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$(\mathrm{Ad}_{f,g})$, $(\mathrm{ad}_{f,g})$ AND LOCALLY $(\mathrm{ad}_{f,g})$ INVARIANT AND CONTROLLABILITY DISTRIBUTIONS*

ARTHUR J. KRENER†

Abstract. In the study of nonlinear control systems, the concepts of an invariant foliation and an invariant distribution play important roles. In this paper we explore various forms of these concepts and show how they occur in the study of controllability, observability and decoupling of nonlinear systems.

Key words. invariant foliation, invariant distribution, nonlinear controllability, nonlinear observability, nonlinear decoupling

1. Introduction. Through the work of many researchers over the past decade it has become clear that concepts from differential topology such as foliation and invariant distribution play a crucial roll in the study of nonlinear systems. These tools were first used in the study of nonlinear controllability and later observability. More recently they have arisen in the study of decoupling and linearization via feedback.

As their use has widened, a greater precision in their application has become necessary. This paper is an attempt at that precision at least as far as my own joint work with R. Hermann [12], A. Isidori, C. Gori-Giorgi and S. Monaco is concerned [7]. These papers use differential topological tools to extend to nonlinear systems the geometric approach to linear systems. Although they are quite successful, they do not have the same logical simplicity and elegance of the corresponding linear theory. This reflects a basic fact of mathematical life, nonlinearities are much messier to deal with, one usually must make strong regularity assumptions and distinguish between a much larger range of phenomena when in their presence.

In this paper we introduce the basic concepts needed for an understanding of controllability, observability and decoupling of nonlinear systems. Some of the theorems contained herein build on and are refinements of those appearing in [12] and [7]. By slightly modifying some definitions we achieve a synthesis of the previous work. From this firm platform we are able to treat controllability distributions in a precise manner and prove several interesting results.

2. Mathematical preliminaries. Throughout this paper we consider nonlinear systems of the form

$$\dot{x} = f(x, u) = g^0(x) + g(x)u, \tag{2.1a}$$

$$y = h(x), \tag{2.1b}$$

$$x(0) = x^0, \tag{2.1c}$$

where x denotes local coordinates on a smooth n-dimensional Hausdorff, paracompact connected manifold M, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, g^0 and $g^1, \cdots, g^m$, the m columns of g, are local coordinate descriptions of smooth vector fields globally defined on M. Smooth means either $\mathscr{C}^\infty$ or $\mathscr{C}^\omega$ (analytic). The definitions of differentiable manifold, etc. can be found in Boothby [18] or Spivak [22].

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† Department of Mathematics, University of California, Davis, California 95616 and Flight Dynamics and Control Branch, NASA Ames Research Center 210-3, Moffet Field, California 94035.

Of course (2.1) is a local description; different descriptions of this type are valid in different coordinate neighborhoods of M. As far as possible we use local coordinate notation; hopefully this will make the paper accessible to a wider audience.

We denote by T_xM and T_x^*M the tangent and cotangent spaces at x, and by TM and T^*M the tangent and cotangent bundles. The ring of smooth real valued functions on M is denoted by $\mathcal{F}(M)$, the space of smooth vector fields (sections of TM) by $\mathcal{X}(M)$ and the space of smooth one forms (sections of T^*M) by $\mathcal{X}^*(M)$. $\mathcal{X}(M)$ and $\mathcal{X}^*(M)$ are real vector spaces and $\mathcal{F}(M)$ modules, and $\mathcal{X}(M)$ is a Lie algebra under the Lie (or Jacobi) bracket. Locally vector fields are represented by column n vectors and one forms by row n vectors. The bilinear pairing between a one form $\omega(x)$ and a vector field $X(x)$ is then the multiplication of $1\times n$ and $n\times 1$ matrices. It defines a function denoted by $\langle \omega, X\rangle \in \mathcal{F}(M)$.

A vector field X defines a flow $\Phi(t,x)$, the solution of the differential equation

$$\frac{\partial}{\partial t}\Phi(t,x) = X(\Phi(t,x)),$$

$$\Phi(0,x) = x.$$

For each x, $t\to\Phi(t,x)$ is a curve defined for t in some open interval depending on x. For some x the curve may escape from the manifold in finite time and hence not be definable for all t. We use the phrase "for all t" to mean "for all t where defined". For each t the map $x\to\Phi(t,x)$ is a smooth diffeomorphism where defined.

A vector field X or its flow $\Phi(t,x)$ acts on functions $\varphi\in\mathcal{F}(M)$, vector fields $Y\in\mathcal{X}(M)$ and one forms $\omega\in\mathcal{X}^*(M)$. The right side of the following are local coordinate descriptions which can be taken as the definitions of the symbols to the left.

