ON THE EXISTENCE OF GLOBALLY (f,g)-INVARIANT DISTRIBUTIONS

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Abstract. (f,g)-invariant distributions have begun to play a role in nonlinear control similar to the role played by (A,B)-invariant subspaces in the theory of linear systems. The importance of (A,B)-invariant subspaces lies in their connection with the existence of feedback laws possessing various special, desirable properties and in this regard the generalization of this connection to the nonlinear theory, i.e. the existence of globally (f,g)-invariant distributions, remains open. We give an example of a locally (f,g)-invariant distribution, with simply-connected leaves, which is not globally (f,g)-invariant. An intermediate kind of distribution - the "input-insensitive" distributions introduced by Hirschorn - are studied. We show that a stronger form of the Bott Vanishing Theory must hold if a locally (f,g)-invariant distribution can be rendered "input-insensitive" by feedback, yielding topological obstructions to input-insensitivity. Next, for input-insensitive distributions we derive explicit, complete topological obstructions in $H^1_{dR}(M)$ to global (f,g)-invariance. This generalizes the local sufficiency criterion given by Hirschorn's Lemma to the global setting as well as various assertions in the literature concerning the sufficiency of the 1-connectedness of M.

1. Introduction

A basic problem in control theory, which illustrates one of the many uses of feedback, is the problem of disturbance decoupling. Suppose, for simplicity, that we consider a nonlinear control system of the form:

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i(t)g_i(x) + \sum_{j=1}^{r} w_j(t)p_j(x) \qquad (1.1a)$$

$$y = h(x) \qquad (1.1b)$$

$$z = k(x) \qquad (1.1c)$$

with state $x \varepsilon M$, where M is a smooth n-manifold on which is defined smooth vector fields

$$f,\ g_i,\ p_j\ \varepsilon\ \Gamma(M,TM) \qquad (1.2)$$

and smooth vector-valued functions

$$h : M \to \mathbb{R}^p, \qquad k : M \to \mathbb{R}^q \qquad (1.3)$$

Taking the p_j's and k to be zero, we recover a nonlinear control system, with control functions $u_i(t)$:

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i(t)g_i(x), \quad y = h(x) \quad . \qquad (1.4)$$

In the present situation, the input w models a disturbance entering the system through the "channels" $p_1,\ldots,p_r$ and the output z represents a vector quantity which we wish to insulate from the disturbance channels by use of (static) feedback:

$$u = \alpha(x) + v = \alpha(x) + \sum_{i=1}^{m} g_i(x)\beta_i(x) \quad . \qquad (1.5)$$

Thus, the (static) disturbance decoupling problem is solvable locally provided after implementing the feedback (1.5),(1.1a-c) becomes, in local coordinates,

$$\begin{aligned} \dot{x}_1 &= \tilde{f}_1(x_1) + \tilde{g}_1(x_1)v \\ \dot{x}_2 &= \tilde{f}_2(x_1,x_2) + \tilde{g}_2(x_1,x_2)v + p_2(x_1,x_2)w \end{aligned} \qquad (1.1a)'$$

$$z = k(x_1) \qquad (1.c)'$$

where

$$\tilde{f}(x) = f(x) + g(x)\alpha(x), \ \tilde{g}(x) = g(x)\beta(x) \qquad (1.6)$$

In the linear case, the solution (in closed form) of the disturbance decoupling problem was one of the elegant applications of the theory of (A,B)-invariant subspaces, initiated by Basile and Marro [1], [2] and fashioned into a powerful tool by Wonham and Morse [11], [14], and [15]. There has recently been a development of the nonlinear generalization, (f,g)-invariant distributions - of this theory (independently in [5] and in [7], [8]), which under certain regularity assumptions gives a local solution to problems such as the disturbance decoupling stated above. In this paper, we aim to initiate the study of the global aspects of this theory.

More precisely, suppose M is a connected n-manifold and $\Delta \subset TM$ is a distribution on M of constant rank d, i.e. the subspace $\Delta_x \subset T_xM$ satisfies

$$d = \dim(\Delta_x), \qquad \forall x \in M \ .$$

Following [7] we define, with respect to the control system (1.4),

<u>Definition 1.1</u>. Δ is a locally (f,g)-invariant distribution provided

(i) $[f(x),\Delta_x] \subseteq \Delta_x + \mathrm{span}\{g_1(x),\dots,g_n(x)\}, \qquad \forall\, x \in M$

(ii) $[g_i(x),\Delta_x] \subseteq \Delta_x + \mathrm{span}_{C^\infty(M)}\{g_1(x),\dots,g_n(x)\}, \quad \forall\, x \in M$

There is another, stronger, definition of local (f,g)-invariance - due to R. Hirschorn - which we will discuss in Section 3. As in the linear case, (f,g) - invariant distributions are of interest because of their intimate connection with feedback laws satisfying certain desirable properties.

