization by Output Injection

and Nonlinear Observers

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Background

1. Because linear differential equations and linear control systems are so much simpler than their nonlinear counterparts, a natural question to ask is when one of the latter is equivalent to one of the former under some group of transformations such as change of state coordinates.

The earliest work in this area goes back to Poincaré [1] who gave a sufficient condition for linearizeability of a single vector field around a critical point by change of state coordinates. Later Hermann [2] gave a formal argument for the linearizability of a finite dimensional semisimple Lie algebra of vector fields around a common critical point by change of state coordinates. Guillemin and Sternberg [3] gave a rigorous proof of this result. Sedwick and Elliott [4] considered the same question and showed that the assumption of semisimplicity could be replaced by that of transitivity.

Krener [5] discussed the question of when a nonlinear system of the form

$$\dot{\xi} = f(\xi) + \sum_{j=1}^{m} g^j(\xi)u_j \qquad \xi \in \mathbb{R}^n \tag{1.1}$$

can be transformed locally into a linear system

$$\dot{x} = Ax + Bu \tag{1.2}$$

by a change of state coordinates,

$$x = x(\xi). \tag{1.3}$$

A necessary and sufficient condition for the existence such a transformation is that the set of vector fields $\{ad^k(f)g^j : k \geq 0,\ j=1,\ldots,m\}$ spans an n dimensional abelian Lie algebra. It is not hard to see that if this hypothesis is satisfied then after reordering $g^1,\ldots,g^m$ if necessary there exists integers

$\varkappa_1 \geq \varkappa_2 \geq \cdots \geq \varkappa_m \geq 0$, $\varkappa_1 + \ldots + \varkappa_m = n$, such that the set $\{ad^k(f)g^j\colon 0 \leq k < \varkappa_j, j=1,\ldots,m\}$ is a basis for the Lie algebra. If the $(\varkappa_1,\ldots,\varkappa_m)$ is smallest such m-tuple with respect to the lexographic ordering, then these are called the controllability indices (or Kronecker indices) of (1.2).

Brockett [6] introduced a wider class of transformations, allowing not only change of state cooridnates but also certain forms of state feedback. The full state feedback problem was considered independently by Hunt and Su [7] and Jakubczyk and Respondek [8]. They gave equivalent necessary and sufficient conditions for there to exist locally a transformation of the form (1.3) and a state feedback of the form

$$(1.4) \qquad u = u(\xi,v) = \alpha(\xi) + \beta(\xi)v$$

carrying (1) into the linear system

$$(1.5) \qquad \dot{x} = Ax + Bv$$

The Hunt and Su conditions are a bit easier to state. There must exist controllability indices $\varkappa_1 \geq \varkappa_2 \geq \cdots \geq \varkappa_m$, $\varkappa_1 + \ldots + \varkappa_m = n$ such that (after reordering $g^1,\ldots,g^m$).

(a) the vector fields $\{ad^k(f)g^j\colon 0 \leq k < \varkappa_j, j=1,\ldots,m\}$ are linearly independent at each x.

(b) for each $j=1,\ldots,m$, the vector fields $\{ad^k(f)g^j\colon 0 \leq k < \varkappa_j - 1\}$ are involutive.

(c) for each $j=1,\ldots,m$, the vector field $ad^{\varkappa_j}(f)g^j$ is linear combination (over the C^∞ functions) of $\{ad^k(f)g^i\colon (k,i) < (\varkappa_j,j)\}$ (this is the lexographic ordering, i.e., $k < \varkappa_j$ or ($k = \varkappa_j$ and $i < j$)).

Meyer and Cicolani [9] have used this result and related work of their own to design advanced flight control systems for high performance aircraft.

Isidori and Krener [10] considered the problems of partial linearization of the dynamics, using transformations of the form (1.3) and (1.4).

2. Observers and Output Injection

Consider a uncontrolled dynamics

$$\dot{x} = Ax \qquad x \in \mathbb{R}^n \tag{2.1}$$

with observations

$$y = Cx \qquad y \in \mathbb{R}^p \tag{2.2}$$

The dual problem to that of control is that of approximation or estimation. From $y(s)$, $s \leq t$ we would like to obtain an estimate of $x(t)$. In a stochastic setting the well-known Kalman-Bucy filter supplies the answer. In a deterministic setting the Luenberger observer is the appropriate tool. One chooses an $n \times p$ matrix G and sets up a differential equation for an approximation $z(t)$ of $x(t)$,

$$\dot{z} = Az - G(y-Cz). \tag{2.3}$$

If $e = x-z$ then

$$\dot{e} = (A+GC)e. \tag{2.4}$$

If G is choosen so that the spectrum of $A+GC$ is in the left half plane then $e(t) \to 0$. The farther left the spectrum is, the faster the convergence. It is well-known that a necessary and sufficient condition to be able to arbitrarily fix the spectrum (up to invariance under complex conjugation) of $A+GC$ is that (C,A) be an observable pair, i.e., the (p.n) x n matrix

$$(2.5)\qquad \begin{pmatrix} C \\ CA \\ \cdot \\ \cdot \\ CA^{n-1} \end{pmatrix}$$

be of full rank n.

