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NONLINEAR OBSERVERS WITH LINEARIZABLE ERROR DYNAMICS*

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Abstract. We present a new method for designing asymptotic observers for a class of nonlinear systems. The error between the state of the system and the state of the observer in appropriate coordinates evolves linearly and can be made to decay arbitrarily exponentially fast.

Key words. nonlinear estimation, nonlinear observer, observable and observer form

1. Introduction. The problem of approximating the state $x \in \mathbb{R}^n$ of a linear system

$$\dot{x} = Ax + Bu, \tag{1.1a}$$

$$y = Cx \tag{1.1b}$$

based on knowledge of the input $u \in \mathbb{R}^n$ and output $y \in \mathbb{R}^p$ has a well-known solution provided only that (C, A) be an observable pair, i.e.

$$\operatorname{rank}\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = n. \tag{1.2}$$

We define $z(t)$, an estimate of $x(t)$, to evolve according to the dynamics

$$\dot{z} = (A+GC)z - Gy + Bu \tag{1.3}$$

where G is an $n \times p$ matrix to be chosen. Then the error $e = x - z$ satisfies

$$\dot{e} = (A+GC)e. \tag{1.4}$$

The observability hypothesis (1.2) ensures that for any set of n complex numbers invariant under complex conjugation there exists a G so that the spectrum of $(A+GC)$ is that set. In particular G can be chosen so that the spectrum is sufficiently to the left in the complex plane so that error decays arbitrarily exponentially fast. See [6] for details.

In this paper we identify a class of nonlinear systems of the form

$$\dot{\xi} = f(\xi, u), \tag{1.5a}$$

$$\psi = h(\xi) \tag{1.5b}$$

for which there exists observers with arbitrary exponential error decay at least locally. We give necessary and sufficient conditions in the form of a constructive algorithm for there to exist changes of coordinates

$$\xi = \xi(x), \tag{1.6a}$$

$$\psi = \psi(y) \tag{1.6b}$$

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in a suitable domain transforming (1.5) into

$$\dot{x} = Ax + \gamma(y, u), \tag{1.7a}$$

$$y = Cx \tag{1.7b}$$

where (C, A) is an observable pair. If (C, A) is in dual Brunovsky canonical form we say that (1.7) is in observer form. A slight modification of (1.3) yields an observer for (1.6)

$$\dot{z} = (A + GC)z - Gy + \gamma(y, u) \tag{1.8}$$

with the same error dynamics (1.4) as before.

If we transform (1.8) back by (1.6), we obtain a differential equation for $\zeta(t) = \xi(z(t))$

$$\dot{\zeta} = \hat{f}(\zeta, \psi, u). \tag{1.9}$$

On a compact subdomain one can achieve arbitrary exponential decay of the error between $\xi(t)$ and $\zeta(t)$ by proper choice of G.

This paper grew out of earlier work of Krener–Isidori [1] who considered the above question when $p = 1$, $\psi = y$ and with no inputs. Essentially we shall reduce the more general question to the multi-output ($p \geqq 1$) version of that. In some loose sense the question which we address is the mathematical dual of that solved by Jakubcyzk-Respondek [2] and Hunt-Su [3]. They considered the problem of linearization of (1.5a) using change of coordinates in the state space and state dependent change of coordinates in the input space (nonlinear state feedback). We refer the reader to [1]-[3] for a fuller discussion of these points.

A referee called to our attention similar work of Bestle and Zeitz [7]. They assumed the existence of the linearizing transformations and showed how the observer could be constructed when $p = 1$.

The paper is organized as follows. Section 2 discusses the observability of a nonlinear system and § 3 develops a key necessary condition (Proposition 3.3), for the existence of an observer form. Section 4 is the heart of the paper, in which the two theorems which reduce the general problem to the multivariable version of [1] are presented. In § 5 the multivariable version of [1] is given. Sections 2–5 consider systems without inputs, while § 6 generalizes to systems with inputs. We close by a series of examples in § 7. The reader may wish to consult these immediately after reading the statements of Theorems 4.1, 4.2 and 5.1 and the associated remarks.

2. Observability. Consider the problem of estimating the current state $\xi(t)$ of the nonlinear system without inputs

$$\dot{\xi} = f(\xi), \qquad \xi \in \mathbb{R}^n, \tag{2.1a}$$

$$\psi = h(\xi), \qquad \psi \in \mathbb{R}^p, \tag{2.1b}$$

$$\xi^0 \approx \xi(0) \tag{2.1c}$$

from knowledge of the past outputs $\psi(s)$, $0 \leqq s \leqq t$, but with no knowledge of the initial state $\xi(0)$ except that it is near ξ^0. Later in § 6 we shall treat systems with inputs. We are not using the term "estimation" in a statistical sense although one could make additional assumptions about (2.1) and formulate the problem as such. Rather we desire that our estimate $\hat{\xi}(t)$ converge to $\xi(t)$ "sufficiently fast" as t increases. The initial displacement $\xi(0) - \xi^0$ represents an error in our current estimate of the state due to the accumulation of past disturbances. The estimate should converge fast enough so that the error becomes negligible in a short length of time and future disturbances are dampened at a rate faster than they arrive. On the other hand if one attempts too high a rate of decay, the inaccuracies of the observations (2.1b) can play havoc with the estimate.

The mathematical extreme of this approach is to estimate $\xi(t)$ by differentiating the output $\psi(t)$ several times. While this is not a practical approach, it does set a limit on the observability inherent in the model (2.1). Of course this requires that (2.1) be sufficiently differentiable or $\mathscr{C}^\infty$, we shall implicitly assume this throughout this paper. If from the knowledge of $\psi(t)$ and its derivatives at t one can uniquely determine $\xi(t)$, then (2.1) is observable.

To make this mathematically precise we must introduce some terminology. Let $\psi_i^{(j)}(t)$ denote the jth time derivative of the ith output. This can be expressed using Lie differentiation of the functions h_i by the vector field f,

$$\psi_i^{(j)}(t) = L_f^j(h_i)(\xi(t)). \tag{2.2}$$

$L_f^j(h_i)(\xi)$ is the jth Lie derivative of h_i by f and a function of ξ defined inductively by

$$L_f^0(h_i)(\xi) = h_i(\xi), \tag{2.3a}$$

$$L_f^j(h_i)(\xi) = \frac{\partial}{\partial \xi}(L_f^{j-1}(h_i)(\xi))f(\xi). \tag{2.3b}$$

The symbol $(\partial/\partial\xi)(h_i)$ stands for the gradient of the function h_i and is a $1 \times n$ vector valued function of ξ. It is the local coordinate description of the one form dh_i, which can also be Lie differentiated by f. For our purposes the following suffices as a definition

$$L_f(dh_i) = d(L_f(h_i)). \tag{2.4}$$

DEFINITION 2.1. This system (2.1) is *observable* at ξ^0 if there exists a neighborhood $\mathscr{U}$ of ξ^0 and p-tuple of integers $(k_1, \cdots, k_p)$ such that

(i) $k_1 \geqq k_2 \geqq \cdots \geqq k_p \geqq 0$ and $\sum_{i=1}^p k_i = n$.

(ii) After suitable reordering of the h_i's, at each $\xi \in \mathscr{U}$ the n row vectors $\{L_f^{j-1}(dh_i): i = 1, \cdots, p; j = 1, \cdots, k_i\}$ are linearly independent.

(iii) If $(l_1, \cdots, l_p)$ satisfies (i) and after suitable reordering the n row vectors $\{L_f^{j-1}(dh_i)(\xi): i = 1, \cdots, p; j = 1, \cdots, k_i\}$ are linearly independent at some $\xi \in \mathscr{U}$ then $(l_1, \cdots, l_p) \geqq (k_1, \cdots, k_p)$ in the lexographic ordering $[(l_1 > k_1)$ or $(l_1 = k_1$ and $l_2 > k_2)$ or $(l_1 = k_1, l_2 = k_2$ and $l_3 > k_3)$ or $\cdots$ or $(l_1 = k_1, \cdots, l_p = k_p)]$.

The integers $(k_1, \cdots, k_p)$ are called the observability indices at ξ^0.

This definition of observability is not the only one which has appeared in the literature. See [4] and [5] for alternatives. It is equivalent to being able to take the n functions $\{L_f^{j-1}(h_i): i = 1, \cdots, p; j = 1, \cdots, k_i\}$ as coordinates in a neighborhood of ξ^0 where no set of lower derivatives would suffice. If we abuse notation by letting $\xi_{ij} = L_f^j(h_i)(\xi)$, then (2.1) becomes

$$\begin{array}{lcl} \psi_1 = \xi_{11} & \cdots & \psi_p = \xi_{p1} \\ \dot{\xi}_{11} = \xi_{12} & & \dot{\xi}_{p1} = \xi_{p2} \\ \dot{\xi}_{12} = \xi_{13} & & \vdots \\ \vdots & & \dot{\xi}_{pk_p} = f_{p(\xi)} \\ \dot{\xi}_{1k_1} = f_{1(\xi)} & \cdots & \end{array} \tag{2.5}$$

where $f_i = L_f^{k_i}(h_i)$.

