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NEW APPROACHES TO THE DESIGN OF NONLINEAR COMPENSATORS*

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1. INTRODUCTION. The standard approach to compensator design for a nonlinear plant is to assume that it will operate over a range where a linear approximation is valid. This can be done either in the state space around a nominal state and control or in the frequency domain over a nominal range of frequencies and amplitudes. A linear model is chosen and a compensator is designed for that model. Among its other goals, the compensator should keep everything close to the nominal thereby preserving the accuracy of the linear approximation.

There are plants for which the above procedure does not work. They have significant nonlinearities over any range of values in which one could reasonably hope to operate. Typical among these are robots and high performance aircraft such as helicopters. We have chosen these two examples for in both cases the dynamics are reasonably well-known. The problems are principally those of control rather than identification.

In this paper we describe a new approach to compensator design for plants with accurate but nonlinear state space models. It is based on the method of approximately linearizing transformations. It uses the same linear model to describe the nonlinear system, but the transformations between the input, output and state variables of the two models are no longer essentially the identity transformation. Instead they are nonlinear transformations which attempt to linearize the nonlinear model.

As one can see from the references, many researchers have contributed to the development of this approach. I would especially like to mention Dr. George Meyer of NASA-Ames Research Center who has been a driving force behind this program. My thinking about nonlinear systems has been strongly influenced by our discussions.

2. A SIMPLIFIED VIEW OF LINEAR STATE SPACE DESIGN. Consider the linear plant

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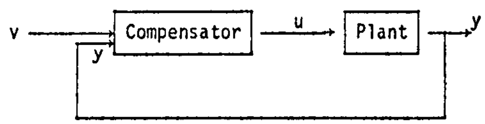
$$\dot{x} = Ax + Bu \tag{2.1a}$$

$$y = Cx \tag{2.1b}$$

$$x(o) \approx x^o = 0$$

We seek to design a compensator of the same linear form which processes the open loop input v and the output y of (2.1) to obtain a suitable input $u(t)$.



Typical goals of compensation are one or more of the following:

(i) stability of the external and internal behavior of the plant,

(ii) the ability to track reference signals from a certain class, i.e., $v(t) = r(t)$ and $e(t) = y(t) - r(t) \to 0$ as t increases,

(iii) modification of the input-output behavior $v(t) \to y(t)$ so that it approximates that of some ideal system,

(iv) reduction of the sensitivity of the plant's input-output behavior to unmeasurable plant parameter variations,

(v) reduction of the sensitivity of the plant's input-output behavior to external noisy (unpredictable) signals and to unmodeled and unmodellable elements of the plant,

(vi) simultaneous on-line identification of the plant dynamics and construction of a suitable compensator to achieve some of the above goals (adaptive control).

For linear, stationary, finite dimensional systems, the theory of compensator design to achieve the first three goals is quite satisfactory. Unfortunately, this is not as true for (iv) and even less so for (v) but the renewed interest in frequency domain techniques hold great promise. As for the last goal, there has been exciting progress in adaptive control over the past several years but considerable work remains to be done.

In this paper we will focus on the design of compensators to achieve stability. This is where the new approach can most easily be employed. To simplify the exposition we shall restrict ourselves to single input/single output systems but almost all that we shall discuss can be extended to multivariable systems.

The standard approach to linear compensator design is to split the problem into two parts. The first problem is to choose a state feedback $u = Fx + Gv$ which stabilizes (2.1a). This assumes that the state x of (2.1) is directly measurable. In other words, after feedback the dynamics will be given by

$$\dot{x} = (A + BF)x + BGv \tag{2.2}$$

and we wish to choose F and G so that spectrum of $(A + BF)$ lies in the left half of the complex plane and the columns of BG have certain properties. It is well-known that the controllability of the pair (A,B), i.e.,

$$\mathrm{rank}(B : \dots : A^{n-1}B) = n \tag{2.3}$$

is a necessary and sufficient condition for the spectral assignability (up to invariance under complex conjugation) of $A + BF$.

If (A,B) is controllable then there are several approaches to finding a suitable F. For example, one can choose F by transforming (2.3) into a standard form called controller form by Kailath [6]. Alternatively one can choose F using the theory of linear quadratic optimal control, as espoused by Athans [1]. In this brief treatment we shall not discuss why and how to choose G.

Having found F and G we still have not solved the stabilizing compensator problem because y and not x is directly measurable. Therefore we must estimate the current value of x from past values of the measurable quantities y and u. Let $\hat{x}(t)$ denote the estimate of $x(t)$ and suppose $\hat{x}(t)$ evolves according to the equation

$$\dot{\hat{x}} = (A + KC)\hat{x} - Ky + Bu \tag{2.4}$$

The error $\tilde{x}(t) = x(t) - \hat{x}(t)$ of this estimate satisfies

$$\dot{\tilde{x}} = (A + KC)\tilde{x} \quad . \tag{2.5}$$

If we choose K so that the spectrum of $(A + KC)$ is in the left half of the complex plane then $\tilde{x}(t) \to 0$ as $t \to \infty$. The spectral assignability (up to complex conjugation) of $A + KC$ is equivalent to the observability of the pair (C,A), i.e.,

$$\mathrm{rank}\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = n \tag{2.6}$$

If (C,A) is observable then there are several approaches to finding a suitable G, including transforming (2.1) to observer from [6] and reformulating the estimation problem as a Kalman filtering problem [1].

Having chosen F, G, and K, the stabilizing compensator is given by (2.4) and

$$u = F\hat{x} + Gv \quad . \tag{2.7}$$

The combined dynamics of x and $\hat{x}$ is most conveniently expressed in $(x,\tilde{x})$ coordinates

$$\begin{pmatrix}\dot{x}\\ \dot{\tilde{x}}\end{pmatrix} = \begin{pmatrix}A + BF & -BF\\ 0 & A + KC\end{pmatrix}\begin{pmatrix}x\\ \tilde{x}\end{pmatrix} + \begin{pmatrix}BG\\ 0\end{pmatrix}v \tag{2.8}$$

where the stability is apparent.