<u>Definition 1.2</u>. Δ is globally (f,g)-invariant provided there exists a globally defined feedback law (1.5) such that the closed-loop system (1.6) satisfies

(i) $[\tilde{f},\Delta_x] \subseteq \Delta_x$ (1.7a)

(ii) $[\tilde{g}_i,\Delta_x] \subseteq \Delta_x, \quad$ for $i = 1,\dots,m.$ (1.7b)

$\forall\, x \in M.$

Clearly, from the form of the feedback (1.5), any globally (f,g)-invariant distribution is locally (f,g)-invariant.

Moreover, one has a local converse (see Section 3, for Hirschorn's converse for the stronger definition).

<u>Lemma 1.3.</u> ([7], [8], [12]) Suppose Δ is a locally (f,g)-invariant distribution, $\overline{\Delta}$ its involutive closure, and the dimensions of $\overline{\Delta}$, $\mathrm{span}\{g_1(x),\dots,g_m(x)\}$, and $\overline{\Delta} \cap \mathrm{sp}\{g_1(x),\dots,g_m(x)\}$ are constant.

Then locally around each x there exists $\alpha(x)$ and an invertible $\beta(x)$ satisfying (1.7a-b) in a neighborhood of x.

In this language then, the conditions for local disturbance decoupling are (roughly) that there should exist an (f,g)-invariant Δ such that

$$\text{span}\{(p_1(x),\ldots,p_r(x)\} \subset \Delta_x \subset \text{Ker } dk(x) \ .$$

(Compare (1.1a-c)' and Lemma 1.3.)

In section 2, we give an example of a distribution Δ on a 3-manifold which is locally, but not globally, (f,g)-invariant. In fact, we give necessary and sufficient conditions for global (f,g)-invariance of an integrable, regular, codimension 1 locally (f,g)-invariant distribution on a compact manifold. As a consequence, we derive necessary conditions on the Stiefel-Whitney numbers of the manifold and distribution for global (f,g)-invariance, yielding our counter-example as a corollary. An example of a codimension 2, globally (f,g)-invariant distribution on $SL(2, \mathbb{R})$ is also given.

In section 3, 23 consider the "input-insensitive" distributions, introduced by Hirschorn [5] in his study of the disturbance decoupling problem. A generalized form of the Bott Vanishing Theorem is derived as a necessary condition that a locally (f,g)-invariant distribution be input-insensitive after feedback. In the simplest case compatible with generalized Bott Vanishing, we describe the complete obstructions to global (f,g)-invariance of an input-insensitive distribution, generalizing the local assertions given by Hirschorn's Lemma. These obstructions lie in $H^1_{dk}(M)$ and therefore, as a special case, one knows the global form of Hirschorn's Lemma - in our setting - for all input-insensitive distributions when M is simply - connected.

Finally, in section 4 we study a particular disturbance decoupling problem, following the algorithm given in [7] for finding a globally (f,g)-invariant distribution. We find that the algorithm breaks down on a submanifold which is Poincaré dual to the obstructions uncovered in Section 2, giving another interpretation to the topological obstructions uncovered here.

2. Global (f,g)-Invariance

We begin this section with 2 examples which give the flavor of the general problem. All manifolds, distributions, fields and functions are C^∞.

Example 2.1. Let $M = SL(2, \mathbb{R})$ and consider the system

$$\dot{x} = f(x) + ug(x)$$

where $f = H$, and $g = X_+$ in the standard Cartan basis for $s\ell_2(\mathbb{R}) \subset \Gamma(M,TM)$. Setting

$$\Delta = sp_{C^\infty(M)}\{H\}$$

one sees that Δ is locally (f,g)-invariant, since

(i) $[f(x),\Delta_x] = \{0\}$

(ii) $[g(x),\Delta_x] \subset sp\{g(x)\}$

Moreover, $\Delta = \overline{\Delta}$ and the hypotheses of Lemma 1.3 are satisfied for all $x \in M$. On the other hand, (f,g)-invariance will follow provided we can find a function $\beta(x)$ such that

$$[\beta(x)g(x),\Delta_x] \subset \Delta_x, \quad \text{for all } x \in M \tag{2.1}$$

Expanding (2.1) we have

$$[\beta(x)g(x),H(x)] = -2\beta(x)g(x)-H(\beta)(x)g(x) \ .$$

Since X_+ and H are everywhere independent, (2.1) implies

$$H(\beta) = 2\beta \ ; \tag{2.2}$$

that is, β is an eigenfunction for the differential operator H with eigenvalue 2. Thus, Δ is globally {f,g} invariant if, and only if, the eigenvalue problem (2.2) has a solution. That (2.2) can be solved, for a smooth function on $SL(2, \mathbb{R})$, follows from the classical representation theory of $s\ell_2(\mathbb{R})$.