Following Kailath [6] we refer to (2.5) as a system in *observable form*. It is not a canonical form relative to the pseudo-group of state and output coordinate changes because different output coordinates (or even different ordering of the outputs) lead to different f_i's.

LEMMA 2.2. *The system* (2.1) *admits an observable form around* ξ^0 *iff* (2.1) *is observable at* ξ^0. *The observability indices at* ξ^0 *are the same as the* k_i*'s of any observable form* (2.5) *at* ξ^0.

Let us consider how one might verify Definition 2.1 and Lemma 2.2 for a system (2.1). Define $\mathscr{E}^0=\{0\}$ and

$$\mathscr{E}^k=\text{Span}\,\{L_f^{j-1}(dh_i)\colon i=1,\cdots,p;j=1,\cdots,k\}$$

where Span indicates all linear combinations over the $\mathscr{C}^\infty$ functions of ξ. Each $\mathscr{E}^k$ is a module of one forms over this ring of functions; such an object is called a *codistribution* or *Pfaffian system*. Let $E^k(\xi)$ denote the space of cotangent vectors obtained by evaluating the one forms of $\mathscr{E}^k$ at ξ. Each $E^k(\xi)$ can be thought of as a space of $1\times n$ vectors. Clearly $\mathscr{E}^{k-1}\subset\mathscr{E}^k$ and $E^{k-1}(\xi)\subset E^k(\xi)$. Let $d_k(\xi)$ denote the codimension of $E^{k-1}(\xi)$ in $E^k(\xi)$.

LEMMA 2.3. *The system* (2.1) *is observable at* ξ^0 *with observability indices* $(k_1,\cdots,k_p)$ *iff* $d_i(\xi)$ *is constant in a neighborhood of* ξ^0 *for* $i=1,\cdots,n$ *and* $d_n(\xi)=n$. *The relation between these sets of integers is given by*

$$d_k=\text{card}\,\{k_i\colon k_i\geqq k\}, \tag{2.6a}$$

$$k_i=\max\,\{k\colon d_k\geqq i\}. \tag{2.6b}$$

The proof amounts to an algorithm for transforming (2.1) to (2.5). It uses the fact that $\mathscr{E}^k$ is invariant under change of output and state coordinates.

Proof. Suppose (2.1) is observable with indices $(k_1,\cdots,k_p)$; then $E^k(\xi)$ has as a basis $\{L_f^{j-1}(dh_i)\colon i=1,\cdots,p;\,j=1,\cdots,\min(k_i,k)\}$ hence is of constant dimension.

On the other hand suppose $d_k(\xi)$ is constant for each k. After reordering the outputs we can assume that the first d_1 of the dh_i's are a basis for $E^1(\xi)$. We can reorder the first d_1 of the outputs so that $L_f(dh_i)$ $i=1,\cdots,d_2$ and dh_i, $i=1,\cdots,d_1$ are a basis for $E^2(\xi)$. We repeat the processes reordering the first d_2 of the outputs so that $L_f^2(dh_i)$: $i=1,\cdots,d_3$ and the previous chosen basis for $E^2(\xi)$ forms a basis for $E^3(\xi)$. In this way we obtain n linearly independent exact one forms. The corresponding functions are the desired coordinates ξ_{ij}. Q.E.D.

3. Necessary conditions. While observable form is useful for deciding the observability of a system, it is not particularly helpful in constructing an observer. Suppose there exist changes of coordinates $x=x(\xi)$ and $y=y(\psi)$ around ξ^0 and $\psi^0=h(\xi^0)$ which transform (2.1) into *observer form*

$$\begin{array}{lcl}
y_1=x_1 & \cdots & y_p=x_{p1}\\
\dot x_{11}=x_{12}+\alpha_{11}(y) & & \dot x_{p1}=x_{p2}+\alpha_{p1}(y)\\
\dot x_{12}=x_{13}+\alpha_{12}(y) & & \vdots\\
\vdots & & \dot x_{pk_p}=\alpha_{pk_p}(y)\\
\dot x_{1k_1}=\alpha_{1k_1}(y) & \cdots &
\end{array} \tag{3.1}$$

The construction of an observer for (3.1) is straightforward. Let z_{ij} evolve according to

$$
(3.2)\qquad
\begin{aligned}
&\dot z_{11} = z_{12} + \alpha_{11}(y) + q_{11}(y_1 - z_{11}) \quad \cdots \quad && \dot z_{p1} = z_{p2} + \alpha_{p1}(y) + q_{p1}(y - z_{p1})\\
&\dot z_{12} = z_{13} + \alpha_{12}(y) + q_{12}(y_1 - z_{11}) && \vdots\\
&\vdots && \dot z_{pk_p} = \alpha_{pk_p}(y) + q_{pk_p}(y_p - z_{p1})\\
&\dot z_{1k_1} = \alpha_{1k_1}(y) + q_{1k_1}(y_1 - z_{11}) \quad \cdots &&
\end{aligned}
$$

where q_{ij} are constants to be chosen.

If $e_{ij} = x_{ij} - z_{ij}$, then

$$
(3.3)\qquad
\begin{aligned}
&\dot e_{11} = e_{12} - q_{11}e_{11} \quad \cdots \quad && \dot e_{p1} = p_{p2} - q_{p1}e_{p1}\\
&\dot e_{12} = e_{13} - q_{12}e_{11} && \vdots\\
&\vdots && \dot e_{pk_f} = -q_{pk_p}e_{p1}\\
&\dot e_{1k_1} = -q_{1k_1}e_{11} \quad \cdots &&
\end{aligned}
$$

The characteristic polynomial of this linear system is

$$
(3.4)\qquad p(\lambda) = \prod_{i=1}^{p} \left(\sum_{j=0}^{k_i} q_{ij}\lambda^{k_i - j} \right)
$$

where $q_{i0} = 1$. Clearly we can set the spectrum arbitrarily so that the error decays exponentially fast at any desired rate.

It is well known that every observable linear system can be transformed into observer form where α_{ij} is linear in y by linear coordinate changes in the state and output [6]. However, even if we allow nonlinear coordinate changes and nonlinear α_{ij}, the analogous result for nonlinear systems does not hold.

PROPOSITION 3.1. *If the system* (2.1) *admits an observer form* (3.1) *at* ξ^0, *it must be observable at* ξ^0 *with observability indices given by the* k_i*'s of* (3.1).

Proof. Let $\Psi = (\partial\psi_i/\partial y_j)$; then

$$
d\psi = \begin{pmatrix} d\psi_1 \\ \vdots \\ d\psi_p \end{pmatrix} = \Psi\, dy = \Psi \begin{pmatrix} dy_1 \\ dy_p \end{pmatrix},
$$

so

$$
\mathscr{E}^1 = \text{Span}\{dy_1, \cdots, dy_p\} = \text{Span}\{dx_{11}, \cdots, dx_{p1}\}.
$$

Assume by induction that

$$
L_f^{j-2}(d\psi) \equiv \Psi L_f^{j-2}(dy) \qquad \text{mod } \mathscr{E}^{j-2};
$$

then

$$
\begin{aligned}
L_f^{j-1}(d\psi) &\equiv \Psi L_f^{j-1}(dy) + L_f(\Psi) L_f^{j-2}(dy) && \text{mod } L_f(\mathscr{E}^{j-2}) = \mathscr{E}^{j-1},\\
L_f^{j-1}(d\psi) &\equiv \Psi L_f^{j-1}(dy) && \text{mod } \mathscr{E}^{j-1}.
\end{aligned}
$$

But

$$
L_f^{j-1}(dy_i) \equiv \begin{cases} dx_{ij+1} & \text{if } j+1 \leqq k_i, \\ 0 & \text{if } j+1 > k_i, \end{cases} \quad \text{mod } \mathscr{E}^{j-1}
$$

so $\mathscr{E}^j$ is spanned by dx_{ij}, $j \leqq k_i$, mod $\mathscr{E}^{j-1}$. From this we see that the dimensions of $E^j(\xi)$ are constant. Q.E.D.

DEFINITION 3.2. Suppose the nonlinear system (2.1) is in observable form (2.5). We denote $\mathcal{P}(\xi)$ the ring of polynomials in ξ with coefficients that are $\mathcal{C}^\infty$ function of ψ. The degree of ξ_{ij} is defined to be $j-1$ and the degree of the monomial $\xi_{i_1 j_1}\cdot\cdots\cdot\xi_{i_n j_n}$ is the sum of the degrees of its factors, $(j_1-1)+\cdots+(j_r-1)$. $\mathcal{P}^k(\xi)$ denotes the polynomials of degree k or less and $\mathcal{P}_0^k(\xi)$ those polynomials of $\mathcal{P}^k(\xi)$ which are generated by elements of $\mathcal{P}^{k-1}(\xi)$. In particular, ξ_{ik+1} is in $\mathcal{P}^k(\xi)$ but not in $\mathcal{P}_0^k(\xi)$.