Notice that spectrum of the overall system is just the disjoint union of the spectra of $(A + BF)$ and $(A + GC)$. This fact we call the *spectral separation principle*. It is the key to the two-part approach to nonlinear design. If F is chosen via a quadratic regulator and K via a Kalman filter then the separation principle of linear quadratic Gaussian optimal control holds so our compensator is optimal in an appropriate sense [1].

3. LINEAR APPROXIMATIONS. Suppose we have an accurate but nonlinear model of a plant of theform

$$\dot{\xi} = f(\xi,\mu) \tag{3.1a}$$

$$\psi = h(\xi) \tag{3.1b}$$

$$\xi(o) \approx \xi^0 \tag{3.1c}$$

where ξ, μ and ψ are the state, input and output vectors. We assume ξ is n-dimensional and μ and ψ are one-dimensional, but as we mentioned before this restriction can be dropped. If (ξ^0,μ^0) is a critical point of the controlled differential equation (3.1a), i.e.

$$f(\xi^0,\mu^0) = 0 \tag{3.2}$$

then we can attempt to approximate (3.1) by

$$\dot{x} = Ax + Bu \tag{3.3a}$$

$$y = Cx \tag{3.3b}$$

$$x(o) \approx 0^0 \tag{3.3c}$$

where $x \approx \xi - \xi^0$, $u = \mu - \mu^0$, $y \approx \psi - \psi^0$ $(\psi^0 = h(\xi^0))$ and

$$A = \frac{\partial f}{\partial \xi}(\xi^0,\mu^0) \qquad B = \frac{\partial f}{\partial \mu}(\xi^0,\mu^0) \qquad C = \frac{\partial h}{\partial \xi}(\xi^0) \quad . \tag{3.4}$$

But in the open loop mode the accuracy of this approximation critically depends on the stability of A. If A is stable and we set

$$u(t) = \mu(t) - \mu^0 \tag{3.5a}$$

then we can expect the errors to decay and

$$x(t) = \xi(t) - \xi^0 + O(\xi(t) - \xi^0)^2 \tag{3.5b}$$

$$y(t) = \psi(t) - \psi^0 + O(\xi(t) - \xi^0)^2 \tag{3.5c}$$

The notation $O(\xi - \xi^0)^2$ denotes a quantity bounded by $M\|\xi - \xi^0\|^2$ for small $\xi - \xi^0$. If A is not stable then the errors will not decay and (3.5b,c) need not hold.

But, of course, if A is not stable then neither is the nonlinear system (3.1) and the first goal of any compensator is to stabilize it. If we choose F, G, and K for (3.3) as described in section 2 and implement the resulting compensator on (3.1) using (3.5a) we have the nonlinear system,

$$\dot{\xi} = f(\xi,\mu^0 + F\hat{x} + Gv) \tag{3.6a}$$

$$\dot{\hat{x}} = (A + BF + KC)\hat{x} - K(h(\xi) - \psi^0) + BGv \quad . \tag{3.6b}$$

The important fact is that the linear approximation to (3.6) around the nominal $(\xi^0,\hat{x}^0,v^0) = (\xi^0,0,0)$ is precisely

$$\dot{x} = Ax + BF\hat{x} + BGv \tag{3.7a}$$

$$\dot{\hat{x}} = (A + BF + KC)\hat{x} - KCx + BGv \quad . \tag{3.7b}$$

This system is clearly stable as seen by its $(x,\tilde{x})$ description (2.8). Hence the closed loop linear system (3.7) approximates the closed loop nonlinear systems at both stable and unstable critical points (ξ^0,μ^0).

But suppose we wish to operate over a large range of values in (ξ,μ) space or equivalently suppose the nonlinearities of $f(\xi,\mu)$ and $h(\xi)$ are so severe that the first order approximations

$$f(\xi,\mu) \approx \frac{\partial f}{\partial \xi}(\xi^0,\mu^0)(\xi - \xi^0) + \frac{\partial f}{\partial \mu}(\xi^0,\mu^0)(\mu - \mu^0)$$

$$h(\xi) \approx \psi^0 + \frac{\partial h}{\partial \xi}(\xi^0)$$

are not very accurate even for moderate values of $\xi - \xi^0$ and $\mu - \mu^0$. Then even in the closed loop mode we cannot expect the linear system to accurately approximate the nonlinear system. The usual way out of this difficulty is to linearize around several *operating points*. An operating point is a (ξ^i,μ^i,ψ^i) where (ξ^i,μ^i) is a critical point of (3.1a) and $\psi^i = h(\xi^i)$. We approximate (3.1) around the operating point (ξ^i,μ^i,ψ^i) by

$$\dot{x}^i = A^i x^i + B^i u^i \tag{3.7a}$$

$$y^i = C^i x^i \tag{3.7b}$$

$$x^i(o) \approx 0 \tag{3.7c}$$

where

$$A^i = \frac{\partial f}{\partial \xi}(\xi^i,\mu^i), \quad B^i = \frac{\partial f}{\partial \mu}(\xi^i,\mu^i), \quad C = \frac{\partial h}{\partial \xi}(\xi^i) \tag{3.8}$$

and

$$x^i(t) \approx \xi(t) - \xi^i, \quad u^i(t) = \mu(t) - \mu^i, \quad y^i(t) \approx \psi(t) - \psi^i \quad . \tag{3.9}$$

One can also linearize around an *operating trajectory*, i.e., a solution $(\xi^i(t),\mu^i(t),\psi^i(t))$ of (3.1a,b) for $t \in [0,\tau^i]$ such that the matrices $A^i(t)$, $B^i(t)$ and $C^i(t)$ given by (3.8) (with $(\xi^i(t),\mu^i(t),\psi^i(t))$ replacing (ξ^i,μ^i,ψ^i)) are essentially constant over $[0,\tau^i]$. Of course every operating point is also an operating trajectory.

