Systems & Control Letters 5 (1984) 181–185
North-Holland

December 1984

Approximate linearization by state feedback and coordinate change

Arthur J. KRENER *

Department of Mathematics, University of California, Davis, CA 95616, USA

Received 29 June 1984

A nonlinear system can always be approximated to first order by linear systems. It has been shown by Jakubczyk and Respondek [4] and Hunt and Su [3] that certain nonlinear systems are the exact transforms of linear systems under nonlinear state coordinate change and nonlinear state feedback. In this paper we give necessary and sufficient conditions for a nonlinear system to be approximated to higher order by the transform of a linear system. The use of this technique in the design of nonlinear compensators has been suggested recently by the author [6].

Keywords: Nonlinear systems, Linearization, Nonlinear feedback, Nonlinear change of coordinates.

1. First order linear approximations

The standard approach to the design of a state feedback law for the nonlinear system

$$\dot{\xi} = f(\xi, \mu) \tag{1.1}$$

is to approximate it by a linear system

$$\dot{x} = Ax + Bu \tag{1.2}$$

in some region around the pair (ξ^0, μ^0) consisting of the nominal state and nominal control. Typically the pair (ξ^0, μ^0) is a rest point of the system $f(\xi^0, \mu^0) = 0$.

The approximation is first order in the state,

$$x = \xi - \xi^0 + \mathrm{O}(\xi - \xi^0)^2 \tag{1.3a}$$

but exact in the control,

$$u = \mu - \mu^0. \tag{1.3b}$$

This fixes the matrices of (1.2) as

$$A = \frac{\partial f}{\partial \xi}(\xi^0, \mu^0), \qquad B = \frac{\partial f}{\partial \mu}(\xi^0, \mu^0). \tag{1.4}$$

The validity of the approximation depends on the size of the second order terms which in turn depends on the stability of A. If the spectrum of A lies far enough to the left in the complex plane and the input $\mu(t)$ remains close to μ^0 then we can expect $\xi(t)$ to remain close to ξ^0. The second order errors in (1.3) can be ignored and the linear system accurately portrays the nonlinear behaviour.

On the other hand if A is unstable then the first goal in any feedback law is to stabilize it. Suppose we use state feedback of the form

$$u = Fx + v \tag{1.5}$$

where v is an open loop control, then (1.2) becomes

$$\dot{x} = \tilde{A}x + Bv \tag{1.6}$$

where

$$\tilde{A} = A + BF. \tag{1.7}$$

It is well known that the controllability of the pair (A, B) is equivalent to arbitrary spectral assignability (up to complex conjugation) of $A + BF$ by choice of F. This suggests the feedback

$$\mu = \mu(\xi, \nu) = F(\xi - \xi^0) + \nu \tag{1.8}$$

for the nonlinear system, where ν is the nonlinear open loop control whose nominal value is $\nu^0 = \mu^0$. The resulting nonlinear system is

$$\dot{\xi} = \tilde{f}(\xi, \nu) \tag{1.9}$$

where

$$\tilde{f}(\xi, \nu) = f(\xi, \mu(\xi, \nu)). \tag{1.10}$$

It is easy to see that the linear approximation to (1.10) at (ξ^0, ν^0) is precisely (1.6). The linear and nonlinear states agree to first order as before (1.3a). The linear and nonlinear open loop controls satisfy

* Research supported in part by NASA under NAG 2-268 and by NSF under MCS-8300884.

a similar relationship,

$$v = \nu - \nu^0 + O(\xi - \xi)^2. \tag{1.11}$$

The stability of $\tilde{A}$ tends to keep $O(\xi - \xi)^2$ negligible. With the accuracy of the approximation of (1.9) by (1.6) assured we can go on with the problem choosing a state feedback to accomplish our desired goals, provided the closed loop system remains stable. We can view this additional feedback as a modification of (1.9) rather than the original system (1.1). In other words we seek a function $\nu(\xi, w)$ (where w is another open loop control) which suitably modifies the behaviour of (1.9).