This introduces the concept of linear output injection which is the dual of linear state feedback.

Two systems (2.1,2) and

$$(2.6)\qquad \dot{x} = \tilde{A}x$$

$$(2.7)\qquad y = \tilde{C}x$$

are equivalent under linear output injection if $C = \tilde{C}$ and there exists G such that $A = A+GC$. If (2.1,2) models a physical plant then in general output injection is not physically realizable on the plant. But this is not important because we can realize output injection on the error dynamics (2.3) of our approximation (2.4).

Now suppose we are trying to observe the state of a nonlinear system,

$$(2.8)\qquad \dot{\xi} = f(\xi)$$

$$(2.9)\qquad y = h(\xi),$$

in general this is a difficult task. However, it is conceivable that (2.8,9) is the result of applying nonlinear output injection to a linear system (2.1,2) yielding the system

$$(2.10)\qquad \dot{x} = Ax + \varphi(y)$$

(2.11) $$y = Cx$$

followed by a nonlinear change of coordinates $\xi = \xi(x)$.

If this is the case and we can find $\varphi(y)$ and $\xi(x)$ then we can construct an observer for (2.10,11) almost as easily as for (2.1,2). Let the approximation z satisfy

(2.12) $$\dot{z} = Az - G(y-Cz) + \varphi(y)$$

then the error $e = x-z$ satisfies

(2.13) $$\dot{e} = (A + GC)e$$

as before. Therefore we would like to solve the following.

Problem Given (2.8,9) when does there exist a change of coordinates $\xi = \xi(x)$ which carries (2.8,9) into (2.10,11).

In the next section we shall answer this question locally for the scalar case, $y \in \mathbb{R}$, a complete answer will appear elsewhere.

The localness of the solution is not a problem at least theoretically for once can always choose the gain G so that the error e goes to zero arbitrarily fast and hence before ξ leaves the locality.

Notice that we only apply output injection (whether linear or nonlinear) to a linear system. In particular nonlinear output injection is not a group of transformations.

3. Linearization by Nonlinear Output Injection. We assume that the nonlinear system (2.8,9) satisfies the following in some neighborhood $\mathcal{U}$ of ξ^o, the point of interest.

Observability Assumption. The one forms $L_f^k(dh)(\xi)$, $k=0,\ldots,n-1$ are linearly independent for $\xi \in \mathcal{U}$.

Notation. $L_f(dh)$ denotes Lie differentiation of the one form dh by the vector field f, and dh is the gradient of h.

$$dh(\xi) = \left(\frac{\partial h}{\partial \xi_1}(\xi), \ldots \frac{\partial h}{\partial \xi_n}(\xi)\right)$$

$$L_f(dh) = f'(\xi)\frac{\partial^2 h(\xi)}{\partial \xi^2} + \frac{\partial h(\xi)}{\partial \xi}\frac{\partial f}{\partial \xi}(\xi)$$

Lie differentiation by f also acts on functions h

$$L_f(h)(\xi) = dh(\xi)f(\xi)$$

and these are related by

$$L_f(dh)(\xi) = d(L_f(h))(\xi).$$

Proposition 1. The nonlinear system (2.8,9) is locally equivalent to a linear system of the form (2.1,2) under a change of state coordinates $x = x(\xi)$ where $x(\xi^o) = 0$ iff $f(\xi^o) = 0$, $h(\xi^o) = 0$ and $L_f^n(dh)$ is a $\mathbb{R}$-linear combination of $L_f^k(dh)$ for $k = 0,\ldots,n-1$.