With this terminology available we can introduce our second necessary condition.

PROPOSITION 3.3. *Suppose the system* (2.1) *admits an observer form at* ξ^0; *then in any observable form* (2.5) *the functions* $f_i(\xi)$ *are in* $\mathcal{P}^{k_i}(\xi)$ *for* $i=1,\cdots,p$.

Proof. Let $\mathcal{P}(x)$ denote the polynomials in x with coefficients that are $\mathcal{C}^\infty$ functions of y; in a similar fashion we define the degree of x_{ij} to be $j-1$. $\mathcal{P}^k(x)$ are the polynomials of degree $\leqq k$ and $\mathcal{P}_0^k(x)$ the subset of $\mathcal{P}^k(x)$ generated by elements of $\mathcal{P}^{k-1}(x)$.

It is easy to see that $L_f(\mathcal{P}^{k-1}(x))\subset\mathcal{P}^k(x)$ and $L_f(\mathcal{P}_0^{k-1}(x))\subset\mathcal{P}_0^k(x)$. For example x_{ik}, $k\leqq k_i$, is of degree $k-1$ and

$$L_f(x_{ik})=\dot{x}_{ik}=\begin{cases}x_{ik+1}+\alpha_{ik}(y), & k<k_i,\\ \alpha_{ik}(y), & k=k_i\end{cases}$$

is clearly of degree at most k. A similar calculation using the Leibniz rule shows that monomials of degree $k-1$ go into monomials of degree k under Lie differentiation by f.

Notice that the changes of coordinates transform $\mathcal{P}^0(\xi)$ onto $\mathcal{P}^0(x)$, i.e., $\mathcal{P}^0(\xi(x))=\mathcal{P}^0(x)$ and $\mathcal{P}^0(\xi)=\mathcal{P}^0(\xi(x))$. Moreover $\mathcal{P}_0^1(\xi)=\mathcal{P}^0(\xi)$ and $\mathcal{P}_0^1(x)=\mathcal{P}^0(x)$ so $\mathcal{P}_0^1(\xi)$ is transformed into $\mathcal{P}_0^1(x)$.

We show induction that $\mathcal{P}^k(\xi)$ is transformed to $\mathcal{P}^k(x)$ and $\mathcal{P}_0^{k+1}(\xi)$ is transformed to $\mathcal{P}_0^{k+1}(x)$ for all k. If $k_i\geqq 2$, then

$$\xi_{i2}=\dot{\xi}_{i1}=\sum_{j=1}^{p}\frac{\partial\psi_i}{\partial y_j}\dot{x}_{j1},$$

(3.5)

$$\xi_{i2}=\sum_{k_j\geqq 2}\frac{\partial\psi_i}{\partial y_j}x_{j2}+p_{i2}(x)$$

where

$$p_{i2}(x)=\sum_{j=1}^{p}\frac{\partial\psi_i}{\partial y_j}\alpha_{j2}(y)\in\mathcal{P}_0^1(x)=\mathcal{P}^0(x).$$

This proves the above statement for $k=1$.

Suppose it is true for $k-1$ and suppose also that the generalization of (3.5) holds, i.e., if $k_i\geqq k$

$$\xi_{ik}=\sum_{k_j\geqq k}\frac{\partial\psi_i}{\partial y_j}x_{jk}+p_{ik}(x), \tag{3.6}$$

where $p_{ik}(x)\in\mathcal{P}_0^{k-1}(x)$. If $k_i\geqq k+1$, then

$$\xi_{ik+1}=L_f(\xi_{ik})=\xi_{ik}=\sum_{k_j\geqq k+1}\frac{\partial\psi_i}{\partial y_j}x_{jk+1}+p_{ik+1}(x) \tag{3.7}$$

where

$$p_{ik+1}(x)=\sum_{k_j\geqq k}\left(\frac{\partial\psi_i}{\partial y_j}\alpha_{jk+1}+L_f\left(\frac{\partial\psi_i}{\partial y_j}\right)x_{jh}\right)+L_f(p_{ik}(x)), \tag{3.8}$$

and hence $p_{ik+1}(x)\in\mathcal{P}_0^k(x)$. From (3.7) the statement follows for $k+1$. Q.E.D.

Actually we can deduce a slightly stronger result from the above argument.

PROPOSITION 3.4. *If a system* (2.1) *admits an observer form* (3.1) *around* ξ^0, *then it admits an observable form* (2.5) *around* ξ^0 *which satisfies* $f_i(\xi) \in \mathcal{P}_0^{k_i}(\xi)$, $i = 1, \cdots, p$.

Proof. Suppose $y = \psi$. Differentiating (3.6) with $k = k_i$ yields

$$f_i(\xi) = \dot{\xi}_{ik_i} = L_f(\xi_{ik_i}) = \sum_{k_j \geqq k_i} \frac{\partial \psi_i}{\partial y_j} \dot{x}_{jk_i} + p_{ik+1}(x) \tag{3.9}$$

where $p_{ik+1}(x) \in \mathcal{P}_0^{k_i}(x)$ is given by (3.8). Since $\psi_i = y_i$ and $x_{ik_i} = \alpha_{ik_i} \in \mathcal{P}^0(x)$, the result follows. Q.E.D.

DEFINITION 3.5. A system (2.5) is in *special observable form* if $f_i(\xi) \in \mathcal{P}_0^{k_i}(\xi)$, $i = 1, \cdots, p$.

Of course a system need not have a special observable form and such forms are not always unique. As we shall see in the next section, they are a very useful intermediate step between the observable and observer forms. We will also give necessary and sufficient conditions for the existence of a special observable form. Notice that if $k_1 = \cdots = k_p$ (e.g. $p = 1$), then any observable form satisfying Proposition 3.3 is special. This is because $\mathcal{P}_0^{k_i}(\xi) = \mathcal{P}^{k_i}(\xi)$.

4. Change of output coordinates and prolongation. Consider a system satisfying the two necessary conditions of Propositions 3.1 and 3.3, namely, that it can be transformed to observable form (2.5) where $f_i(\xi) \in \mathcal{P}^{k_i}(\xi)$. Suppose we take the obvious approach and compare (2.5) and (3.1) to obtain differential equations for $\psi(y)$ and $\alpha(y) = (\alpha_{ij}(y))$. For simplicity assume $p = 1$ and hence $k_1 = n$. This approach yields

$$\begin{aligned} \xi_1 &= \psi(x_1), \\ \xi_2 &= \dot{\xi}_1 = \frac{d\psi}{dx_1}(x_2 + \alpha_1), \\ \xi_3 &= \dot{\xi}_2 = \frac{d\psi}{dx_1}\left(x_3 + \alpha_2 + \frac{d\alpha_1}{dx_1}(x_2 + \alpha_1)\right) + \frac{d^2\psi}{dx_1}(x_2 + \alpha_1), \\ &\vdots \\ f_n(\xi) &= \dot{\xi}_n = \frac{d\chi}{dx_1}(\alpha_n + \cdots) + \cdots + \frac{d^n\psi}{dx_1^n}(x_2 + \alpha_1)^n. \end{aligned} \tag{4.1}$$

The result is an nth order system of nonlinear ordinary differential equations for the $1 + n$ unknowns $\psi, \alpha_1, \cdots, \alpha_n$. If $p > 1$ the situation is even worse, for we obtain a k_1th order system of nonlinear partial differential equations for the $p + n$ unknown ψ_i and α_{ij}. Clearly a better approach is needed for all but the smallest value of p and n.

Our approach will be to separate the problem into two parts. The first step is to derive a first order linear differential equation which essentially determines the change of output coordinates $\psi(y)$ if it exists. Once we have this, then we can use the method of Krener and Isidori [1] to decide if the system can be transformed into observer form and to compute the change of coordinates and $\alpha(y)$. This latter task we postpone to § 5. The rest of this section will be devoted to proving the following.

THEOREM 4.1. *Consider a system in special observable form. If it can be transformed to observer form, then the Jacobian* $\Psi = (\Psi_i^j) = (\partial\psi_i/\partial y_j)$ *of the change of output coordinates must satisfy*

$$\Psi_i^j = 0 \quad \text{if } k_j > k_i \tag{4.2a}$$

and

$$\frac{\partial}{\partial \psi_l} \Psi_i^j = \frac{1}{k_i} \sum_{r=1}^{p} f_{i;lk_i;r2} \Psi_r^j. \tag{4.2b}$$

We can normalize $\Psi(\psi)$ by specifying that

$$\Psi(\psi^0) = I. \tag{4.2c}$$

A different initial condition amounts to a linear change of state and output coordinates.