The usual approach is to find a feedback $v(x, w)$ which achieves the desired behaviour in the linear system (1.6). This can be transformed into $\nu(\xi, w)$ by (1.3a) and (1.11) ignoring the second order errors. The hope is that the nonlinear system will inherit the desired behaviour from the linear system despite these errors.

2. Higher order linear approximations

There are systems where the nonlinearities are so severe that the method described in the previous section fails. The second order terms $O(\xi - \xi)^2$ cannot be neglected over the full range of the systems. The method of linearizing transformations can overcome these difficulties. One seeks a change of state coordinate and a state feedback which transforms (1.1) into a linear system. Variations on this question have been treated by Krener [5], Brockett [1], Jakubczyk and Respondek [4] and Hunt and Su [3]. Meyer and Cicolani [7] have applied this to automatic flight control systems. Even before the general theory was developed, Freund [2] was using this method in robotics. Krener [6] has suggested a new approach to compensator design based on this and related work.

We describe an approximate version of this approach, which is computationally much easier. For simplicity, we restrict our attention to a nonlinear system where the control μ enters the dynamics in a linear fashion,

$$\begin{aligned}\dot{\xi} &= g^0(\xi) + g(\xi)\mu \\ &= g^0(\xi) + \sum_{j=1}^{m} g^j(\xi)\mu_j,\end{aligned} \tag{2.1a}$$

$$\xi(0) \approx \xi^0. \tag{2.1b}$$

The state ξ lies in $\mathbb{R}^n$ and the control μ in $\mathbb{R}^m$. The nominal operating point (ξ^0, $\mu^0 = 0$) is presumed to be a critical point ($g^0(\xi^0) = 0$), although this can be weakened.

The first step is to compute the linearization of (2.1) at ξ^0, namely

$$\dot{x} = Ax + Bu, \tag{2.2a}$$

$$A = \frac{\partial g^0}{\partial \xi}(\xi^0), \qquad B = g(\xi^0). \tag{2.2b}$$

One seeks a change of coordinates $x = x(\xi)$ and $u = u(\xi, \mu)$ which transforms (2.2) into (2.1) with an error of order $O(\xi - \xi)^3$. Clearly only the lower order terms of the coordinate changes come into play, so we can assume that

$$\begin{aligned}x_i = \xi_i - \xi_i^0 &+ \tfrac{1}{2} \sum_{k,l=1}^{n} \frac{\partial^2 x_i}{\partial \xi_k \partial \xi_l}(\xi^0)(\xi_k - \xi_k^0)(\xi_l - \xi_l^0) \\ &+ O(\xi - \xi^0)^3,\end{aligned} \tag{2.3a}$$

$$\begin{aligned}u_j = \mu_j &+ \tfrac{1}{2} \sum_{k,l=1}^{n} \frac{\partial^2 u_j}{\partial \xi_k \partial \xi_l}(\xi^0, 0)(\xi_k - \xi_k^0)(\xi_l - \xi_l^0) \\ &+ \sum_{k=1}^{n} \sum_{h=1}^{m} \frac{\partial^2 u_j}{\partial \xi_k \partial \mu_h}(\xi^0, 0)(\xi_k - \xi_k^0)\mu_h \\ &+ O(\xi - \xi^0, \mu)^3.\end{aligned} \tag{2.3b}$$