Proof. Suppose the change of coordinates exists, then in the x coordinates

$$L_f^k(dh) = CA^k.$$

By the Cayley Hamilton theorem there exists $a_1,\ldots,a_n \in \mathbb{R}$ so that

$$CA^n = \sum_{k=1}^{n} a_k CA^{k-1}$$

hence

$$L_f^n(dh) = \Sigma\, a_k\, L_f^{k-1}(dh). \tag{3.1}$$

On the other hand suppose (3.1) is satisfied, define new coordinates $x_1, \ldots, x_n$ by

$$x_k(\xi) = L_f^{k-1}(h)(\xi). \tag{3.2}$$

By the observability assumption this is a valid change of coordinates and $x_k(\xi^o) = 0$. Then

$$\dot{x}_k = L_f(L_f^{k-1}(h)) = \begin{cases} x_{k+1} & \text{if } k < n \\ \sum_{j=1}^{n} a_j L_f^{j-1}(h) = \Sigma a_j x_j & \text{if } k = n \end{cases}$$

$$y = x_1$$

So (3.2) transforms (2.8,9) into (2.1,2) where

$$A = \begin{pmatrix} 0 & 1 & & 0 \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ 0 & 0 & & 1 \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} \qquad C = (1\ 0\ \ldots\ 0)$$

Q.E.D.

One can recast the above result in a way that is reminiscent of the work of Guillemin and Sternberg [3].

Proposition 2. The nonlinear system (2.8,9) is locally equivalent to a linear system (2.1,2) under change of state coordinates $x = x(\xi)$ where $x(\xi^o) = 0$ iff $f(\xi^o) = 0$, $h(\xi^o) = 0$ and the vector field $g(\xi)$ defined by

$$L_g L_f^k(h) = \begin{cases} 0, & 0 \leq k < n-1 \\ 1 & k = n-1 \end{cases} \tag{3.3}$$

satisfies

$$[g, ad^k(f)g] = 0 \qquad k=1,3,\ldots,2n-1. \tag{3.4}$$

Proof. Suppose such a g exists. By (3.3) and the observability assumption it follows that $g(\xi), [f,g](\xi), \ldots, ad^{n-1}(f)g(\xi)$ spans $\mathbb{R}^n$. The Jacobi identity for the Lie bracket of vector fields is

$$\left[g^1[g^2,g^3]\right] = \left[[g^1,g^2]g^3\right] + \left[g^2[g^1,g^3]\right].$$

From this it follows that (3.4) is equivalent to

$$\left[ad^k(f)g,\ ad^\ell(f)g\right] = 0 \qquad 0 \leq k + \ell \leq 2n \ . \tag{3.5}$$

By a standard result we can choose local coordinates x such that $x(\xi^o) = 0$ and for $k = 0,\ldots,n-1$

$$\frac{\partial}{\partial x_{n-k}} = (-1)^k ad^k(f)g \ .$$

By (3.3)

$$\frac{\partial h}{\partial x_{n-k}} = \sum_{i=0}^{k} (-1)^i \binom{k}{i} L_f^i L_g L_f^{k-i}(h)$$

$$= \begin{cases} 0 & 0 \le k < n-1 \\ 1 & k = n-1. \end{cases}$$

This implies that h is linear in the x coordinates,

(3.6) $$y = Cx = (10\ldots0) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

To compute the local coordinate description of the vector field f, suppose $f = \sum_{i=1}^{n} f_i(x) \frac{\partial}{\partial x_i}$ and $0 \le k < n-1$. By definition of the coordinates

$$\left[f, \frac{\partial}{\partial x_{n-k}}\right] = \left[f, (-1)^k ad^k(f)g\right] = (-1)^k ad^{k+1}(f)g = -\frac{\partial}{\partial x_{n-k-1}}$$

and in local coordinates

$$\left[f, \frac{\partial}{\partial x_{n-k}}\right] = -\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_{n-k}} \frac{\partial}{\partial x_i}$$

so

$$\frac{\partial f_i}{\partial x_{n-k}} = \begin{cases} 1 & i = n-k-1 \\ 0 & 0 \text{ otherwise} \end{cases} .$$

For $k,\ell = 0,\ldots,n-1$ by (3.5)

(3.7) $$\left[\frac{\partial}{\partial x_{n-k}} [f, \frac{\partial}{\partial x_{n-\ell}}]\right] = \left[(-1)^k ad^k(f)g, (-1)^\ell ad^{\ell+1}(f)g\right] = 0$$

and in local coordinates

$$\left[\frac{\partial}{\partial x_{n-k}}\left[f, \frac{\partial}{\partial x_{n-\ell}}\right]\right] = - \sum_{i=1}^{n} \frac{\partial^2 f_i}{\partial x_{n-k} \partial x_{n-\ell}}$$

so

$$\frac{\partial^2 f_i}{\partial x_{n-k} \partial x_{n-\ell}} = 0 \quad .$$

Since $f_i(\xi^o) = 0$, it is a linear function of x,

$$(3.8) \qquad \begin{pmatrix} f_1(x) \\ \cdot \\ \cdot \\ \cdot \\ f_n(x) \end{pmatrix} = Ax = \begin{pmatrix} a_1 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & & \cdot & & & \\ \cdot & & & \cdot & & \\ \cdot & & & & \cdot & \\ a_{n-1} & 0 & \cdot & \cdot & \cdot & 1 \\ a_n & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$$