Notation. We are using the semicolon to denote partial differentiation, e.g.,

$$f_{i;sk_i} = \frac{\partial f_i}{\partial \xi_{sk_i}}, \qquad f_{i;sk_i;l2} = \frac{\partial^2 f_i}{\partial \xi_{l2}\, \partial \xi_{sk_i}}.$$

Remark 4.1. Note that $f_{i;sk_i;l2} \in \mathcal{P}^0(\xi)$ since the system is assumed to be in special observable form. Therefore the differential equation (4.2b) lives not on the state space but on the output space as it should.

Remark 4.2. The equations (4.2b) are locally solvable iff the mixed partial conditions are satisfied

$$\frac{\partial}{\partial \psi_n} \frac{\partial}{\partial \psi_l}(\Psi_i^j) = \frac{\partial}{\partial \psi_l} \frac{\partial}{\partial \psi_n}(\Psi_i^j).$$

Since Ψ must be invertible, this reduces to

$$f_{i;lk_i;r2;m1} + \sum_{s=1}^{p} \frac{1}{k_2} f_{i;lk_i;s2} f_{s;mk_s;r2} = f_{i;mk_i;r2;l1} + \sum_{s=1}^{p} \frac{1}{k_s} f_{i;mk_i;s2} f_{s;lk_s;r2} \tag{4.3}$$

for $i, m, l, r = 1, \cdots, p$.

Moreover, a solution of (4.2b) need not automatically satisfy (4.2a). This imposes additional necessary conditions on the f_i's, namely that their partials have the same block upper triangular structure as Ψ, i.e.

$$f_{i;lk_i;j2} = 0 \quad \text{if } k_j > k_i. \tag{4.4}$$

Remark 4.3. Suppose Ψ is a solution to (4.2); then $\psi(y)$ satisfies

$$\frac{\partial \psi_i}{\partial y_j} = \Psi_i^j. \tag{4.5}$$

Equation (4.5) is integrable iff the mixed partials commute. This is equivalent to

$$f_{i;sk_i;r2} = f_{i;rk_i;s2}. \tag{4.6a}$$

It is useful to choose the solution so that ψ^0 transforms to $y^0 = 0$, i.e.,

$$\psi(0) = \psi^0 = h(\xi^0). \tag{4.6b}$$

Remark 4.4. Proposition 4.1 deals with a system in special observable form. Of course any system which can be transformed into observer form must satisfy $f_i(\xi) \in \mathcal{P}^{k_i}(\xi)$ in any observable form. To bring it to special observable form requires a change of output coordinates $\tilde{\psi} = \tilde{\psi}(\psi)$ as described below.

Let $Y^1(\psi), \cdots, Y^p(\psi)$ be vector fields defined on the output space whose coordinate descriptions relative to ψ are given by

$$L_{Y^j}(\psi_i) = \begin{cases} 1 & \text{if } i = j, \\ f_{i;jk_i+1} & \text{if } k_j > k_i, \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

Notice that $f_{i;jk_i+1}$ is a function of ψ alone since $f_i(\xi) \in \mathcal{P}^{k_i}(\xi)$.

THEOREM 4.2. *Consider a system is observable form satisfying $f_i(\xi) \in \mathcal{P}^{k_i}(\xi)$. It is in special observable form relative to the transformed output coordinates $\tilde{\psi} = \tilde{\psi}(\psi)$ iff*

$$L_{Y^j}(\tilde{\psi}_i) = 0 \quad \text{for } k_j > k_i. \tag{4.8}$$

There exists such a change of coordinates $\tilde{\psi}(\psi)$ iff the distributions

$$\mathcal{U}^i = \text{Span}\,\{Y^j(\psi): k_j > k_i\}$$

are involutive for $i = 1, \cdots, p$.

Recall that a *distribution* is a family of vector fields closed under addition and multiplication by $\mathcal{C}^\infty$ functions. It is *involutive* if the Lie bracket of any two vector fields from the distribution is again in the distribution. The Lie bracket is defined in local coordinates by

$$[Y^i, Y^j] = \frac{\partial Y^j}{\partial \psi} Y^i - \frac{\partial Y^i}{\partial \psi} Y^j;$$

note that it is again a vector field on the same space, in this case the output space.

Proof of Theorem 4.1. We start with a system in special observable form. For the time being assume all of the observability indices are the same, $k_1 = k_2 = \cdots = k_p = k$, so that $n = p \cdot k$. Let $g^j(\xi)$ be the vector field on the state space which is the unit vector in the ξ_{jk}-direction, for $j = 1, \cdots, p$. Equivalently these p vector fields are characterized by the equations

$$L_{g^j} L_f^l(\psi_i) = \begin{cases} 0, & 0 \leqq l < k-1, \\ \delta_i^j, & l = k-1, \end{cases} \tag{4.9a}$$

where δ_i^j is the Kronecker δ symbol.

We introduce the *ad*-notation for repeated Lie brackets

$$ad^0(-f)g^j = g^j, \qquad ad^{l+1}(-f)g^j = [-f, ad^l(-f)g^j]$$

and the pairing of a one-form $\omega(\xi)$ and vector field $X(\xi)$

$$\langle \omega, X\rangle(\xi) = \omega(\xi) X(\xi).$$

In local coordinates the right side is the product of $1 \times n$ and $n \times 1$ vector valued functions of ξ. The Leibniz formula holds for this pairing under Lie differentiation

$$L_f\langle \omega, X\rangle = \langle L_f(\omega), X\rangle + \langle \omega, [f, X]\rangle.$$

Therefore (4.9a) is equivalent to

$$\begin{aligned} \langle L_f^{l-r}(d\psi_i), ad^r(-f)g^j\rangle &= L_{ad^r(-f)g^j} L_f^{l-r}(d\psi_i) \\ &= \begin{cases} 0, & 0 \leqq r \leqq l < k-1, \\ \delta_i^j, & 0 \leqq r \leqq l = k-1. \end{cases} \end{aligned} \tag{4.9b}$$

Suppose the system can be transformed into observer form by change of state and output coordinates. Let B^j be the vector field on the state space which is the unit vector in the x_{jk} direction and $\bar{g}^j(\xi)$ be the representation of this vector field in ξ coordinates. Let $\bar{f}(\xi)$ and $\bar{\alpha}(\xi)$ be the representations in ξ coordinates of the vector fields represented in x coordinates by Ax and $\alpha(y)$ where Ax is the linear part of the right side of the differential equation of (3.1). In other words if x is the x_{ij}'s in lexographic ordering, then

(4.10a)
$$A=\left(\begin{array}{ccc|c|ccc} 0 & 1 & & & & & \\ & \ddots & 1 & 0 & & 0 & \\ & & 0 & & & & \\ \hline & 0 & & \ddots & & 0 & \\ \hline & & & & 0 & 1 & \\ & 0 & & 0 & & \ddots & 1 \\ & & & & & & 0 \end{array}\right).$$

The output is given by $y=Cx$ and $B=(B^1,\cdots,B^p)$ where

(4.10b)
$$B=\left(\begin{array}{c|c|c} 0 & & \\ \vdots & 0 & 0 \\ 0 & & \\ 1 & & \\ \hline 0 & \ddots & \\ \hline & & 0 \\ 0 & 0 & \vdots \\ & & 0 \\ & & 1 \end{array}\right).$$

(4.10c)
$$C=\left(\begin{array}{c|c|c} 1\ 0\ \cdots\ 0 & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & & 1\ 0\ \cdots\ 0 \end{array}\right).$$

The ith diagonal blocks of A, B and C are of dimensions $k_i\times k_i$, $k_i\times 1$ and $1\times k_i$ respectively.

Viewing $y_i=y_i(\xi)$ as a function of ξ we have

$$\begin{aligned} L_{\bar g^j}L_{\bar f}^{l}(y_i) &= \langle L_{\bar f}^{l-r}(dy_i),\ ad^r(-\bar f)\bar g^j\rangle \\ &= L_{ad^r(-\bar f)\bar g^j}L_{\bar f}^{l-r}(dy_i) \\ &= \begin{cases} 0, & 0\leqq r\leqq l<k-1, \\ \delta_i^j, & 0\leqq r\leqq l=k-1. \end{cases} \end{aligned}$$

From the proof of Proposition 3.1 we see that both $g(\xi)=(g^1(\xi),\cdots,g^p(\xi))$ and $\bar g(\xi)=(\bar g^1(\xi),\cdots,\bar g^p(\xi))$ annihilate the codistribution $\mathscr{E}^{k-1}$ which is of codimension p. Moreover

$$L_f^{k-1}(d\psi)\equiv \Psi L_{\bar f}^{k-1}(dy)\quad \operatorname{mod}\ \mathscr{E}^{k-1}$$

where $\Psi=(\Psi_i^j)=(\partial\psi_i/\partial y_j)$, hence $\langle L_f^{k-1}(d\psi),\bar g(\xi)\rangle=\Psi$ and so