Operating trajectories tend to appear naturally in applications. If $A^i(t)$, $B^i(t)$ and $C^i(t)$ change significantly then one can overcome this difficulty by splitting up the time interval into several intervals thereby introducing additional operating trajectories. Around an operating trajectory one approximates (3.1a,b) by (3.7a,b), (3.8) and (3.9).

An *operating regime* is the domain of (ξ,μ) space around an operating point or a trajectory where the linear model is valid. For each operating regime we choose controller gains F^i, G^i and an observer gain K^i so that $(A^i + B^iF^i)$ and $(A^i + K^iC^i)$ have the desired spectral properties and B^iG^i has the appropriate columns. In this regime one would like to implement the control law $u^i = F^ix^i + G^iv$ but since x^i is not directly measurable we must replace it by an estimate $\hat{x}^i$ from an observer with gain K^i. In this operating regime we have a combined plant and compensator modeled by

$$\dot{\xi} = f(\xi,\mu^i + F^i\hat{x}^i + G^iv) \tag{3.10a}$$

$$\psi = h(\xi) \tag{3.10b}$$

$$\dot{\hat{x}}^i = (A^i + B^iF^i + K^iC^i)\hat{x}^i - K^i(\psi - \psi^i) + BG^iv \quad . \tag{3.10c}$$

If we stay close to the operating point or trajectory we can expect the first order variations to satisfy

$$\dot{x}^i = A^ix^i + B^iF^i\hat{x}^i + B^iG^iv \tag{3.11a}$$

$$y^i = C^ix^i \tag{3.11b}$$

$$\dot{\hat{x}}^i = (A^i + B^iF^i + K^iC^i)\hat{x}^i - K^iC^ix^i + B^iG^iv \quad . \tag{3.11c}$$

If we have chosen F^i, G^i and K^i appropriately, then the performance in the i-th regime will be satisfactory.

But the problem of transitions between operating regimes remain to be solved. When do we switch compensators? Will the system be unstable between regimes? This is the problem of *gain scheduling*.

If the system is only mildly nonlinear then there will be a few large operating regimes. Transitions will not present a great problem since they will occur relatively infrequently. Presumably they can be handled in an open loop, manual mode without automatic compensation. But for systems with severe nonlinearities the transitions occur frequently and an alternate, automatic approach is needed. In the following sections we will describe a method that increases the accuracy of the approximations and the size of the operating regimes. In some cases it reduces to a simple operating regime.

Before closing we would like to point out that simultaneous stabilization is not a viable solution to the problem of frequent transitions between operating regimes. The simultaneous stabilization approach is to seek a single F and a single K so that $(A^i + B^iF)$ and $(A^i + KC^i)$ have the desired spectral properties in all operating regimes. But notice that one is still faced with the problem of deciding when to use the i-th model (3.10), i.e., when are we operating in the i-th regime? Moreover even if $(A^i + B^iF)$ and $(A^i + KC^i)$ are stable for all i, this does not guarantee the system will be stable in its transitions between operating regimes. The time varying linear differential equation

$$\dot{x} = A(t)x \tag{3.12}$$

can be thought of as nonlinear system with no inputs. It is well-known that the spectrum of $A(t)$ can be in the left half plane for all t (hence no compensation is needed at any constant linearization) yet the differential equation (3.12) is unstable [2, p. 158].

4. LINEARIZING TRANSFORMATIONS. Nonlinear transformations can destroy the linearity of a system of the form

$$\dot{x} = Ax + Bu \quad . \tag{4.1}$$

Under the nonlinear change of state coordinates $x = x(\xi)$ and the nonlinear feedback $u = u(\xi,\mu)$, (4.1) becomes the nonlinear system

$$\dot{\xi} = f(\xi,\mu) \tag{4.2}$$

where

$$f(\xi,\mu) = \frac{\partial \xi}{\partial x}(x(\xi))\,(Ax(\xi) + Bu(\xi,\mu)) \quad . \tag{4.3}$$

This suggests a mathematical question. When can such transformations be used to linearize a nonlinear system? For various classes of transformations, this question was answered by Krener [7], Brockett [3], Jakubczyk-Respondek [5], and Hunt-Su [4].

Actually the practical application of this technique in robotics predated the mathematical studies. More recently it has been used in the design of flight control systems for high performance aircraft [14].

Suppose $x = x(\xi)$ and $u = u(\xi,\mu)$ transforms (4.2) into (4.1) in some domain in (ξ,μ) space and the operating point (ξ^0,μ^0) is transformed into $(x^0,u^0) = (0,0)$. If $u = Fx$ stabilizes (4.1) around $x = 0$ then $\mu = \mu(x(\xi),Fx(\xi))$ will stabilize (4.2) around $\xi^0 = \xi(x=0)$. This stability is not just local but it is as global as the domain of definition of the transformations $x = x(\xi)$ and $u = u(\xi,\mu)$ will allow. If these are globally defined then global stability is ensured, but generally this is too much to hope for.

We will follow the Hunt-Su approach to finding the linearizing transformations. Their work is equivalent to the earlier results of Jakubczyk-Respondek, but the Hunt-Su formulation is preferable.

The first step is to linearize the way the control enters (4.1). Frequently the nonlinear model comes in this form, i.e.,

$$\dot{\xi} = g^0(\xi) + g(\xi)\mu \quad . \tag{4.4}$$

If this is not the case then one might be able to find vector fields $g^0(\xi)$ and $g(\xi)$ and a function $\nu = \nu(\xi,\mu)$ so that the relation

$$g^0(\xi) + g(\xi)\nu = f(\xi,\mu) \tag{4.5}$$

(almost) holds. If this is so then (4.5) can be interpreted as defining a preliminary transformation of the control variables. If no such vector fields can be found then one can consider the control μ as an additional state ξ_{n+1} and define a new control $\nu = \dot{\mu}$. Since one is introducing an additional integrator into the system model, one can expect somewhat more sluggish response. But this technique does accomplish the desired linearization and it also smooths the plant input μ, which may be desirable in some cases.