We can express $\dot{x}_i$ as a function of ξ and μ with an error term of order $O(\xi - \xi^0, \mu)^3$ in two ways. The first is to substitute (2.3) into the right side of (2.2). The other is to time differentiate the right side of (2.3) with the help of (2.1). If we equate the second order terms in these expressions we obtain a system of $n^2(n+1)/2 + n^2m$ linear equations in the $n^2(n+1)/2 + mn(n+1)/2 + m^2n$ unknown second partial derivatives found in (2.3), namely

$$\begin{aligned}&\frac{\partial^2 g_i^0}{\partial \xi_k \partial \xi_l}(\xi^0) + 2 \sum_{p=1}^{n} \frac{\partial^2 x_i}{\partial \xi_k \partial \xi_p}(\xi^0) \frac{\partial g_p^0}{\partial \xi_k}(\xi^0) \\ &\quad = \sum_{p=1}^{n} \frac{\partial g_i^0}{\partial \xi_p} \frac{\partial^2 x_p}{\partial \xi_k \partial \xi_l}(\xi^0) + \sum_{j=1}^{m} g_i^j(\xi^0) \frac{\partial^2 u_j}{\partial \xi_k \partial \xi_l}\end{aligned} \tag{2.4a}$$

for $1 \leqslant i \leqslant n$ and $1 \leqslant k \leqslant l \leqslant n$,

$$\frac{\partial g_i^l}{\partial \xi_k} + \sum_{l=1}^{n} \frac{\partial^2 x_i}{\partial \xi_k \partial \xi_l}(\xi^0) g_l^j(\xi^0) = \sum_{h=1}^{m} g_i^h(\xi^0) \frac{\partial^2 u_h}{\partial \xi_k \partial \mu_j}(\xi^0, 0) \tag{2.4b}$$

for $1 \leqslant i, k \leqslant n$ and $1 \leqslant j \leqslant m$.

In Section 3 we shall give necessary and sufficient conditions for the solvability of (2.4). Suppose it is solvable, typically the solution will not be unique but that need not concern us. Notice that (2.3) agrees with the identity transformations to first order and hence it is easy to invert them at least to second order,

$$\xi_i - \xi_i^0 = x_i - \tfrac{1}{2} \sum_{k,l=1}^{n} \frac{\partial^2 x_i}{\partial \xi_k \partial \xi_l}(\xi^0) x_k x_l + \mathrm{O}(x)^3, \tag{2.5a}$$

$$\mu_j = u_j - \tfrac{1}{2} \sum_{k,l=1}^{n} \frac{\partial^2 u_j}{\partial \xi_k \partial \xi_l}(\xi^0, 0) x_k x_l - \sum_{k=1}^{n} \sum_{h=1}^{m} \frac{\partial^2 u_j}{\partial \xi_k \partial \mu_h}(\xi^0, 0) x_k u_h + \mathrm{O}(x, u)^3. \tag{2.5b}$$

By ignoring the third order terms in (2.3) and (2.5) we can pass back and forth between the nonlinear system (2.1) and its linear approximation (2.2) with at most third order error. A stabilizing state feedback

$$u = Fx + v \tag{2.6}$$

which produces the desired performance for the linear system (2.2) can be transformed into a nonlinear feedback by substituting (2.3a) into (2.6) and then substituting the result into (2.5b). Hopefully this feedback will give the desired performance for the nonlinear system (2.1).

If it does not, then the above method can be generalized to find transformations between (2.1) and (2.2) with fourth order error. The third partials of these transformations must satisfy a set of linear equations found as before by computing $\dot{x}$ as a function of ξ and μ in two ways. Needless to say, the computations are somewhat messy. Since they are all polynomials they can be done automatically using a symbolic manipulation package such as MACSYMA.

One can iterate this procedure to find transformations with any degree of accuracy provided certain conditions, which we present in the next section, are met. At each stage the previous solution can be taken as the lower order terms of the transformations, so only the highest terms need to be computed.

The resulting transformations and their approximate inverses are polynomial hence they can be easily stored and evaluated in real time. The actual computation of the transformations need only be done once and off-line. Since the higher order transformations agree with the previous ones in their lower order terms, we can continue to iterate this procedure until satisfactory performance of the closed loop nonlinear system is achieved.