(4.10)
$$\bar g=g\Psi.$$

Next we show by induction that

(4.11)
$$ad^l(-f)\bar g^j=ad^l(-\bar f)\bar g^j\quad \text{for } j=1,\cdots,p,\quad l=0,\cdots,k-1.$$

Suppose (4.11) holds for $l-1$; then since $f(\xi)=\bar f(\xi)+\bar\alpha(\xi)$, we have

$$\begin{aligned} ad^l(-f)\bar g^j &= -[\bar f+\bar\alpha,\ ad^{l-1}(-\bar f)\bar g^j] \\ &= ad^l(-\bar f)\bar g^j-[\bar\alpha,\ ad^{l-1}(-\bar f)\bar g^j]. \end{aligned}$$

In x coordinates $ad^{l-1}(-\bar{f})\bar{g}^j = A^{l-1}B^j$ and $-[\bar{a}, ad^{l-1}(\bar{f})\bar{g}^j]$ is $(\partial\alpha/\partial y)CA^{l-1}B=0$. Moreover since $ad^l(-f)\bar{g}^s(\xi)$ is the constant vector field A^lB^s in x coordinates for $s=1,\cdots,p$ and $l=0,\cdots,k-1$, it follows that

$$[ad^j(-f)\bar{g}^r, ad^l(-f)\bar{g}^s]=0 \tag{4.12}$$

for $r, s=1,\cdots,p$ and $j, l=0,\cdots,k-1$. (Note: A^l is the lth power of A, B^s the sth column of B.)

The Leibniz rule applied to (4.10) yields

$$ad^{k-1}(-f)\bar{g}^s = \sum_{l=1}^{k-1}\sum_{\sigma=1}^{s}\binom{k-1}{l} ad^l(-f)g^\sigma L_{-f}^{k-1-l}(\Psi^s_\sigma), \tag{4.13a}$$

$$ad^{k-2}(-f)\bar{g}^r = \sum_{j=1}^{k-2}\sum_{\rho=1}^{r}\binom{k-2}{j} ad^j(-f)g^\rho L_{-f}^{k-2-j}(\Psi^r_\rho). \tag{4.13b}$$

From (4.12) we see that

$$0=\langle L_f(d\psi_i), [ad^{k-2}(-f)\bar{g}^r, ad^{k-1}(-f)\bar{g}^s]\rangle \tag{4.14}$$

and if we expand the right side using (4.13), most of the terms drop out because of (4.9). We are left with

$$\begin{aligned}0= \sum_{\rho,\sigma=1}^{p} \{&\langle L_f(d\psi_i), [ad^{k-2}(-f)g^\rho, ad^{k-1}(-f)g^\sigma]\rangle\Psi^r_\rho\Psi^s_\sigma\\ &+(k-1)\langle L_f(d\psi_i), ad^{k-2}(-f)g^\sigma\rangle L_{ad^{k-2}(-f)g^\rho}L_{-f}(\Psi^s_\sigma)\Psi^r_\rho\\ &-\langle L_f(d\psi_i), ad^{k-2}(-f)g^\rho\rangle L_{ad^{k-1}(-f)g^\sigma}(\Psi^r_\rho)\Psi^s_\sigma\}.\end{aligned} \tag{4.15}$$

From (4.13) and the identity $L_f^k(d\psi_i)=f_i$ we obtain

$$\begin{aligned}\sum_{\rho,\sigma=1}^{p} f_{i;\sigma k;\rho 2}\Psi^r_\rho\Psi^s_\sigma &= \sum_{\rho,\sigma=1}^{p}\left\{(k-1)\frac{\partial}{\partial\psi_\rho}(\Psi^s_\sigma)\Psi^r_\rho+\frac{\partial}{\partial\psi_\sigma}(\Psi^r_\rho)\Psi^s_\sigma\right\}\\ &=(k-1)\sum_{\rho=1}^{p}\frac{\partial^2\psi_\rho}{\partial y_r\,\partial y_s}+\sum_{\sigma=1}^{p}\frac{\partial^2\psi_\sigma}{\partial y_s\,\partial y_r}\\ &=k\sum_{\rho,\sigma=1}^{p}\frac{\partial\Psi^r_\rho}{\partial\psi_\sigma}\Psi^s_\sigma.\end{aligned} \tag{4.16}$$

Multiplication by Ψ^{-1} yields the desired result (4.2).

Now suppose the observability indices are not all the same $k_1 \geqq \cdots \geqq k_p$. By hypothesis the system is in special observable form relative to the output ψ. Moreover by the proof of Proposition 3.4 the system will also be in special observable form relative to the output y.

Theorem 4.2 implies that the change of coordinates $y=y(\psi)$ satisfies the equations (4.8) with $y=\tilde{\psi}$, i.e.,

$$\frac{\partial y_i}{\partial\psi_j}=0 \quad \text{for } k_j>k_i$$

because $f_{i;jk_i+1}=0$. This implies (4.2a).

To show (4.2b) we prolong the system, i.e., define a new system similar to the old but with all observability indices equal to the largest index k_1 of the old. We do this in such a way that the new system is transformable to observer form by ψ and α iff the original system is also. Moreover the form of the differential equations (4.2b) for Ψ is left invariant.

In order to simplify the exposition, we will restrict to the case where there are two distinct observability indices $k_1 = k$ and $k_2 = k-1$ of multiplicities p_1 and p_2. The general case follows by repeated application of this technique. Let y_1 denote the first p_1 ouptuts and y_2 the last p_2 outputs; each ξ_{ij} is a p_i vector, etc. The original system and its transformed version are

$$\psi_i = \xi_{i1}, \qquad y_i = x_{i1},$$

$$\text{(4.17)} \qquad \dot{\xi}_{ij} = \begin{cases} \xi_{ij+1}, & 1 \leqq j < k_i, \\ f_i, & j = k_i, \end{cases} \qquad \dot{x}_{ij} = \begin{cases} x_{ij+1} + \alpha_{ij}(y), & 1 \leqq j \leqq k_i, \\ \alpha_{ik_i(y)}, & j = k_i, \end{cases}$$

for $i = 1$ and 2.

The prolonged system and its transformed version are in different variables but the same function $\psi(\cdot)$ and $\alpha(\cdot)$ should accomplish the transformation,

$$\bar{\psi}_i = \bar{\xi}_{i1}, \qquad \bar{y}_i = \bar{x}_{i1},$$

$$\text{(4.18)} \qquad \dot{\bar{\xi}}_{ij} = \begin{cases} \bar{\xi}_{ij+1}, & 1 \leqq j < k_1, \\ \bar{f}_i, & j = k_1, \end{cases} \qquad \dot{\bar{x}}_{ij} = \begin{cases} \bar{x}_{ij} + \alpha_{ij}(\bar{y}), & 1 \leqq j \leqq k_1, \\ \alpha_{ik}(\bar{y}), & j = k_1, \end{cases}$$

where

$$\text{(4.19a)} \qquad \alpha_{2k} = 0,$$

$$\text{(4.19b)} \qquad \bar{f}_1 = f_1,$$

$$\text{(4.19c)} \qquad \bar{f}_2 = \frac{\partial}{\partial \psi_2}(\Psi_2^2)\bar{\xi}_{22}\Psi_2^{2^{-1}}(\bar{\xi}_{2k_1} - \bar{f}_2) + \sum_{i=1}^{2}\sum_{j=1}^{k_i} f_{2,ij}\bar{\xi}_{ij+1}.$$

Of course the functions on the right side of (4.19) are to be evaluated on the new (barred) variables. Recall also that by Proposition 4.2

$$\Psi_2^1 = \frac{\partial \psi_2}{\partial y_1} = 0.$$

The claim is that (4.17) holds if (4.18) does. To see this notice that the straightforward approach (4.1) described at the beginning of this section yields almost the same set of differential equations. The only difference occurs at the k_2 and $k_1 = k_2 + 1$ time derivatives of ψ_2 and $\bar{\psi}_2$. At the k_2th derivative we have

$$\text{(4.20a)} \qquad f_2(\xi) = \dot{\xi}_{2k_2} = \Psi_2^2(\alpha_{2k_2} + \cdots) + \cdots,$$

$$\text{(4.20b)} \qquad \bar{\xi}_{2k_1} = \dot{\bar{\xi}}_{2k_2} = \Psi_2^2(\bar{x}_{2k_1} + \alpha_{2k_2} + \cdots).$$

Assuming that (4.17) holds and (4.18) holds up to this equation, then comparing (4.20) and the earlier equations yields

$$\text{(4.21)} \qquad \bar{\xi}_{2k_1} = \Psi_2^2\bar{x}_{2k_1} + f_2(\bar{\xi}).$$

Now (4.18) will hold if the derivative of this is consistent with

$$\text{(4.22)} \qquad \dot{\bar{\xi}}_{2k_1} = \bar{f}_2(\bar{\xi}).$$

But differentiating (4.21) yields

$$\text{(4.23)} \qquad \dot{\bar{\xi}}_{2k_1} = \frac{\partial}{\partial \psi_2}(\Psi_2^2)\bar{\xi}_{22}x_{2k_1} + \sum_{i=1}^{2}\sum_{2j=1}^{k_i} f_{2;ij}\xi_{ij+1}$$

as desired.