Henceforth we restrict our remarks to the nonlinear system (4.4). The linearizing feedback will be of the form $u = \alpha(\xi) + \beta(\xi)\mu$. Suppose we have a scalar function

$$\psi = h(\xi) \tag{4.6}$$

which we can think of as a pseudo-output for (4.4).

The Krener-Isidori [8] concept of a zero for such a system is defined in terms of invariant distributions, but for such a system π can be described as follows. The plant (4.4) and (4.6) has no (finite) zeroes if one must time differentiate (4.6) n times before the control appears (n = dimension of ξ). These time derivatives are conveniently expressed using the concept of the Lie derivative of a function by a vector field.

$$\frac{d}{dt}\psi(t) = L_{g^0}(h)(\xi(t)) + L_g(h)(\xi(t))\mu(t)$$

where

$$L_{g^0}(h)(\xi) = \frac{\partial h}{\partial \xi}(\xi)g^0(\xi) \quad .$$

$$L_g(h)(\xi) = \frac{\partial h}{\partial \xi}(\xi)g(\xi) \quad .$$

If $L_g(L^j_{g^0}(h))(\xi) = 0$ for $0 \le j < k$ then

$$\left(\frac{d}{dt}\right)^k \psi(t) = L^k_{g^0}(h)(\xi(t)) + L_g L^{k-1}_{g^0}(h)(\xi(t))\mu(t)$$

where

$$L^j_{g^0}(h) = L_{g^0}(L^{j-1}_{g^0}(h)) \quad .$$

Therefore the no zero condition is just that

$$L_g L^k_{g^0}(h) = 0 \qquad 0 \le k < n-1 \quad . \tag{4.7}$$

It is easy to see that every linear system (4.1) admits a linear output

$$y = Cx \tag{4.8}$$

such that there are no zeroes. In this case the no zero condition (4.7) reduces to

$$CA^kB = 0 \qquad 0 \le k < n-1 \quad .$$

Hence we conclude that a necessary condition for the nonlinear system (4.4) to be transformable into a linear system (4.1) is that it admits an output (4.6) which satisfies that no zero condition (4.7). This is almost sufficient, all we need is a controllability condition which we describe in a moment.

Using some elementary Lie theory, we can transform the no zero condition (4.7) into an undetermined first order partial differential equation for the output $h(\xi)$. We define the Lie bracket $[g^0,g]$ of vector fields by

$$[g^0,g](\xi) = \frac{\partial g}{\partial \xi}(\xi)g^0(\xi) - \frac{\partial g^0}{\partial \xi}(\xi)g(\xi)$$

and the ad-notation for repeated brackets

$$ad^0(g^0)g = g$$
$$ad^j(g^0)g = [g^0, ad^{k-1}(g^0)g] \quad .$$

If we are seeking to linearize (4.4) in a domain which includes the operating point $(\xi^0,0)$ (μ^0 has been normalized to 0) then the controllability condition is that the linear approximation to (4.4) at this point be controllable in the linear sense. This translates into the condition that

$$\{g(\xi^0),\dots,ad^{n-1}(g^0)g(\xi^0)\} \text{ are linearly independent.} \tag{4.7}$$

The no zero condition is equivalent to existence of a function $h(\xi)$ satisfying the $n-1$ partial differential equations

$$L_{ad^k(g^0)g}(h) = 0 \qquad 0 \le k < n-1 \tag{4.8a}$$

and of course h should not be constant so

$$L_{ad^{n-1}(g^0)g}(h) \ne 0 \quad . \tag{4.8b}$$

This system is solvable iff the mixed partial conditions are satisfied. These can be expressed as follows.

$$\{g,\dots,ad^{n-2}(g^0)g\} \text{ are involutive} \tag{4.9}$$

i.e., there exists functions $c_k^{ij}(\xi)$ such that for $i,j = 0,\dots,n-2$

$$[ad^i(g^0)g, ad^j(g^0)g] = \sum_{k=0}^{n-2} c_k^{ij}(\xi)\, ad^k(g^0)g \quad .$$

If the Hunt-Su conditions (4.7) and (4.9) are satisfied then one can find h satisfying (4.8) but h is not unique. Given any such h, the linearizing transformations are defined by

$$x_i = L_{g^0}^{i-1}(h) \tag{4.10a}$$

$$u = L_{g^0}^n(h) + L_g L_g^{n-1}(h)\mu = \alpha + \beta\mu \quad . \tag{4.10b}$$

and the resulting linear system is in Brunovsky form, i.e.,

$$\dot{x}_i = \begin{cases} x_{i+1} & i < n \\ u & i = n \end{cases} \quad . \tag{4.11}$$

Notice that if $n = 2$ then condition (4.2) is trivially satisfied, hence every suitably controllable nonlinear system with $n = 2$ and $m = 1$ can be linearized by this technique. In robotics models typically $m = 2n$ (one applied torque for each two states consisting of an angle and an angular velocity), and the couplings between the joints enter as additional torques. This allows the application of linearizing transformations which brings the system to Brunovsky form. The desired pseudo-outputs (generalizing h) are the joint angles so the linearizing transformations are obvious.

Recently Krener, Isidori and Respondek [10] have considered the problem of partial linearization. If a weakened form of the Hunt-Su conditions are satisfied then one can transform the nonlinear system (4.4) into a partially linear system of the form

$$\dot{x}_i = \begin{cases} x_{i+1} & 1 \le i < \rho \\ u & i \quad \rho \\ \phi_i(x) + \psi_i(x)u & \rho < i \le n \end{cases} \quad . \tag{4.12}$$

The index of linearity of a nonlinear system at a critical point $(\xi^0,0)$ is the largest such ρ, and this can be easily calculated.

Typically, there are many such partial linearizations of maximal linear size ρ, but they differ in their stability properties. See [10] for the details.