3. Necessary and sufficient conditions

In this section we derive necessary and sufficient conditions for the existence of transformations

$$x = x(\xi), \tag{3.1a}$$

$$u = u(\xi, \mu) = \alpha(\xi) + \beta(\xi)\mu, \tag{3.1b}$$

which transform the nonlinear system (2.1) into a nearly linear system

$$\dot{x} = Ax + Bu + \mathrm{O}(\xi, \mu)^{\rho+1}. \tag{3.2}$$

The transformations (3.1) exist and are invertible from some open domain of (x, u) space onto an open domain of (ξ, μ) space. In particular, $(x, u) = (0, 0)$ is mapped to $(\xi, \mu) = (\xi^0, \mu^0)$.

Jakubczyk and Respondek [4] and Hunt and Su [3] gave necessary and sufficient conditions when there is no error term. At the other extreme, as was shown in Section 1, it is always possible to find such transformations when $\rho = 1$. The identity transformations (or any other linear transformations) suffice.

There are two 'natural' choices for the linear system (3.2), one is the first order linear approximation (2.2), the other is the Brunovský form of this linear system. Most researchers have focused on the latter, but the former is more appropriate in applications since then the linear and nonlinear coordinates can agree to first order [6].

But as far as existence of an approximate linearization is concerned, the choice of linear form is immaterial. Any higher order approximate linearization of (2.1) must have the same controllability indices as its first order approximation (2.2). If there exist transformations (3.1) carrying (2.1) into its linear approximation (2.2) with an error of $O(\xi, \mu)^{\rho+1}$ then these can be followed by linear transformations to obtain the Brunovský form of (2.2). On the other hand if (3.1) transforms (2.1) into any linear form with error $O(\xi, \mu)^{\rho+1}$ then

$$\bar{x} = \frac{\partial x}{\partial \xi}(\xi^0)^{-1} x(\xi), \tag{3.3a}$$

$$\bar{u} = \beta^{-1}(\xi^0)\Big(\alpha(\xi) - \frac{\partial \alpha}{\partial \xi}(\xi^0)(\xi - \xi^0) + \beta(\xi)\mu\Big) \tag{3.3b}$$

transform (2.2) into its linear approximation (3.2) with the same order error. This is because

$$\bar{x} = \xi - \xi^0 + O(\xi - \xi^0)^2, \tag{3.4a}$$

$$\bar{u} = \mu + O(\xi - \xi^0, \mu)^2. \tag{3.4b}$$

To state the necessary and sufficient conditions we need some notation. The *Lie bracket* of vector fields is another vector field defined by

$$[g^0, g^j](\xi) = \frac{\partial g^j}{\partial \xi}(\xi) g^0(\xi) - \frac{\partial g^0}{\partial \xi}(\xi) g^j(\xi).$$

The *ad notation* is used for repeated Lie brackets,

$$\mathrm{ad}^0(g^0) g^j = g^j,$$

$$\mathrm{ad}^k(g^0) g^j = [g^j, \mathrm{ad}^{k-1}(g^0) g^j].$$

A *distribution* $\mathscr{D}$ is a module of vector fields (over the C^∞ functions). $\mathscr{D}$ has an *order ρ local basis* around ξ^0 if there exist vector fields $X^1, \ldots, X^d$ which are linearly independent at ξ^0 and such that for every $Y \in \mathscr{D}$ there exists functions c_k such that

$$Y = \sum_{k=1}^{d} c_k X^k + O(\xi - \xi^0)^{\rho+1}. \tag{3.5}$$

The integer d is the *order ρ dimension* of $\mathscr{D}$ at ξ^0. Such a distribution is said to *order ρ involutive* at ξ^0 if there exist functions c_k^{ij} such that

$$[X^i, X^j] = \sum_{k=1}^{d} c_k^{ij} X^k + O(\xi - \xi^0)^{\rho}. \tag{3.6}$$

Such a distribution is said to be *order ρ integrable* at ξ^0 if there exist $n - d$ independent functions $h_{d+1}, \ldots, h_n$ such that

$$\langle \mathrm{d}h_k, X^j \rangle = O(\xi)^{\rho+1}. \tag{3.7}$$

The classical Frobenius theorem can be generalized as follows.