On the other hand suppose (4.18) holds. The differential equation on the right restricts to the hyperplane given by $x_{2k_1}=0$. This transforms to the hypersurface given by $\bar{\xi}_{2k_1}=f_2(\bar{\xi})$. The restricted systems are precisely those of the original (4.17). Q.E.D.

This completes the proof of Theorem 4.1 in the course of which we have used Theorem 4.2.

Proof of Theorem 4.2. We start with a system in observable form satisfying $f_i(\xi)\in\mathcal{P}^{k_i}(\xi)$ and at least two distinct observability indices, (if the observability indices are all the same there is nothing to prove.) Similar to before we define vector fields $g^1(\xi),\cdots,g^p(\xi)$ to be the unit vectors in the $\xi_{1k_1},\cdots,\xi_{pk_p}$ directions, i.e.,

$$L_{g^j}L_f^l(\psi_i)=\begin{cases}0, & 0\leqq l<k_i-1,\\ \delta_i^j, & l=k_i-1.\end{cases}\tag{4.24}$$

As we noted before in § 2 the codistributions

$$\mathscr{E}^k=\text{Span}\,\{L_f^l(dh_i)\colon 1\leqq i\leqq p;0\leqq l<k\}$$

are invariant under changes of state and output coordinates. The vector fields $g^1,\cdots,g^p$ and their brackets under f span the dual distributions, $\mathscr{D}^k=\mathscr{E}^{k\perp}$, given by

$$\mathscr{D}^k=\text{Span}\,\{ad^l(-f)g^j\colon k_j>k, 0\leqq l<k_j-k\};$$

hence these are also invariant. Moreover the distribution obtained by bracketing the elements of $\mathscr{D}^{k_i}$ with f up to k_i times is also invariant; we denote this by $\tilde{\mathscr{D}}^{k_i}$

$$\tilde{\mathscr{D}}^{k_i}=\{ad^l(-f)g^j\colon k_j>k_i;0\leqq l<k_j\}.$$

It is straightforward to verify that

$$L_{ad^l(-f)g^j}(\psi_i)=\begin{cases}1 & \text{if } i=j \text{ and } l=k_j-1,\\ f_{i:jk_i+1} & \text{if } k_j>k_i \text{ and } l=k_j-1,\\ 0 & \text{otherwise,}\end{cases}$$

so the image of $\tilde{\mathscr{D}}^{k_i}$ under $d\psi$ is precisely the distribution $\mathscr{Y}^i$ on the output space. This shows that $\mathscr{Y}^i$ is independent of the output coordinates.

Now suppose we wish to choose output coordinates $\tilde{\psi}$ so that relative to these coordinates we have the special observable form; then $d\tilde{\psi}_i$ must annihilate $\mathscr{Y}^i$. Hence $\tilde{\psi}_i$ must satisfy (4.8).

But $\mathscr{Y}^1\subseteq\mathscr{Y}^2\subseteq\cdots\subseteq\mathscr{Y}^p$ so $d\tilde{\psi}_i,\cdots,d\tilde{\psi}_p$ also must annihilate the $p-i$ dimensional distribution $\mathscr{Y}^i$; hence $\mathscr{Y}^i$ must be involutive.

On the other hand if each $\mathscr{Y}^i$ is involutive, then we can choose p independent functions $\tilde{\psi}_1,\cdots,\tilde{\psi}_p$ such that $d\tilde{\psi}_i\perp\mathscr{Y}^i$. This is the desired output coordinate change. Q.E.D.

5. Sufficient condition. Let us review the previous sections. We start with a nonlinear system (2.1) around some nominal operating point ξ^0 for which we desire to build an observer. We first check that it is observable at ξ^0 by attempting to transform it into observable form (2.5); then we check if $f_i(\xi)\in\mathcal{P}^{k_i}(\xi)$ as described in Proposition 3.3. Next we attempt to make a change of output coordinates to get it into special observable form as described in Theorem 4.2. If this can be achieved, then we attempt to solve equation (4.6) using Theorem 4.1 to find the output $y=y(\psi)$. If we are able to accomplish all of this, we have the system in the form

$$\dot{\xi}=f(\xi),\tag{5.1a}$$

$$y=h(\xi),\tag{5.1b}$$

$$\xi(0) \approx \xi^0 \tag{5.1c}$$

which we wish to transform by change of state coordinates $\xi = \xi(x)$ into

$$\dot{x} = Ax + \alpha(y), \tag{5.2a}$$

$$y = \mathscr{C}x, \tag{5.2b}$$

$$y(0) \approx 0 \tag{5.2c}$$

where A, C are as in (4.10) but with possibly varying block sizes determined by the observability indices $k_1, \cdots, k_p$. The diagonal blocks of A and C are $k_i \times k_i$ and $1 \times k_i$ respectively.

The scalar output ($p = 1$) version of this problem was solved by Krener and Isidori [1]; the following theorems are straightforward generalizations.

THEOREM 5.1. *Let $\bar{g}^1(\xi), \cdots, \bar{g}^p(\xi)$ be vector fields defined by the equations*

$$L_{\bar{g}^j} L_f(y_i) = \begin{cases} 0, & 0 \leqq l \leqq k_i - 1, \\ \delta_i^j, & l = k_i - 1. \end{cases} \tag{5.3}$$

There exists a change of coordinates transforming (5.1) to (5.2) iff

$$[ad^k(-f)\bar{g}^i, ad^l(-f)\bar{g}^j] = 0 \tag{5.4}$$

for $i, j = 1, \cdots, p$, $k = 0, \cdots, k_i - 1$; $l = 0, \cdots, k_j - 1$. The appropriate coordinates $x = (x_{ik})$ are such that the vector field $ad^{k_i - k}(-f)\bar{g}^i$ is the unit vector in the x_{ik} direction,

$$L_{ad^{k_i-k}(-f)\bar{g}^i}(x_{jl}) = \delta_j^i \delta_l^k. \tag{5.5}$$

The appropriate functions $\alpha = (\alpha_{jl})$ can be computed by applying the state coordinate transformation to (5.1) and comparing the result with (5.2) or by solving the equations

$$\frac{\partial \alpha_{jl}}{\partial y_i} = L_{ad^{k_i}(-f)\bar{g}^i}(x_{jl}). \tag{5.6}$$

These are always solvable if (5.4) holds.

Remark 5.1. The repeated Lie brackets of vector fields X^1, X^2 and X^3 satisfy the Jacobi identity

$$[X^1[X^2, X^3]] = [[X^1, X^2]X^3] + [X^2[X^1, X^3]]. \tag{5.7}$$

This leads to considerable redundancy in the conditions (5.4). Suppose the conditions hold for any $k < k_i$, $l < k_j$ and $k + l < r$.

If $k + l = r$, applying (5.7) we obtain

$$\begin{aligned} [ad^k(-f)g^i, ad^l(-f)g^j] = &-[ad^{k-1}(-f)g^i, ad^{l+1}(-f)g^j] \\ &+ [f[ad^{k-1}(-f)g^i, ad^l(-f)g^j]] \end{aligned} \tag{5.8}$$

but the second on the right is zero by assumption. Hence for each i, j and r we need check (5.4) for only one value of k and l summing to r. Moreover because of the skewsymmetry of the bracket, (5.8) is skewsymmetric in i and j if r is even and symmetric if r is odd. Therefore for even r we need only check for $i < j$ and for odd r for $i = j$.

In particular if $p = 1$, (5.4) need only be checked for $k = l - 1$ and $l = 1, \cdots, n - 1$.

The condition that $f_i(\xi) \in \mathscr{P}^{k_i}(\xi)$ and the basic differential equation (4.2) are implied by (5.4) and as we shall see in the examples are sometimes equivalent to (5.4).

Remark 5.2. Suppose we have a system in special observable form relative to the output ψ and we have computed Ψ, the solution of (4.2). It is not necessary to compute $\psi(y)$ to verify (5.4).

If $g^1, \cdots, g^p$ are the vector fields defined by (4.24), then they are related to $\bar{g}^1, \cdots, \bar{g}^p$ by (4.10). With the help of (4.13) we can convert (5.4) into a family of differential equations which Ψ must satisfy.

Proof of Theorem 5.1. Suppose there exists a change of coordinates $\xi = \xi(x)$ transforming (5.1) to (5.2), then the vector fields $\bar{g}^1, \cdots, \bar{g}^p$ are transformed to constant vector fields in the x_{ik_i} direction. Let B be as in (4.10) with block sizes determined by the observability indices, the diagonal blocks are $k_i \times 1$.