5. APPROXIMATE LINEARIZATIONS. There are several difficulties with the method of linearizing transformations described in the last section. The

first is computational. One must solve a first order partial differential equation (4.8) to find the linearizing transformations (4.10). While this can be done off-line, the resulting transformations must be stored in a way that allows fast and accurate real time evaluation.

Another difficulty is that the linear coordinates (x,u) given by (4.10) usually have no natural meaning. (Robotics is an exception to the statement.) Typically the nonlinear coordinates (ξ,μ) have a natural physical meaning; e.g., positions, velocities, angles, angular velocities, applied forces, applied torques, etc. In the course of deriving the nonlinear model (4.4) certain physically reasonable simplifications may have been made. What do those simplifications mean in (x,u) coordinates. If there are limits on the state or the controls in (ξ,μ) coordinates how do they transform into (x,u) coordinates?

If stability is the only goal of the design process then the method of linearizing transformations can be quite useful. But suppose one has a nonlinear servo problem, i.e., there is a natural output

$$\psi = k(\xi) \tag{5.1}$$

associated to the system (4.4) and one wishes to design a controller so that the closed loop systems tracks signals from a class of reference outputs. In other words, we wish to define a compensator that accepts a function $\rho(t)$ from a given class and then drives (4.4) and (5.1) so that $|\rho(t) - \psi(t)| \to 0$ as $t \to \infty$. If the output (5.1) is a linearizing output, i.e., it satisfies the no zero condition (4.8) then the method of linearizing transformations can be used to change the problem into a linear servo problem. But this is not usually the case.

If the reference output $\rho(t)$, which $\psi(t)$ is to track, is a smooth function then by differentiating $\rho(t)$ and comparing it with (5.1) and (4.4) one might be able to calculate the desired state trajectory $\xi(t)$ and control $\mu(t)$ (invert the system). But no such $\xi(t)$ and $\mu(t)$ may exist or they may be very large. This is very often the case because $\rho(t)$ is usually not given as a smooth function but rather as a piecewise smooth function consisting of line segments, circular arcs, etc.

We briefly describe an alternate approach which partially mitigates these difficulties. The full details can be found in [12].

If the nonlinear system (4.4) satisfies the Hunt-Su conditions (4.7) and (4.9) then it can be transformed to Brunovsky form (4.10). One can follow this by a linear change of state coordinates and a linear state feedback to get a different linear form. In other words, if a system can be linearized then there are many possible linear forms. Moreover, even for a fixed linear form, the linearizing transformations that achieve that form are not unique.

This is equivalent to the nonuniqueness of the solution h to (4.8). Which transformation and linear form are most natural?

One answer to this question is suggested by the first order linear approximation technique as described in section 3. We seek a change of coordinates $\bar{x} = \bar{x}(\xi)$ and $\bar{u} = \bar{u}(\xi,\mu)$ which transforms (4.4) into

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}\bar{u} \tag{5.2a}$$

satisfying

$$\bar{A} = \frac{\partial g^0}{\partial \xi}(\xi^0) \qquad \bar{B} = g(\xi^0) \tag{5.2b}$$

and

$$\bar{x} = \xi - \xi^0 + O(\xi)^2 \qquad \bar{u} = \mu + O(\xi,\mu)^2 \tag{5.2c}$$

Suppose $x(\xi)$, $\alpha(\xi)$ and $\beta(\xi)$ are defined by (4.10) and transform (4.4) to Brunovsky form (4.1). It is easy to see that $\bar{x}(\xi)$, $\bar{\alpha}(\xi)$ and $\bar{\beta}(\xi)$ transforming (4.4) into (5.2) are given by.

$$\bar{x}(\xi) = \left(\frac{\partial x}{\partial \xi}(\xi^0)\right)^{-1} x(\xi) \quad , \tag{5.3a}$$

$$\bar{\alpha}(\xi) = \beta^{-1}(\xi^0)(\alpha(\xi) - \frac{\partial \alpha}{\partial \xi}(\xi^0)(\xi - \xi^0) \quad , \tag{5.3b}$$

$$\bar{\beta}(\xi) = \beta^{-1}(\xi^0)\beta(\xi) \quad . \tag{5.3c}$$

The functions transform the nonlinear system into its linear approximating system at $(\xi^0, \mu^0 = 0)$. The nonlinear and linear coordinates agree to first order. However, they are not unique but depend on the choice of h satisfying the partial differential equations (4.8). These formulas (5.3) are not a convenient way of computing the transformations or their inverses which are also needed.

Since the linear and nonlinear coordinates agree to first order (5.2c), we know some of the terms in a Taylor series expansion of $\bar{x}(\xi)$, $\bar{\alpha}(\xi)$ and $\bar{\beta}(\xi)$ at ξ^0, $\mu^0 = 0$. The higher terms can be computed term by term. We briefly describe the computation of the second order term for a scalar input system. The interested reader is referred to [12] for a discussion of the general case.

Suppose

$$\bar{x}_i(\xi) = \xi_i - \xi_i^0 + \frac{1}{2}\sum_{k,\ell} \bar{x}_{i;k\ell}(\xi_k - \xi_k^0)(\xi_\ell - \xi_\ell^0) + O(\xi - \xi^0)^3 \tag{5.4a}$$

$$\bar{\alpha}(\xi) = \frac{1}{2}\sum_{k,\ell} \bar{\alpha}_{;k\ell}(\xi_k - \xi_k^0)(\xi_\ell - \xi_\ell^0) + O(\xi - \xi^0)^3 \tag{5.4b}$$

$$\bar{\beta}(\xi) = 1 + \sum_k \bar{\beta}_{;k} (\xi_k - \xi_k^o) + O(\xi - \xi^o)^2 \tag{5.4c}$$

where

$$\bar{x}_{i;k\ell} = \partial^2\bar{x}_i/\partial\xi_k\partial\xi_\ell(\xi^o) , \quad \bar{\alpha}_{;k\ell} = \partial^2\bar{\alpha}/\partial\xi_k\partial\xi_\ell(\xi^o)$$

and

$$\bar{\beta}_{;k} = \partial\bar{\beta}/\partial\xi_k(\xi^o) \quad .$$

Of course $\bar{x}_{i;k\ell} = \bar{x}_{i;\ell k}$ and $\bar{\alpha}_{;k\ell} = \bar{\alpha}_{;\ell k}$.