Theorem (Frobenius with remainder). *Let $\mathscr{D}$ be a distribution with order ρ basis $\{X^1, \ldots, X^d\}$ at ξ^0. $\mathscr{D}$ is order ρ integrable at ξ^0 iff $\mathscr{D}$ is order ρ involutive at ξ^0.*

Certainly this theorem has appeared before in some form or other, but we shall sketch a proof in lieu of a reference.

Proof. Suppose $\mathscr{D}$ is order ρ integrable at ξ^0, then

$$\begin{aligned}\langle \mathrm{d}h_k, [X^i, X^j] \rangle &= X^i(\langle \mathrm{d}h_k, X^j \rangle) \\ &\quad - X^j(\langle \mathrm{d}h_k, X^i \rangle) \\ &= O(\xi)^\rho.\end{aligned}$$

By nonsingularity there exists c_k^{ij} satisfying (3.6).

On the other hand suppose (3.6) is satisfied. Let $X^{d+1}, \ldots, X^n$ be vector fields which are linearly independent of $X^1, \ldots, X^d$ at ξ^0 and let $\varphi^i(s, \xi)$ denote the flow of X^i. Consider the map

$$\Phi(s_1, \ldots, s_n) = \varphi^1(s_1, \varphi^2(s_2, \ldots, \varphi^n(s_n, \xi^0) \cdots)).$$

This map is locally invertible hence we can consider $s_i = s_i(\xi)$. Define $h_k(\xi) = s_k$. By repeated use of the Lie–Taylor series

$$\varphi^i(s_i)_* X^j = \sum_{k=1}^{\infty} \mathrm{ad}(X^i) X^j \frac{(s_i)^k}{k!},$$

it can be shown that (3.7) holds.

Given the nonlinear system (2.1) define distributions

$$\mathscr{D}^k = C^\infty \operatorname{span}\{\mathrm{ad}^l(g^0) g^j : 0 \leqslant l < k,\ j = 1, \ldots, m\}.$$

We can now state a weaker version of the Jakubczyk–Respondek and Hunt–Su linearization result.

Theorem. *The nonlinear system (2.1) can be transformed into the order ρ linear system (3.2) where*

(A, B) is a controllable pair with controllability indices $k_1 \geqslant \cdots \geqslant k_n$ iff

(i) $\mathscr{D}^k$ has an order ρ local basis at ξ^0 consisting of

$$\left\{\operatorname{ad}^l(g^0)g^j\colon 0 \leqslant l < \min(k_j, k);\ j = 1, \ldots, m\right\},$$

(ii) $\mathscr{D}^{k_j - 1}$ is order ρ involutive at ξ^0 for $j = 1, \ldots, m$.

Proof. Suppose the transformations (3.1) exist; let $\bar{g}^0(\xi)$ and $\bar{g}^1(\xi), \ldots, \bar{g}^m(\xi)$ denote the transforms of Ax and $B^1, \ldots, B^m$ into ξ coordinates. Define distributions

$$\bar{\mathscr{D}}^k = C^\infty \operatorname{span}\{\operatorname{ad}^l(\bar{g}^0)\bar{g}^j\colon 0 \leqslant l < k,\ j = 1, \ldots, m\}.$$

It is straightforward to verify that $\bar{\mathscr{D}}^k$ satisfies (i) and (ii) with phrase "order ρ" deleted. Moreover from the form of (3.1) one can verify that $\mathscr{D}^k$ and $\bar{\mathscr{D}}^k$ agree to order ρ at ξ^0, i.e., any vector field of one agrees with a vector field of the other to order ρ. Hence (i) and (ii) follow.