$SL(2, \mathbb{R})$ acts transitively on $\mathbb{R}^2-\{0\}$ by linear transformation; thus, it suffices to solve (2.2) on $\mathbb{R}^2-\{0\}$ for the linear vector field $\tilde{H}$ induced on $\mathbb{R}^2$ by H. Now, as a differential operator

$$\tilde{H} : (\mathbb{R}^2)^* \to (\mathbb{R}^2)^*$$

inducing the action

$$\tilde{H} : \mathcal{P}^k(\mathbb{R}^2)^* \to \mathcal{P}^k(\mathbb{R}^2)^*$$

on homogeneous polynomials of degree k. Taking k = 2, it follows ([6])

that

$$\mathrm{spec}(\tilde{H}) = \{2,0,-2\}$$

as a differential operator on quadratic forms. Indeed,

$$\tilde{H}(x^2) = 2x^2 \quad . \tag{2.3}$$

Pulling back x^2 to $SL(2,\mathbb{R})$ along the projection

$$\pi : SL(2,\mathbb{R}) \to \mathbb{R}^2-\{0\} \simeq SL(2,\mathbb{R})/\left\{\begin{bmatrix}1 & b\\ 0 & 1\end{bmatrix}\right\}$$

we obtain $\beta = \pi^* x^2 \in C^\infty(SL(2,\mathbb{R}))$ satisfying

$$H\beta = 2\beta \quad . \tag{2.2)'$$

Remark. The solution in (2.2)' is, however, unsatisfactory from a control theoretic perspective since the new control vector field

$$\tilde{g}(x) = g(x)\beta(x)$$

vanishes wherever $\beta(x)$ vanishes. In general, i.e. for $m \geq 1$, we ask that $\beta(x)$ be invertible for all $x \in M$, so that we can maintain "full open-loop control", see [7].

Returning to our example, one knows that any β obtained from the finite dimensional representations on $\mathcal{P}^{2k}(\mathbb{R}^2)^*$ and satisfying (2.2) must vanish on $SL(2,\mathbb{R})$. Note, however, that (2.2) is the infinitesimal form of

$$\beta(h_\theta x) = e^{i2\theta}\beta(x) \tag{2.2)'$$

for $h_\theta \in SO(2) \subset SL(2,\mathbb{R})$. Consider, then, the representation ρ_2 in the principal series of $SL(2,\mathbb{R})$ associated to the character $e^{i2\theta}$ on $SO(2)$. This representation space is the space of L^2-sections of the line bundle L on the Poincare upper half plane P ([4])

$$SL(2,\mathbb{R}) \to SL(2,\mathbb{R})/SO(2) = P \tag{2.4}$$

associated to (2.4) by the character $e^{i2\theta}$. Since $L \to P$ is trivializable, we can choose a nowhere vanishing section $\overline{\beta} : P \to L$ giving rise to a nowhere vanishing function

$$\beta : SL(2,\mathbb{R}) \to \mathbb{R}$$

satisfying (2.2)' and hence (2.2).

Therefore, Δ is globally (f,g)-invariant.

Example 2.2. Let $M = U(2)/O(2)$ and consider the distribution Δ tangent to the foliation of M by the fibering

$$\det^2 : M \to S^1 . \qquad (2.5)$$

Now, (2.5) is a fiber bundle with fiber 2 copies of $SU(2)/SO(2;\mathbb{R})$ identified; that is, (2.5) is a fiber bundle over S^1 with fiber S^2 glued by the antipodal map. Thus,

(i) $(\det^2)^{-1}(\theta)$ is the leaf of a codimension 1 distribution on M, with tangent bundle $\Delta \subset TM$;

(ii) $(\det^2)^* d\theta$ is a 1-form ω on M, with $\ker \omega = \Delta$.