We can compute the time derivative of $\bar{x}_i$ as a function of ξ and μ in two ways. One way is to use (5.2a) and (5.2b) to obtain $\dot{\bar{x}}_i$ as a function of $\bar{x}$ and $\bar{u}$ and then (5.4) to express this can function if ξ and μ. The other way is to differentiate (5.4a) using (4.4). When the results of these two ways are equated ignoring terms of order $O(\xi,\mu)^3$ one obtains a system of linear equations

$$g^o_{i;k\ell} + 2\sum_\rho \bar{x}_{i;k\rho}g^o_{\rho j\ell} = \sum_\rho g^o_{i;\rho}\bar{x}_{\rho;k\ell} + g_i\alpha_{;k\ell} \tag{5.5a}$$

$$g_{i;k} + \sum_\ell \bar{x}_{i;k\ell}g^j_\ell = g_i\bar{\beta}_{;k} \tag{5.5b}$$

for $i,k = 1,\dots,n$ and $\ell = k,\dots,n$.

$$(g_i = g_i(\xi^o) , \quad g^o_{i;k} = \frac{\partial g_i}{\partial \xi_k}(\xi^o) , \quad g^o_{i;k\ell} = \frac{\partial^2 g^o_i}{\partial\xi_k\partial\xi_\ell}(\xi^o), \quad \text{etc.})$$

We have $n^2(n+1)/2 + n^2$ linear equations in $n^2(n+1)/2 + n(n+1)/2 + n$ unknowns. The solvability is assured if the Hunt-Su conditions (4.7) and (4.9) are satisfied but the solution need not be unique. (Actually a weaker form of (4.9) suffices in this case. See [12].)

Any solution to (5.5) defines quadratic functions $\bar{x}(\xi)$, $\bar{\alpha}(\xi)$ and $\bar{\beta}(\xi)$ which transform (4.4) into its linear approximating system (5.2) with an error of order $O(\xi,\mu)^3$ not $O(\xi)^2$. The inverse transformations $\xi(\bar{x})$ and $\mu(\bar{x},\bar{u})$ are easily computed to second order. They transform the linear approximating system (5.2) into (4.4) with a similar third order error.

If the third order error is not acceptable one can compute additional terms in the Taylor series expansion to make the error higher order. A symbolic computation package such as MACSYMA greatly simplifies the computation. The resulting transformations are polynomial with leading terms the identity hence one can easily compute polynomials of the same degree which invert them to the desired order of accuracy. Since the transformations and their approximate inverses are polynomial they can be easily stored and evaluated quickly and accurately in real time.

Finally, notice that one is always working with the same linear system (5.2) regardless of the order of accuracy. Hence the control law as expressed in linear coordinates remains the same even as the order of accuracy is increased. The lowest order of accuracy with errors of order $O(\xi)^2$ is just the standard first order approximation approach as described in section 3.

6. OBSERVERS WITH LINEAR ERROR DYNAMICS. In section 2 we briefly discussed the theory of observers for systems of the form (2.1). We would like to describe a similar theory for a certain class of nonlinear systems of the form

$$\dot{\xi} = f(\xi,\mu) \tag{6.1a}$$

$$\psi = h(\xi) \tag{6.1b}$$

$$\xi(o) \approx \xi^o \quad . \tag{6.1c}$$

Suppose we can find a change of state and output coordinates $\xi = \xi(x)$ and $\psi = \psi(y)$ which transforms (6.1) into

$$\dot{x} = Ax + \gamma(\xi,\mu) \tag{6.2a}$$

$$y = Cx \quad . \tag{6.2b}$$

Not every system (6.1) can be so transformed, necessary and sufficient conditions for the existence of such a transformation can be found in Krener-Respondek [11]. Related work is in Krener-Isidori [9].

Before we describe these conditions, let us note that it is easy to construct an observer for (6.2). We choose a K so that the spectrum of $(A + KC)$ is in the left half of the complex plane. The estimate $\hat{x}(t)$ of $x(t)$ evolves according to the dynamics

$$\dot{\hat{x}} = (A + KC)\hat{x} - Ky + \gamma(y,\mu) \quad . \tag{6.3a}$$

The error $\tilde{x}(t) = x(t) - \hat{x}(t)$ satisfies

$$\dot{\tilde{x}} = (A + KC)\tilde{x} \tag{6.3b}$$

hence it converges to zero exponentially fast.

The estimate $\hat{\xi}(t)$ of $\xi(t)$ is just the transform of $\hat{x}(t)$ so it evolves according to

$$\dot{\hat{\xi}} = \hat{f}(\hat{\xi},\psi,\mu) \tag{6.4a}$$

$$\hat{\psi} = h(\hat{\xi}) \tag{6.4b}$$

where

$$\hat{f}(\hat{\xi},\psi,\mu) = \frac{\partial\xi}{\partial x}(\hat{\xi})\ ((A + KC)x(\hat{\xi}) - Ky(\psi) + \gamma(\psi,\mu)) \quad . \tag{6.5}$$

This defines an observer for the nonlinear system (6.1) such that the error when measured in appropriate coordinates has linear dynamics (6.3b).

We briefly sketch the development of necessary and sufficient conditions for the existence of the transformations taking (6.1) to (6.2). Of course, we are only interested in transforming (6.1) to an observable pair (C,A) so we require an observability condition. Namely that the linear approximation to (6.1) at (ξ^0,μ^0) be observable. This can be expressed using Lie differentiation by the condition that

$$\{L^k_{f^0}(h)(\xi) : k = 0,\dots,n-1\} \text{ are independent functions around } \xi^0 \quad . \tag{6.6}$$

Here $f^0(\xi) = f(\xi,\mu^0)$.