Then $ad^l(-f)g^j = A^l B^j$ for $j = 1, \cdots, p$ and $l = 0, \cdots, k_{j-1}$ so clearly (5.4) holds. (Note: A^l is lth power of A, B^j the jth column of B.)

On the other hand suppose (5.4) holds. These are the integrability conditions for the set of partial differential equations (5.5) so there must exist coordinates x in which $ad^{k_i-1}(-f)\bar{g}^j$ is the unit vector in the x_{jl} direction.

We wish to compute $\dot{x}_{jl} = L_f(x_{jl})$.

If $1 \leq k \leq k_i$,

$$\frac{\partial \dot{x}_{jl}}{\partial x_{ik}} = L_{ad^{k_i-k}(-f)\bar{g}^i} L_f(x_{jl}) = L_{ad^{k_i-k+1}(-f)\bar{g}^i}(x_{jl}) + L_f L_{ad^{k_i-k}(-f)\bar{g}^i}(x_{jl}).$$

From (5.5) we see that

$$\frac{\partial \dot{x}_{jl}}{\partial x_{ik}} = \begin{cases} \delta^i_j \delta^{k+1}_l & \text{if } k > 1, \\ L_{ad^k(-f)\bar{g}^i}(x_{jl}) & \text{if } k = 1. \end{cases}$$

But $x_{i1} = y_i$ so

$$\dot{x}_{jl} = \begin{cases} x_{jl+1} + \alpha_{jl}(y), & l < k_j, \\ \alpha_{jk_j}(y), & l = k_j \end{cases}$$

where α_{jl} is the solution of (5.6).

These are first order partial differential equations so they are solvable if the mixed partials agree.

$$\frac{\partial}{\partial y_r} \frac{\partial \alpha_{jl}}{\partial y_i} = L_{ad^{k_r-1}(-f)\bar{g}^r} L_{ad^{k_i}(-f)\bar{g}^i}(x_{jl})$$

$$= L_{[ad^{k_i-1}(-f)\bar{g}^r, ad^{k_i}(-f)\bar{g}^i]}(x_{jl}) - L_{ad^{k_i}(-\bar{f})\bar{g}^i} L_{ad^{k_r-1}(-f)\bar{g}^r}(x_{jl}).$$

The second term on the right is zero by (5.5). Skew symmetry, the Jacobi identity (5.7) and (5.4) yield

$$[ad^{k_r-1}(-f)\bar{g}^r, ad^{k_i}(-f)\bar{g}^i] = [ad^{k_i-1}(-f)\bar{g}, ad^{k_r-1}(-f)\bar{g}^i]$$

so the mixed partials agree. Q.E.D.

6. Systems with inputs. The previous method can be easily generalized to handle systems with inputs

(6.1a) $$\dot{\xi} = f(\xi, u),$$

(6.1b) $$\psi = h(\xi),$$

(6.1c) $$\xi^0 \approx \xi(0),$$

(6.1d) $$\psi^0 = h(\xi^0).$$

We seek a change of output and state coordinates which transforms (6.1) into

(6.2a) $$\dot{x} = Ax + \gamma(y, u),$$

(6.2b) $$y = Cx,$$

(6.2c) $$x^0 = x(\xi^0) = 0,$$

(6.2d) $$y^0 = u(\psi^0) = 0.$$

If A, C is in dual Brunovsky form as given by (4.10), we say that (6.2) is in observer form. The system

(6.3a) $$\dot{z} = (A + GC)z + \gamma(y, u) - Gy,$$

(6.3b) $$z(0) = 0$$

tracks (6.2) with the error $e = x - z$ having dynamics

(6.3c) $$\dot{e} = (A + GC)e,$$

(6.3d) $$e(0) = x(\xi(0)) \approx 0.$$

Once again by proper choice of G we can insure that $e(t)$ goes to zero with arbitrary exponential decay.

To reduce this problem to the one considered previously we first choose a nominal input, either a constant u^0 or a function of ψ, $u^0(\psi)$. From a mathematical point of view the choice is immaterial. But of course the mathematical model is never an exact description of the real world; to reduce the effect of modeling errors the nominal control should be typical or average in some sense of the controls that will be employed.

We then rewrite (6.1a) as

(6.4) $$\dot{\xi} = f^0(\xi) + f^1(\xi, u)$$

where

(6.5a) $$f^0(\xi) = f(\xi, u^0(\psi(\xi))),$$

(6.5b) $$f^1(\xi, u) = f(\xi, u) - f^0(\xi)$$

and proceed as before with the unforced system (6.6) and (6.1b, c, d)

(6.6) $$\dot{\xi} = f^0(\xi).$$

If this can be transformed into observer form (6.7) and (6.2b, c, d)

(6.7) $$\dot{x} = Ax + \alpha(y)$$

by change of state and output coordinates, then all one need check is that $f^1(\xi, u)$ is transformed into a vector field of the form

(6.8) $$\beta(y, u).$$

If this is possible, then $\gamma(y, u) = \alpha(y) + \beta(y, u)$ and the problem is solved. If unforced system (6.6) and (6.1b, c, d) cannot be transformed into observer form or $f^1(\xi, u)$ does not transform into (6.8) then the original system (6.1) cannot be transformed into observer form (6.2).

As we remarked before the choice of the nominal control $u^0(\psi)$ is immaterial; a system (6.1) can be transformed into observer form (6.2) iff every unforced closed loop version (6.6) can be transformed into observer form by the same changes of coordinates.

Two nonlinear coordinate changes which transform a system (with or without inputs) into observer form differ by a linear change of coordinates. For such systems the output feedback $u = u^0(\psi)$ affects neither the observability nor the observability indices. However it is possible that (6.1) does not have an observer form yet one or more unforced closed loop versions do. If there is more than one, then typically these will involve different coordinate changes and perhaps even different observability indices.

7. Examples. We consider several simple cases of the above method for transforming a system into observer form.

Example 7.1. $p = 1$, $n = k_1 = 2$. In observable form we have the system

$$\begin{aligned}\psi &= \xi_1,\\ \dot{\xi}_1 &= \xi_2, \qquad \xi^0 = \begin{pmatrix}\xi_1^0\\ \xi_2^0\end{pmatrix}, \quad \psi^0 = \xi_1^0.\\ \dot{\xi}_2 &= f_1(\xi),\end{aligned} \tag{7.1}$$

Proposition 3.3 requires that $f_2 \in \mathcal{P}^2(\xi)$; hence

$$f_2(\xi) = a(\xi_1) + b(\xi_1)\xi_2 + c(\xi_1)\frac{\xi_2^2}{2}. \tag{7.2}$$

If this holds then since there is only one output the system is in special observable form, i.e., $f_2 \in \mathcal{P}_0^2(\xi)$. The differential equation (4.2) for $\Psi = d\psi/dy$ is

$$\frac{d\Psi}{d\psi} = \frac{1}{2} f_{1;2;2}\Psi(\psi) = \frac{1}{2}c(\psi)\Psi(\psi); \tag{7.3}$$

the solution is

$$\Psi(\psi) = \exp\left(\int_{\psi^0}^{\psi} \frac{c}{2}(\nu)\, d\nu\right) \tag{7.4}$$

where we have normalized the constant of integration so that $\Psi(\psi^0) = 1$. Next we check condition (5.4) using the identities (4.13)

$$\overset{g}{\begin{pmatrix}0\\1\end{pmatrix}}\overset{ad(-f)g}{\begin{pmatrix}1\\ f_{1;2}\end{pmatrix}}\overset{\bar{g}}{\begin{pmatrix}0\\ \Psi\end{pmatrix}}\overset{ad(-f)\bar{g}}{\begin{pmatrix}\Psi\\ f_{1;2}\Psi - \Psi^{(1)}\xi_2\end{pmatrix}}\overset{[\bar{g},\, ad(-f)\bar{g}]}{\begin{pmatrix}0\\ f_{1;2;2}\Psi - 2\Psi^{(1)}\end{pmatrix}}. \tag{7.5}$$

Equation (7.3) implies that (5.4) holds and hence the system can be transformed into observer form.

The required change of output coordinates is obtained by integrating (7.4) to

$$y(\psi) = \int_{\psi^0}^{\psi} \exp\left(-\int_{\psi^0}^{\mu} \frac{c}{2}(\nu)\, d\nu\right) d\mu \tag{7.6}$$

where $\psi^0 = \xi_1^0$ and the limits of integration have been chosen so that $y(\psi^0) = 0$. Since $\psi = \xi_1$ and $y = x_1$ this gives half of the change of state coordinates. The other coordinate $x_2(\xi)$ must satisfy (5.5) which reduce to

$$L_{\bar{g}(x_2)} = 1, \; L_{ad(-f)\bar{g}(x_2)} = 0. \tag{7.7}$$

From (7.5) this becomes

$$\frac{\partial x_2}{\partial \xi_1} = -\Psi^{-1}\left(\frac{b + c\xi_2}{2}\right), \qquad \frac{\partial x_2}{\partial \xi_2} = \Psi^{-1}. \tag{7.8}$$

These are easily integrated to obtain

$$x_2(\xi) = \Psi^{-1}\xi_2 - \xi_2^0 - \int_{\xi^0}^{\xi_1} \Psi^{-1}(\nu)b(\nu)\,d\nu \tag{7.9}$$

where the constant of integration has been chosen so that $x_2(\xi^0) = 0$.