On the other hand suppose (2.8,9) is equivalent to a linear system under change of state coordinates. Let $B \in \mathbb{R}^{n \times 1}$ satisfy

$$CA^jB = \begin{cases} 0 & 0 \leq j < n-1 \\ 1 & j = m-1 \end{cases} \quad .$$

Let g be the vector field given in x coordinates by the vector B, then it is straightforward to verify (3.3) and (3.4).

Q.E.D.

The linear system (3.6,8) is said to be in <u>observable companion</u> form. By linearly injecting the output into the dynamics, we replace A by $A+GC$. The latter matrix has the same form as A but by appropriate choice of G the first column can be set to zero (or anything else).

<u>Proposition 3.</u> The nonlinear system (2.8,9) is locally equivalent to a linear system with nonlinear output injection (2.10,11) under change of state coordinates $x = x(\xi)$ where $x(\xi^o) = 0$ if $h(\xi^o) = 0$ and the vector field $g(\xi)$ defined by

$$(3.9) \qquad L_g L_f^k(h) = \begin{cases} 0 & 0 \leq k < n-1 \\ 1 & k = n-1 \end{cases}$$

satisfies

$$(3.10) \qquad [g, ad^k(f)g] = 0 \qquad k=1,3,\ldots,2n-3.$$

<u>Proof.</u> As might be expected the proof is very similar to that of Proposition 2. Suppose such a g exists, then (3.5) holds for $0 \leq k+\ell \leq 2n-2$. This allows us to set up coordinates as before with y given by (3.6). Once again for $0 \leq k < n-1$

$$\frac{\partial f_i}{\partial x_{n-k}} = \begin{cases} 1 & i = n-k-1 \\ 0 & \text{otherwise} \end{cases} .$$

Therefore f_i is the sum of a linear function of $x_2,\ldots,x_n$ and and arbitrary function of $x_1 = y$.

$$\begin{pmatrix} f_1(x) \\ \cdot \\ \cdot \\ \cdot \\ f_n(x) \end{pmatrix} = Ax+\varphi(y) = \begin{pmatrix} 0 & 1 & \cdot & \cdot & \cdot & 0 \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & 0 & \cdot & \cdot & \cdot & 1 \\ 0 & 0 & & & & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} + \begin{pmatrix} \varphi_1(y) \\ \cdot \\ \cdot \\ \cdot \\ \varphi_n(y) \end{pmatrix} .$$

On the other hand suppose (2.8,9) is equivalent to a linear system with nonlinear output injection (2.10,11) under change of state coordinates. Let $B \in \mathbb{R}^{n \times 1}$ satisfy

$$CA^jB = \begin{cases} 0 & 0 \leq j < n-1 \\ 1 & j = n-1 \end{cases} .$$

If we define $y_j = L_f^{j-1}(h)$ then it straightforward calculation to see that

$$y_{j+1} = CA^j x + \psi_j(y_1,\ldots,y_j)$$

for some function ψ_j. Viewing B as a vector field,

$$L_B(h) = L_B(y_1) = CB = 0$$

and by induction for $j=1,\ldots,n-1$

$$L_B L_f^j(h) = L_B(y_{j+1}) = CA^jB + \sum_{i=1}^{j} \frac{\partial\psi_j}{\partial y_i} L_B(y_i)$$

$$= CA^jB = \begin{cases} 0 & j < n-1 \\ 1 & j = n-1 \end{cases}$$

This shows that if g is the vector given in x coordinates by B then g satisties (3.9).

Next we show by induction that in x coordinates

$$ad^j(f)g = \begin{cases} (-1)^j A^k B & k < n \\ (-1)^n A^n B + \frac{\partial\varphi}{\partial y} & k = n \end{cases} .$$

Clearly this holds for $j=0$, suppose it holds for $j-1$. Then

$$ad^j(f)g = [Ax + \varphi(y), (-1)^{j-1}A^{j-1}B]$$

$$= (-1)^j (A^jB + \frac{\partial\varphi}{\partial y} CA^{j-1}B)$$

$$= \begin{cases} (-1)^j A^j B & j < n \\ (-1)^n A^n B + \frac{\partial\varphi}{\partial y} & j = n \end{cases} .$$

From this it follows that (3.10) holds.

Q.E.D.

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