Choosing $f \in \Delta$ and $g = X$, where X satisfies

$$\langle X,\omega\rangle = 1$$

one obtains a control system

$$\dot{x} = f(x) + u(t)g(x)$$

and a distribution Δ which is locally (f,g)-invariant, since

$$\Delta_x + sp\{g(x)\} = T_x M .$$

Moreover, all the hypothesis of Lemma 1.3 hold, so that, for all x, locally a feedback law $\beta(x)$ exists satisfying

$$[\beta(y)g(y),\Delta_y] \subset \Delta_y \quad \text{for y in a neighborhood of x} .$$

We claim, however, that no globally defined, nowhere zero, $\beta(x)$ exists such that

$$[\beta(x)g(x),\Delta_x] \subset \Delta_x$$

holds for all $x \in M$. This claim follows from a stronger assertion for codimension 1 integrable distribution on compact manifolds which are regular, in the sense that the leaf space M/Δ is a smooth manifold such that

$$\pi : M \to M/\Delta$$

is a smooth submersion. We may take the drift-term f to be zero.

Proposition 2.3. Suppose Δ is a regular integrable codimension 1 distribution on a compact manifold M and that $g \in \Gamma(M,TM)$ is transverse to the leaves of Δ. If there exists a nowhere vanishing $\beta(x)$ satisfying

$$[\beta g,\Delta] \subset \Delta , \qquad (2.6)$$

then the leaf-space map $\pi : M \to M/\Delta$ is a trivial fiber bundle; i.e. in particular $M \simeq S^1 \times L$, for L a leaf of Δ. Conversely, if π is a trivial fiber bundle and L is simply-connected, then such a β exists.

Proof. Since Δ is regular and M is compact, the leaf space is S^1, and we have

$$\pi : M \to S^1, \quad \text{with} \quad \pi^{-1}(\theta) \simeq L$$

and g (or $-g$) satisfies

$$\langle \pi^* d\theta, g\rangle > 0 \ . \tag{2.7}$$

If β exists satisfying (2.6), then integrating $\dot{x} = \beta(x)g(x)$ at time T with initial data $x \in \pi^{-1}(\theta_1)$ gives a diffeomorphism Φ_T such that

$$\Phi_T : \pi^{-1}(\theta_1) \to \pi^{-1}(\theta_2), \quad \text{for some } \theta_2 \in S^1$$

since $\beta(x)g(x) \neq 0$. There exists a minimum time $T > 0$ such that

$$\Phi_T : \pi^{-1}(\theta_1) \simeq \pi^{-1}(\theta_1)$$

and $\pi : M \to S^1$ is formed by gluing $L = \pi^{-1}(\theta_1)$ over S^1 by Φ_T. Since Φ_T is homotopic to Φ_o, it follows that $\pi : M \to S^1$ is trivial; in particular, $M \simeq L \times S^1$.

The converse follows from the proof of Lemma 1.3 offered in [7], where it is noted that if

$$\pi : M \to M/\Delta$$

is a smooth fiber bundle with local cross-section N_α, then a solution $\beta_\alpha(x)$ to (2.6) can be found in a neighborhood of N_α by specifying the initial data $\beta_\alpha(x_o)$, $x_o \in N_\alpha$, to a Cauchy problem on this neighborhood. If a global cross-section N exists and the leaves are simply-connected, then a well-defined global solution $\beta(x)$ can be defined by the monodromy principle. Q.E.D.

Our claim now follows from

Lemma 2.4. $U(2)/O(2)$ is not diffeomorphic to $S^2 \times S^1$.

Proof. We will show that $U(2)/O(2)$ and $S^2 \times S^1$ have different DeRhan Cohomology spaces; in particular, we will prove that $U(2)/O(2)$ is not orientable. By the Kunneth formula,

$$H^*_{dk}(S^2 \times S^1) \simeq H^*_{dk}(S^2) \otimes H^*_{dR}(S^1)$$

Thus,

$$\beta_i(S^2 \times S^1) = 1, \quad \text{for } i = 0,\dots,3 \quad .$$

As for U(2)/O(2), by the Leray-Serre Theorem the fibration (2.5) induces a spectral sequence converging to $H^*_{dR}(U(2)/O(2))$ and since the base is 1-dimensional, this sequence abuts at the E^2 term, where

$$E^2_{p,q} \simeq H^q(S^1; \mathscr{H}^p(s^2))$$

In particular,

$$H^*_{dR}(U(2)/O(2)) \simeq H^*_{dR}(S^1) \otimes I$$

where $I \subset H^*_{dR}(S^2)$ is the subring invariant under the antipodal map α. Since α is orientation reversing,

$$I = H^o_{dR}(S^2) \simeq \mathbb{R}$$

so that

$$H^*_{dR}(U(2)/O(2)) \simeq H^*_{dR}(S^1) \ ,$$

yielding

$$\beta_i(U(2)/O(2)) = 1, \text{ for } i = 0,1, \quad \beta_i(U(2)/O(2)) = 0, \quad i \geqslant 2 \ .$$

Q.E.D.