Finally we compute α. Comparing the time derivative of (7.6) with (7.9) yields

$$\alpha_1(\xi_1) = \xi_2^0 + \int_{\xi_1^0}^{\xi_1} \Psi^{-1}(\nu)b(\nu)\,d\nu. \tag{7.10}$$

Time differentiating (7.9) yields

$$\alpha_2(\xi_1) = \Psi^{-1}(\xi_1)a(\xi_1). \tag{7.11}$$

Notice $\alpha_i = \alpha_i(\psi) = \alpha_i(\psi(y))$ as desired.

Hence we have that an $n = 2, p = 1$ system can be transformed to observer form iff it satisfies Proposition 3.3.

Example 7.2. $n = 20, k_1 = \cdots = k_p = 2$. The analysis is very similar to previous example. In observable form we have for $i = 1, \cdots, p$.

$$\begin{aligned} \psi_i &= \xi_{i1}, \\ \dot{\xi}_{i1} &= \xi_{i2}, \\ \xi_{i2} &= f_i(\xi). \end{aligned} \tag{7.12}$$

Let

$$\xi_1 = \begin{pmatrix} \xi_{11} \\ \vdots \\ \xi_{1p} \end{pmatrix}, \quad g_2 = \begin{pmatrix} \xi_{12} \\ \vdots \\ \xi_{p2} \end{pmatrix}, \quad x_1 = \begin{pmatrix} x_{11} \\ \vdots \\ z_{1p} \end{pmatrix}, \quad x_2 = \begin{pmatrix} x_{21} \\ \vdots \\ x_{2p} \end{pmatrix}.$$

Again f_i must be quadratic in ξ_{i2} so

$$f_i(\xi) = a_i(\xi_1) + b_i(\xi_1)\xi_2 + \tfrac{1}{2}\xi_2' c_i(\xi_1)\xi_2$$

where a_i is a scalar and $b_i = (b_{ij})$ and $c_i = (x_{ikl})$ are $1 \times p$ and symmetric $p \times p$ matrix valued functions. The partial differential equations (4.2) become

$$\frac{\partial}{\partial \psi_k}\Psi_i^j = \frac{1}{2}\sum_{l=1}^{p} c_{ikl}(\psi)\Psi_l^j(\psi). \tag{7.13}$$

It is convenient to define $p \times p$ matrix valued functions

$$\Gamma^k(\psi) = (\Gamma_{ij}^k(\psi)) = (\tfrac{1}{2}c_{ikj}(\psi))$$

for $k = 1, \cdots, p$. We rewrite (7.13) as

$$\frac{\partial}{\partial \psi_k}\Psi = \Gamma^k \Psi. \tag{7.14}$$

If the integrability conditions

$$\frac{\partial \Gamma^k}{\partial \psi_l} - \frac{\partial \Gamma^l}{\partial \psi_k} = \Gamma^l \Gamma^k - \Gamma^k \Gamma^l \tag{7.15}$$

are satisfied, we have the solution

$$\Psi(\psi) = \exp\left(\sum_{i=1}^{p} \int_{\psi_i^0}^{\psi_i} \Gamma^k(\nu_i)\,d\nu_i \right). \tag{7.16}$$

Again (7.14) insures us that (5.4) holds, so the changes of coordinates exist. The change of output is obtained by integrating (7.15) via the line integral

$$y(\psi) = \int_{\psi^0}^{\psi} \Psi^{-1}(\nu)\, d\nu. \tag{7.17}$$

This is also half of the state coordinate change; the other half $x_2(\xi)$ satisfies (5.5) which becomes similar to

$$\frac{\partial x_2}{\partial \xi_1} = -\Psi^{-1}\left(b + \sum_{k=1}^{p} \Gamma^k \xi_{k2}\right), \qquad \frac{\partial x_2}{\partial \xi_2} = \Psi^{-1} \tag{7.18}$$

where b is the $p \times p$ matrix whose i, jth entry is the jth entry of b_i. The solution (7.18) is

$$x_2(\xi) = \Psi^{-1}\xi_2 - \xi_2^0 - \int_{\xi_1^0}^{\xi_1} \Psi^{-1}(\nu) b(\nu)\, d\nu \tag{7.19}$$

where the last term is a line integral. The computation of α is as before and given by the vector versions of (7.10) and (7.11).

Hence a $2p = n$, $k_1 = \cdots = k_p$ system can be transformed to observer form iff it satisfies Proposition 3.3 and the integrability conditions (7.15).

Example 7.3. $p = 1$, $n = k_1 = 3$. In observable form we have the system

$$\begin{aligned} \psi &= \xi_1, \\ \dot{\xi}_1 &= \xi_2, \\ \dot{\xi}_2 &= \xi_3, \\ \dot{\xi}_3 &= f_1. \end{aligned} \tag{7.20}$$

Proposition 3.3 requires that f_1 be of the form

$$\begin{aligned} f_1(\xi) = a(\xi_1) + b(\xi_1)\xi_2 + c(\xi_1)\frac{\xi_2^2}{2} + d(\xi_1)\frac{\xi_2^3}{3} \\ + (\rho(\xi_1) + \sigma(\xi_1)\xi_2)\xi_3. \end{aligned} \tag{7.21}$$

The basic differential equation (4.2) is

$$\frac{d\Psi}{d\psi} = \frac{1}{3} f_{1;3;2}\Psi = \frac{1}{3}\sigma(\psi)\Psi \tag{7.22}$$

and the solution is

$$\Psi(\psi) = \exp \int_{\psi^0}^{\psi} \frac{\sigma(\nu)}{3}\, d\nu. \tag{7.23}$$

If we use (4.13), then after a laborious calculation (5.4) reduces to the two differential equations

$$\frac{d\sigma}{d\xi_1} = \frac{3}{2} d + \frac{2}{3}\sigma^2, \tag{7.24a}$$

$$\frac{d\rho}{d\xi_1} = c + \rho\sigma. \tag{7.24b}$$

Hence a $p = 1$, $n = 3$ system can be transformed to observer form iff Proposition 3.3 and equations (7.24) are satisfied. The rest of the calculations proceed as before.

Example 7.4. $p=2$, $n=3$, $k_1=2$, $k_2=1$. In observable form we have

$$\psi_1=\xi_{11},\qquad \psi_2=\xi_{21},$$
$$\dot{\xi}_{11}=\xi_{12},\qquad \dot{\xi}_{21}=f_2(\xi),\qquad \psi=\xi_1=\begin{pmatrix}\xi_{11}\\ \xi_{21}\end{pmatrix}, \tag{7.25}$$
$$\dot{\xi}_{12}=f_1(\xi),$$

where by Proposition 3.3

$$f_1(\xi)=a(\xi_1)+b(\xi_1)\xi_{12}+c(\xi_1)\frac{\xi_{12}^2}{2}, \tag{7.26a}$$

$$f_2(\xi)=\rho(\xi_1)+\sigma(\xi_1)\xi_{12}. \tag{7.26b}$$

First we transform this to special observable form by Theorem 4.2. We seek $\tilde{\psi}_2(\psi)$ such that

$$L_{Y^1}(\tilde{\psi}_2)=0 \quad \text{where } Y^1=\begin{pmatrix}1\\ f_{2;12}\end{pmatrix}=\begin{pmatrix}1\\ \sigma(\xi_1)\end{pmatrix} \tag{7.27}$$

or

$$\frac{\partial\tilde{\psi}_2}{\partial\psi_1}+\sigma(\psi)\frac{\partial\tilde{\psi}_2}{\partial\psi_2}=0.$$

This is always solvable. In observable form (2.5) relative to the new outputs $\tilde{\psi}_1=\psi_1$ and $\tilde{\psi}_2$ we have

$$\tilde{f}_1(\tilde{\xi})=\tilde{a}(\tilde{\xi}_1)+\tilde{b}(\tilde{\xi}_1)\tilde{\xi}_{12}+\tilde{c}(\tilde{\xi}_1)\frac{\tilde{\xi}_{12}^2}{2}, \tag{7.28a}$$

$$\tilde{\phi}_2(\tilde{\xi})=\tilde{\rho}(\xi_1). \tag{7.28b}$$

At this point the presence of the second output is immaterial and we proceed essentially as in Example 7.1 carrying $\tilde{\psi}_2=\tilde{\xi}_{21}$ as a parameter. Hence a $p=2$, $n=3$, $k_1=2$, $k_2=1$ system is transformable to observer form iff it satisfies Proposition 3.3.

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