On the other hand suppose (i) and (ii) hold. Let $l_1 \geqslant \cdots \geqslant l_p$ be the distinct controllability indices of multiplicities $m_1, \ldots, m_p$. Let $l = l_1$, by controllability, $\mathscr{D}^l$ is of codimension zero. $\mathscr{D}^{l-1}$ is of codimension m_1 and by (ii) is order ρ involutive. Therefore we can find m_1 independent functions $h_1(\xi), \ldots, h_{m_1}(\xi)$ which annihilate it to order ρ as in (3.7).

If $l-1$ is not a controllability index then $\mathscr{D}^{l-2}$ has twice the codimension of $\mathscr{D}^{l-1}$. Of course $h_i(\xi)$ annihilates $\mathscr{D}^{l-2}$ but it is easy to verify that the *Lie derivative* of h_i by g^0 defined by

$$L_{g^0}(h_i)(\xi) = \frac{\partial h_i}{\partial \xi}(\xi) g^0(\xi)$$

does so also since $g^0(\xi) = 0$. So $\mathscr{D}^{l-2}$ is order ρ involutive.

If $l-1$ is also a controllability index of multiplicity m_2 then $\mathscr{D}^{l-2}$ must be order ρ involutive by (ii). So in addition to $h_1, \ldots, h_{m_1}$ and $L_{g^0}(h_1), \ldots, L_{g^0}(h_{m_1})$ there must be m_2 independent functions $h_{m_1+1}, \ldots, h_{m_1+m_2}$ annihilating it to order ρ.

One continues on in this fashion finding functions $h_1, \ldots, h_m$ annihilating the various $\mathscr{D}^k$'s. These functions and their Lie derivatives define the desired linearizing coordinates (3.1a)

$$x_j = L_{g^0}^{k-1}(h_i)$$

where $k_1 + \cdots + k_{i-1} < j \leqslant k_1 + \cdots + k_i$ and $k = j - k_1 - \cdots - k_{i-1}$. In these coordinates, (2.1) becomes

$$\dot{x}_j = \begin{cases} x_{j+1} + \mathrm{O}(\xi, \mu)^{\rho+1} \\ \quad \text{if } j \neq k_1 + \cdots + k_i, \\ L_{g^0}^{k_i}(h_i) + L_g L_{g^0}^{k_i-1}(h_i)\mu + \mathrm{O}(\xi, \mu)^{\rho+1} \\ \quad \text{if } j = k_1 + \cdots + k_i. \end{cases}$$

The linearizing feedback (3.1b) is given by

$$u_i = L_{g^0}^{k_i}(h_i) + L_g L_{g^0}^{k_i-1}(h_i).$$

References

[1] R.W. Brockett, Feedback invariants for nonlinear systems, *Proceedings, IFAC Congress*, Helsinki (1978).

[2] E. Freund, Fast nonlinear control with arbitrary pole-placement for industrial robots and manipulators, in: Bradley et al., Eds., *Robot Motion* (MIT Press, 1982) 147–167.

[3] L.R. Hunt and R. Su, Linear equivalents of nonlinear time varying systems, *Proceedings MTNS Symposium*, Santa Monica (1981) 119–123.

[4] B. Jakubczyk and W. Respondek, On the linearization of control systems, *Bull. Acad. Polon. Sci. Ser. Math. Astron. Physics* **28** (1980) 517–522.

[5] A.J. Krener, On the equivalence of control systems and the linearization of nonlinear systems, *SIAM J. Control* **11** (1973) 670–676.

[6] A.J. Krener, New approaches to the design of nonlinear compensators, *Proceedings of Berkeley Ames Conf. on Nonlinear Problems, Aerodynamics and Flight Control* (Math. Sci. Press, 1984).

[7] G. Meyer and L. Cicolani, A formal structure for advanced automatic flight control systems, NASA TN D-7940 (1975).