<u>Remark</u>. From the long exact sequence of (2.5), it follows that

$$\pi_i(U(2)/O(2)) \simeq \pi_i(U(2)) \simeq \pi_i(S^3) \qquad i \geqslant 3$$

$$\pi_2(U(2)/O(2)) \simeq \pi_1(U(2)/O(2)) \simeq Z \quad .$$

On the other hand

$$\pi_i(S^2 \times S) \simeq \pi_i(S^2) \qquad i \geqslant 2$$

$$\pi_1(S^2 \times S^1) \simeq \pi_1(S^1) \quad .$$

Thus, by the Hopf isomorphism, $\pi_i(S^3) \simeq \pi_i(S^2)$ $i \geqslant 3$, it follows that U(2)/O(2) and $S^2 \times S^1$ have the same homotopy groups. This accounts for our choice of proof, from which it also follows that

$$H^*(U(2)/O(2); \mathbb{Z}_p) \simeq H^*(S^2 \times S^1; \mathbb{Z}_p)$$

if, and only if, p = 2.

One can rephrase the necessary condition in Proposition 2.3 in terms of Stiefel-Whitney classes. If $\beta(x)$ exists, then

$$H^*(M;\mathbb{Z}_2) \simeq H^*(L;\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(S^1;\mathbb{Z}_2) \tag{2.8}$$

and via (2.8) the total Stiefel-Whitney class (see e.g. [10])

$$w(L) = 1 + w_1(TL) + \dots + w_{n-1}(TL)$$

sits in $H^*(M;Z_2)$. In these terms,

Corollary 2.5. If Δ is an integrable codimension 1, regular distribution on a compact n-manifold M which is globally g-invariant, then

$$w(M) - w(L) = 0 .$$

In the example given above,

$$H^*(U(2)/O(2);\mathbb{Z}_2) \simeq \mathbb{Z}_2[w_1,w_2]/(w_1^2,w_2^2)$$

with deg $w_i = i$ and an elementary calculation gives

$$w(M) = 1 + w_1 \quad .$$

On the other hand,

$$w(L) = 1 \in H^*(S^2;\mathbb{Z}_2) \quad .$$

In this sense

$$w_1 \in H^*(U(2)/O(2);\ \mathbb{Z}_2) \tag{2.9}$$

is an obstruction to global {g}-invariance.

Corollary 2.6. With Δ,f,g, and M = U(2)/O(2) as above, Δ is a locally (f,g)-invariant distribution, satisfying the regularity condition of Lemma 1.2, which is not globally (f,g)-invariant.

3. Input Insensitive Distributions.

We now present a stronger version of Definition 1.1, originally considered by Hirschorn in his analysis of the disturbance decoupling problem [5] and more recently dubbed "input insensitive distributions" by Nijmeier and van der Schaft. The notation is as in §1.

Definition 3.1. ([5]) A distribution Δ is input insensitive provided

(i) $[f(x),\Delta_x] \subseteq \Delta_x + \text{span}\{g_1(x),\dots,g_m(x)\}$ (3.1a)

(ii) $[g_i(x),\Delta_x] \subseteq \Delta_x$, for $i = 1,\dots,m$ (3.1b)

for all $x \in M$.

One of the fundamental results in [5] is then:

Hirschorn's Lemma: 3.2. Suppose Δ is an involutive, input insensitive distribution and that the dimensions of Δ_x, $\text{span}\{g_1(x),\ldots,g_m(x)\}$, and $\Delta_x \cap \text{span}\{g_1(x),\ldots,g_m(x)\}$ are constant over M. Then, in a neighborhood of each $x \in M$ there exists $\alpha_i(x)$ such that

$$[f(x) + \sum_{i=1}^{m} g_i(x)\alpha_i(x),\Delta_x] \subseteq \Delta_x \tag{3.2}$$

We investigate the following strategy for proving that locally (f,g)-invariant distribution is globally (f,g)-invariant: first, find a feedback law rendering Δ input-insensitive and, second, to give explicitly a complete obstruction to global invariance for input insensitive distribution, generalizing Hirschorn's Lemma.