If this condition is satisfied and the transformation to (6.2) does exist, then (6.6) guaranteed that (C,A) is an observable pair. Without loss of generality we can ssume that (C,A) is indeed Brunovsky form, i.e.,

$$C = (10,\dots,0) \qquad A = \begin{pmatrix} 0 & 1 & \dots & 0 \\ & & \ddots & \\ & & & 1 \\ 0 & & & 0 \end{pmatrix} \quad .$$

We define a pseudo-input vector

$$B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad .$$

This is a vector field on the state space which is constant when expressed in x coordinates. Its repeated Lie brackets by the vector field Ax are the vector fields $(-A)^kB$ which are also constant in x coordinates. The bracket of constant vector fields is just 0, so $B,\dots,(-A)^{n-1}B$ are a commuting family of vector fields and a basis for $\mathbb{R}^n$. Moreover, the Lie derivatives of the output function $y = Cx$ by these vector fields are just $C(-A)^kB$ which equal zero for $k = 0,\dots,n-1$. Let $g(\xi)$ be the transform of B into ξ coordinates then after some calculations one can show that g must satisfy

$$[ad^k(f^0)g, ad^\ell(f^0)g] = 0 \qquad 0 \le k, \ell \le n-1 \tag{6.7}$$

and

$$L_g L^k_{f^0}(h) = 0 \qquad k = 0,\dots,n-2 \tag{6.8}$$

A further calculation shows that $(-A)^kB$ and $\gamma(y,\mu)$ commute for each constant control μ and $k = 0,\dots,n-2$. In ξ coordinates these become the additional conditions that

$$[ad^k(f^0)g,\ f^\mu(\xi)] = 0 \tag{6.9}$$

where $f^\mu(\xi) = f(\xi,\mu) - f(\xi,\mu^0)$, μ is constant. We have sketched half of the proof of the following:

THEOREM. Krener-Respondek [10]. *There exists transformations $x = x(\xi)$ and $y = y(\psi)$ in some neighborhood of (ξ^0,μ^0) which carry (6.1) into (6.2) with (C,A) observable iff the observability condition (6.6) holds and there exists a vector field $g(\xi)$ satisfying (6.7), (6.8) and (6.9).*

On the other hand, if g satisfied (6.7), (6.8), and (6.9) then we can find coordinates x such that

$$\frac{\partial}{\partial x_k} = ad^{n-k}(f^0)g \qquad k = 1,\dots,n \tag{6.10}$$

It is straight forward to verify that $x = x(\xi)$ and $y = x_1(\xi)$ accomplish the desired transformation from (6.1) to (6.2).

This theorem is reminiscent of the nonzero condition for the linearizability of a nonlinear system by change of state coordinates and state feedback. To accomplish that linearization there must exist a pseudo-output $h(\xi)$ such that the nonlinear system has no zeroes. To accomplish this linearization there must exist a pseudo-input $g(\xi)$ such that the nonlinear system has no zeroes.

The theorem as stated is not very useful. In [11] an algorithm is presented for finding $g(\xi)$. One starts with the unique vector field $\bar{g}(\xi)$ which satisfies (6.8) and an additional constraint, i.e.,

$$L_{\bar{g}} L^k_{f^0}(h) = \begin{cases} 0 & 0 \le k < n-1 \\ 1 & k = n-1 \end{cases} \tag{6.11}$$

It can be shown that if g exists then it is a rescaling of $\bar{g}$, $g = \bar{g}\Phi$ where Φ is a function only of ψ. (It turns out that $\Phi = \partial\psi/dy$.) Moreover Φ must satisfy a linear first order differential equation,

$$\frac{d\Phi}{d\psi} = \frac{1}{n}\lambda(\psi)\Phi(\psi) \tag{6.12}$$

where λ is defined by

$$\lambda(\xi) = \langle dh, [ad^{n-2}(f^o)\bar{g}, ad^{n-1}(f^o)\bar{g}]\rangle(\xi) \tag{6.13}$$

A necessary condition is that $\lambda(\xi)$ be a function of only ψ.

The procedure is to calculate $\bar{g}(\xi)$ by (6.11) and then calculate λ by (6.13). If λ is only a function of ψ then we solve (6.12) for $\Phi(\psi)$. This yields $g(\xi) = \bar{g}(\xi)\Phi(\psi)$, which must satisfy (6.7) and (6.9) for the transformations to exist. The multidimensional output problem $(p > 1)$ is considerably more difficult since (6.12) becomes a partial differential equation [11].

Because of the complexities of the method we have just described, a simpler approach is needed. In [13] observers are constructed with error dynamics that are approximately linear, up to some order. This is in the spirit of the approximate linearization method of section 5. We briefly sketch the construction of an observer with error dynamics that is linear up to second order.

We restrict our attention to a neighborhood of an operating point, $f(\xi^o,\mu^o) = 0$. Suppose A, B, C are the matrices of the linear approximations at this point, so are given by (3.4). The second order Taylor series expansions of $x(\xi)$, $y(\psi) = y(h(\xi))$, and $\gamma(\psi,\mu)$ are assumed to be of the form

$$x_i(\xi) = \xi_i - \xi_i^o + \frac{1}{2}\sum_{k,\ell}\frac{\partial^2 x_i}{\partial\xi_k\partial\xi_\ell}(\xi^o)(\xi_k - \xi_k^o)(\xi_\ell - \xi_\ell^o) = O(\xi)^3 \tag{6.14a}$$