Remark. Note that Example 2.2 shows that not every locally (f,g)-invariant distribution can be made input insensitive by feedback.

We now consider the question: For an involutive locally (f,g)-invariant distribution Δ, does there exist (invertible) $\beta(x)$ such that Δ is input-insensitive with respect to $g(x)\beta(x)$?

Set $\nu = TM/\Delta$ to be the normal bundle, of fiber dimension $r = n-d$, of the distribution Δ and consider - under the regularity conditions assumed in Lemmas 1.3 - 3.2 - the subbundle $Q \subset \nu$ where Q is defined as

$$Q = \Delta + \text{span}\{g_1,\ldots,g_m\}/\Delta \quad .$$

Denote the fiber diemsnion of Q by q. Following Bott [3], we may define a "basic connection"

$$\nabla : \Gamma(M,TM) \times \Gamma(M,\nu) \to \Gamma(M,\nu) \tag{3.3}$$

as follows. If $X \in \Gamma(M,\Delta)$, $Z \in \Gamma(M,\nu)$

$$\nabla(X,Z) = \pi[X,\tilde{Z}] \tag{3.4}$$

where

$$\pi : TM \to \nu$$

is the canonical projection and

$$\pi(\tilde{Z}) = Z.$$

Choosing a Riemannian metric on TM, we can extend (3.4) to a connection defined for all $X \in \Gamma(M,TM)$. Computing, as one may, the Pontrjagin classes $p_{4k}(\nu)$ in terms of the curvature of such a basic connection ∇, Bott proves that integrability of Δ implies that ∇ is "flat" in the direction $\Delta \subset TM$, i.e.

Bott Vanishing Theorem, 3.3. Δ integrable implies

$$\mathrm{Pont}^i(\nu) = 0 \quad \text{for } i > 2r . \tag{3.5}$$

This asserts the vanishing of all products of the classes $p_{4k_1}(\nu),\ldots,p_{4k_\ell}(\nu)$ of total degree $\geq 2r$.

Note, however, that if there exists an invertible $\beta(x)$ such that (2.1) is satisfied, then ∇ may be defined - as a flat connection - in the additional directions $g(x)\beta(x)$, giving additional vanishing in (3.5). More precisely, suppose $\Delta_x \cap \mathrm{span}\{g(x)\beta(x)\}$ is constant rank in $x \in M$, and

$$q' = \dim(\Delta_x + \mathrm{span}\{g(x)\beta(x)\}/\Delta_x) \tag{3.6}$$

where

$$[g(x)\beta(x),\Delta_x] \subset \Delta_x$$

is satisfied. Then, we have

Theorem 3.4. If $\beta(x)$ renders Δ input-insensitive, then

$$\mathrm{Pont}^i(\nu) = \{0\} \quad , \text{ for } i > 2r-2q' \tag{3.7}$$

In particular, if (x) has full rank, we have

$$\mathrm{Pont}^i(\nu) = \{0\} \ , \quad \text{for } i > 2r-2q \ . \tag{3.8}$$

Theorem 3.4 can, of course, be generalized to give obstructions to global (f,g)-invariance, as was shown in [9].

In Example 2.2, ν is trivial; thus, (3.8) is not the only obstruction to input-insensitivity, or to global (f,g)-invariance.

We now turn to the simplest case compatible with the vanishing conditions (3.8); viz. we assume that Δ is input-insensitive and that

$$Q = \Delta + \mathrm{span}\{g_1, \ldots, g_m\}/\Delta$$

is a trivial q-bundle. We assume, by reordering, that

$\pi(g_1),\ldots,\pi(g_q)$ span Q, where

$$\pi : \Delta + \mathrm{span}\{g_1,\ldots,g_m\} \to Q$$

is the canonical projection. Under these conditions:

Theorem 3.5. To f, $g_1,\ldots,g_q$ and Δ as above one can canonically assign globally defined closed 1-forms $\omega_1,\ldots,\omega_q$ on M such that is globally (f,g)-invariant if, and only if, each ω_i is exact on M. In particular, if $H_1(M;\mathbb{Q}) = \{0\}$ then each input-insensitive distribution, with Q trivial, is globally (f,g)-invariant.

Thus, for simply-connected manifolds as well as for manifolds - such as SO(3) or $\mathrm{Grass}_{\mathbb{R}}(m,m+p)$ - with $H_1(M;\mathbb{Z})$ all torsion, input insensitivity and Q trivial imply global (f,g)-invariance.

Proof. Our starting point will be analogous to the proof [5] of Hirschorn's Lemma.

For $x \in M$ choose a local connecting frame $X_1 = \frac{\partial}{\partial x_1},\ldots, X_d = \frac{\partial}{\partial x_d}$ for Δ in a neighborhood of x - as we can by Frobenius' Theorem - and consider the equation

$$[f,X_i] = \sum_{j=1}^{d} h_j^i X_j + \sum_{j=1}^{q} k_j^i g_j \tag{3.9}$$

which hold since Δ is locally (f,g)-invariant (compare [5]). Note ([5], p. 6) that

$$X_i(k_j^\ell) = X_\ell(k_j^i) \tag{3.10}$$

so that the locally defined 1-forms

$$\omega_1 = \sum_{i=1}^{d} g_1^i \, dx \ , \ \ldots, \ \omega_q = \sum_{i=1}^{d} g_q^i dx_i \tag{3.11}$$

are closed.

If there exists locally defined functions $\alpha_1,\ldots,\alpha_m$ such that

$$[f(x) + \sum_{j=1}^{m} \alpha_j(x) g_j(x), X_i] \subseteq \Delta_x \tag{3.12}$$

then by equating coefficients mod Δ in (3.9) one has for each $i=1,\ldots,d$

$$\Pi\left(\sum_{j=1}^{q} (X_i\alpha_j - k_j^i) g_j\right) = 0 \quad \text{in } Q$$

That is,

$$X_i \alpha_j = k_j^i, \qquad i = 1,\ldots,d; \quad j = 1,\ldots,q \quad . \qquad (3.13)$$

(3.10) is a necessary condition for (3.13), while a sufficient condition is that the forms $\omega_1,\ldots,\omega_q$ be exact; which, of course, holds locally by Poincaré's Lemma.

Suppose we choose a different connecting frame

$$\tilde{X}_i = \frac{\partial}{\partial x_i} \quad , \qquad i = 1,\ldots,d$$

for Δ in a neighborhood of x. This leads, possibly, to a new choice of 1-forms

$$\tilde{\omega}_j = \sum_{i=1}^{d} \tilde{g}_j^i d\tilde{x}_i \qquad (3.14)$$

If $\tilde{x} = f(x)$ and $\mathrm{Jac}(f) = [f_{ij}]$, then an elementary computation shows that

$$\tilde{\omega}_k = [\tilde{g}_k^1,\ldots,\tilde{g}_k^d] \begin{bmatrix} d\tilde{x}_1 \\ \vdots \\ d\tilde{x}_d \end{bmatrix}$$

$$= [g_k^1,\ldots,g_k^d][f_{ij}]^t \begin{bmatrix} d\tilde{x}_1 \\ \vdots \\ d\tilde{x}_d \end{bmatrix}$$

$$= [g_k^1,\ldots,g_k^d] \begin{bmatrix} dx_1 \\ \vdots \\ dx_d \end{bmatrix} = \omega_k \quad .$$

Therefore, the 1-forms $\omega_1,\ldots,\omega_q$ are independent of the choice of connecting frame $X_1,\ldots,X_d$ for Δ; i.e., $\omega_1,\ldots,\omega_q$ are globally defined closed 1-forms which are canonically associated for f,g, and Δ and a choice of frame for Q. Moreover, a feedback law $\alpha(x)$ satisfying (3.12) exists if, and only if, each ω_i is exact - or if, and only if, the classes

$$[\omega_1],\ldots,[\omega_q] \in H^1_{dR}(M) \simeq H^1(M; \mathbb{R})$$

vanish. Q.E.D.

4. A Distrubance Decoupling Problem.

As in Example 2.2, we take $M = U(2)/O(2)$ and consider the control system

$$\dot{x} = f(x) + u(t)g(x) + \sum_{i=1} w_i(t)p_i(x) \tag{4.1a}$$

$$y = h(x) \tag{4.1b}$$

$$z = k(x) \tag{4.1c}$$

where f, q, p, h, and k are defined as follows. Setting $\omega = (\det^2)^* d\theta$ where

$$\det^2 : U(2)/O(2) \to S^1 \ , \tag{4.2}$$

choose $f \in \Gamma(M,TM)$ so that $f \in \ker \omega$; choose g so that $\langle\omega,g\rangle > 0$; choose $p_1,\ldots,p_4$ so that $\text{span}\{p_1,\ldots,p_4\} = \ker \omega$; define

$$k(x) = \text{Re}(\det^2(x));$$

and let $h \in C^\infty(M)$ be arbitrary.