$$\begin{aligned} y(\xi) &= \psi - \psi^o + \frac{1}{2}\frac{d^2y(\psi^o)}{d\psi^2}(\psi - \psi^o)^2 + O(\psi)^3 \\ &= C(\xi - \xi^o) + \frac{1}{2}\frac{d^2y(\psi^o)}{d\psi^2}(C(\xi - \xi^o)^2 \\ &\quad + \frac{1}{2}\sum_{k,\ell}\frac{\partial^2 h}{\partial\xi_k\partial\xi_\ell}(\xi^o)(\xi_k - \xi_k^o)(\xi_\ell - \xi_\ell^o) + O(\xi)^3 \end{aligned} \tag{6.14b}$$

$$\begin{aligned} \gamma_i(\psi,\mu) &= B(\mu - \mu^o) + \frac{1}{2}\frac{\partial^2\gamma_i}{\partial\psi^2}(\psi^o,\mu^o)(\psi - \psi^o)^2 \\ &\quad + \frac{\partial^2\gamma_i}{\partial\psi\partial\mu}(\psi^o,\mu^o)(\psi - \psi^o)(\mu - \mu^o) \\ &\quad + \frac{1}{2}\frac{\partial^2\gamma_i}{\partial\mu^2}(\psi^o,\mu^o)(\mu - \mu^o) + O(\psi,\mu)^3 \\ &= B(\mu - \mu^o) + \frac{1}{2}\frac{\partial\gamma_i}{\partial\psi^2}(\psi^o,\mu^o)(C(\xi - \xi^o))^2 \\ &\quad + \frac{\partial^2\gamma_i}{\partial\psi\partial\mu}(\psi^o,\mu^o)(\psi - \psi^o)(\mu - \mu^o) \\ &\quad + \frac{1}{2}\frac{\partial^2\gamma_i}{\partial\mu^2}(\psi^o,\mu^o)(\mu - \mu^o)^2 + O(\xi,\mu)^3 \end{aligned} \tag{6.14c}$$

If the Krener-Respondek conditions (6.6-9) are satsified then we can find the second order partial derivatives

$$\frac{\partial x_i}{\partial\xi_k\partial\xi_\ell}(\xi^o),\quad \frac{d^2y}{d\psi}(\psi^o),\quad \frac{\partial^2\gamma_i}{\partial\psi^2}(\psi^o,\mu^o),\quad \frac{\partial^2\gamma_i}{\partial\psi\partial\mu}(\psi^o,\mu^o),\quad \frac{\partial^2\gamma_i}{\partial\mu^2}(\psi^o,\mu^o)$$

so that (6.14) transforms (6.1) into (6.2) with an error of order $O(\xi,\mu)^3$. Linear equations for these unknown derivatives can be found by computing $\dot{x}_i$ in two ways as in section 5 and equating the coefficient of the various monomials. A symbolic manipulation package such as MACSYMA is admirably suited for this computation.

The solvability of these linear equations is weaker than the Krener-Respondek conditions and much easier to check [13]. Therefore it is preferable to derive these linear equations and check their solvability. If a solution can be found it can be used in (6.5) to define the second order observer. The accuracy of the approximations is improved if (6.5) is rewritten as

$$\hat{f}(\xi,\psi,\mu) = f(\hat{\xi},\mu) + \frac{\partial\xi}{\partial x}(\hat{\xi})((\gamma(\psi,\mu) - \gamma(\hat{\psi},\mu) - K(y(\psi) - y(\hat{\psi}))) \tag{6.15}$$

before substituting (6.14).

7. NONLINEAR COMPENSATORS. The first step in designing a nonlinear compensator for (3.1) around an operating point (ξ^o,μ^o) is to compute its linear approximation (3.3). A linear compensator for this system is designed in the

standard fashion using LQG or whatever methodology appeals to the designer. The compensator is described by the three gain matrices F, G, and K. The resulting plant and compensator is modeled by (3.6).

Suppose the nonlinearities of the plant are severe enough to impair the performance of the compensated plant over parts of its operating range. One can use the linearizing techniques that we have described to compensate for the nonlinearities and in effect automatically schedule the controller and observer gains.

Suppose the state of the present plant is directly measurable and we can find transformations $\bar{x} = \bar{x}(\xi)$ and $\bar{u} = \bar{\alpha}(\xi) + \bar{\beta}(\xi)(\mu - \mu^2)$ which transform (3.1) into (5.2) with an error of order higher than two. The standard control law is

$$\mu = \mu^0 + F(\xi - \xi^0) + Gv \tag{7.1}$$

is based on the assumptions that

$$u = Fx + Gv \tag{7.2a}$$

$$u = \mu - \mu^0 \tag{7.2b}$$

$$x = \xi - \xi^0 \quad . \tag{7.2c}$$

But it is preferable to use

$$\mu = \mu^0 + \bar{\beta}^{-1}(\xi)(F\bar{x}(\xi) - \bar{\alpha}(\xi) + Gv) \tag{7.3}$$

based on the assumptions that

$$\bar{u} = F\bar{x} + Gv \tag{7.4a}$$

$$\bar{u} = \bar{\alpha}(\xi) + \bar{\beta}(\xi)(\mu - \mu^0) \tag{7.4b}$$

$$\bar{x} = \bar{x}(\xi) \tag{7.4c}$$

because then the closed loop linear and nonlinear to higher order under the state transformation (7.4c).

If the state is not directly measurable then we try to construct a nonlinear observer (6.4) with error dynamics that are linear to order higher than two. We use the same observer gain K as before. The state estimate $\hat{\xi}(t)$ replaces $\xi(t)$ in (7.3). The stability of the overall configuration follows from a theorem of Vidyasagar [15] using Lyapunov functions.

8. CONCLUSION. We have described a new approach to nonlinear compensator design based on the use of approximately linearizing transformations. It is an extension of the standard approach using linear approximations. Parts of the method have been widely used in nonlinear controllers for robots and aircraft. To our knowledge the total package has never been fully employed. Although the basic skeleton of the theory is pretty well understood, there is considerable theoretical work left to be done, e.g., the design of nonlinear servomechanisms and nonlinear adaptive controllers.

Perhaps more important at this time is a study of the computational aspects of the method and how well it performance in simulated and actual applications.

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