As in §2, $\Delta = \ker \omega$ is locally (f,g)-invariant and, moreover, satisfies

$$\text{span}\{p(x)\} = \Delta_x \subset \ker dk(x) \ . \tag{4.3}$$

Therefore ([7]), the disturbance decoupling problem is solvable locally on M.

Moreover (since ker dk(x) is the maximal (f,g)-invariant distribution contained in ker dk(x)), one can show from the result obtained in §2 that there does not exist any nowhere vanishing $\beta(x)$ which decouples the output z from the disturbance channels $p_1,\ldots,p_4$.

One can also see this by following the algorithm in [7] for a $\beta(x)$ which solves the disturbance decoupling problem. Form the "1×1 matrix"

$$a(x) = ad_g(k) \tag{4.4}$$

and consider the equation ([7], 4.14b)

$$a(x)\beta(x) = \text{constant} \neq 0 \tag{4.5}$$

Now (4.5) is solvable except for those x such that $ad_g(k)(x) = 0$; i.e. except for x lying on the submanifolds

$$S_+^2 = (\det^2)^{-1}(1), \quad S_-^2 = (\det^2)^{-1}(-1) \tag{4.6}$$

Alternatively, the algorithm (4.5) "breaks down" on the submanifolds (4.6), which also satisfy

$$0 \neq [S_+^2] = [S_-^2] \in H_2(U(2)/O(2); \mathbb{Z}_2) \tag{4.7}$$

by the argument given in the proof of Lemma 2.4. Indeed, in light of the fact that one cannot globally disturbance decouple the system (4.1a-c) it is interesting to note that

$$[S_+^2]^* = [S_-^2]^* = \omega_1(M) \in H^1(U(2)/O(2); \mathbb{Z}_2) .$$

That is, the submanifolds where the algorithm [7] fails are Poincaré dual to the topological obstructions given in Corollary 2.5. We expect to have more to say about this in a later paper.

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Bibliography

[1] G. Basile and G. Marro, "L'invarianze respetto ai disturbi studianta nello spazio deli stati," in Proc. 70th Ann. Meeting Elec. Eng. Assoc. Italy, 1969, Paper 1.4.01.

[2] G. Basile and G. Marro, "Controlled and conditioned invariant subspaces in linear system theory," J. Opt. Theory and Appl. 3 (1969) 306-315.

[3] R. Bott, "On a topological obstruction to integrability," Proc. Symp. Pure Math. 16 (1970) 127-131, Providence, R.I.

[4] J. Dieudonné, Special Functions and Linear Representations of Lie Groups, CBMS Regional Conf. Proc. No. 42, Providence, R.I. 1980.

[5] R. Hirschorn, "(A,B)-invariant distributions and disturbance decoupling of nonlinear systems," SIAM J. Control and Opt. 19 (1981), 1-19.

[6] J. Humphreys, Introduction to Lie Algebras and Representation Theory, Springer-Verlag, Berlin-Heidelberg - New York, 1972.

[7] A. Isidori, A.J. Krener, C. Gori-Giorgi and S. Monaco, "Nonlinear decoupling via feedback: a differential geometric approach," IEEE Trans. Aut. Control AC-26 (1981) 331-345.

[8] A. Isidori, A.J. Krener, C. Gori-Giorgi and S. Monaco, "Locally (f,g) invariant distributions," Systems and Control Letters 1 (1981) 12-15.

[9] A.J. Krener, "(f,g)-invariant distributions, connections, and Pontryagin classes," Proc. of the 20th IEEE Conf. on Decision and Control, San Diego, CA, 1981, 1322-1325.

[10] J.W. Milnor and J.D. Stasheff, _Characteristic Classes_, Annals of Math. Studies No. 76, Princeton, N.J. 1974.

[11] A.S. Morse and W.M. Wonham, "Status of noninteracting control," _IEEE Trans. Aut. Control_ 16 (1971) 568-581.

[12] H. Nijmeijer, "Controlled invariance for affine control systems," _Int. J. Control_ 34 (1981) 825-833.

[13] H. Nijmeijer and A.J. van der Schaft, "Controlled invariance for nonlinear control systems," _IEEE Trans. Aut. Control_

[14] W.M. Wonham, _Linear Multivariable Control: A Geometric Approach 2nd Ed._ Springer-Verlag, N.Y. 1979.

[15] W.M. Wonham and A.S. Morse, Decoupling and pole-assignment in linear multivariable systems: a geometric approach," _SIAM J. Control_ 8 (1970) 1-18.