(2.2a) $$\mathrm{Ad}_X^t(\varphi)(x) := \Phi(t)^*\varphi(x) := \varphi(\Phi(t,x)),$$

(2.2b) $$L_X(\varphi)(x) := \langle d\varphi, X\rangle(x),$$

(2.3a) $$\mathrm{Ad}_X^t(Y)(x) := (\Phi(-t)_* Y)(x) := \left.\frac{\partial\Phi(-t,z)}{\partial z}\right|_{z=\Phi(t,x)} Y(\Phi(t,x)),$$

(2.3b) $$\mathrm{ad}_X(Y)(x) := L_X Y(x) := [X,Y](x) := \frac{\partial Y}{\partial x}(x)X(x) - \frac{\partial X}{\partial x}(x)Y(x),$$

(2.4a) $$\mathrm{Ad}_X^t(\omega)(x) := \Phi(t)^*\omega(\Phi(t,x)) := \omega(\Phi(t,x))\left.\frac{\partial\Phi(t,z)}{\partial z}\right|_{z=x},$$

(2.4b) $$\mathrm{ad}_X(\omega)(x) := L_X(\omega)(x) := \left(\frac{\partial\omega'}{\partial x}(x)X(x)\right)' + \omega(x)\frac{\partial X}{\partial x}(x).$$

We use $'$ to denote transpose and $\partial/\partial x$ to denote partial differentiation. It is always applied to a column vector yielding a matrix with i the row and j the column index as in

$$d\varphi(x) := \frac{\partial\varphi}{\partial x}(x) = \left(\frac{\partial\varphi}{\partial x_j}\right),$$

$$\frac{\partial Y}{\partial x}(x) := \left(\frac{\partial Y_i}{\partial x_j}(x)\right),$$

$$\frac{\partial\omega'}{\partial x}(x) := \left(\frac{\partial\omega_i}{\partial x_j}(x)\right).$$

Equation (2.3b) defines the Lie bracket of vector fields. It is standard mathematical notation to denote (2.2b) by $X\varphi$ or $X(\varphi)$. We shall not employ these but instead use $X\varphi$ for X multiplied by φ.

The operator L_X of the above formulas is called Lie differentiation for

(2.5a) $$L_X(\varphi)(x) = \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_X^t(\varphi)(x),$$

(2.5b) $$L_X(Y)(x) = \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_X^t(Y)(x),$$

(2.5c) $$L_X(\omega)(x) = \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_X^t(Y)(x).$$

The following Taylor series expansions are called Lie series:

(2.6a) $$\mathrm{Ad}_X^t(\varphi)(x) \approx \sum_{k+0}^{\infty} \frac{t^k}{k!} L_X^k(\varphi)(x),$$

(2.6b) $$\mathrm{Ad}_X^t(Y)(x) \approx \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathrm{ad}_X^k(Y)(x),$$

(2.6c) $$\mathrm{Ad}_X^t(\omega)(x) \approx \sum_{k=0}^{\infty} \frac{t^k}{k!} L_X^k(\omega)(x),$$

where $\mathrm{ad}_X^k(Y) = [X, \mathrm{ad}_X^{k-1}(Y)]$.

Further identities are

(2.7a) $$\mathrm{Ad}_X^t(\langle \omega, Y\rangle)(x) = \langle \mathrm{Ad}_X^t(\omega), \mathrm{Ad}_X^t(Y)\rangle(x),$$

(2.7b) $$L_X\langle \omega, Y\rangle(x) = \langle L_X(\omega), Y\rangle(x) + \langle \omega, L_X(Y)\rangle(x),$$

(2.8a) $$\mathrm{Ad}_X^t([Y, Z])(x) = [\mathrm{Ad}_X^t(Y), \mathrm{Ad}_X^t(Z)](x),$$

(2.8b) $$[X[Y, Z]](x) = [[X, Y]Z](x) + [Y[X, Z]](x) \text{ (Jacobi identity)},$$

and

(2.9a) $$\mathrm{Ad}_X^t(d\varphi)(x) = d(\mathrm{Ad}_X^t(\varphi))(x),$$

(2.9b) $$L_X(d\varphi)(x) = d(L_X(\varphi))(x).$$

A fundamental geometric concept in the study of nonlinear systems is the following.

DEFINITION. A distribution $\mathcal{D}$ is a submodule of $\mathcal{X}(M)$. We denote by $D(x)$ the subspace of T_xM obtained by evaluating the elements of $\mathcal{D}$ at x. The union $D = \bigcup_{x\in M} D(x)$ of these subspaces is called the *singular subbundle* of TM associated to $\mathcal{D}$. (By definition all singular subbundles of TM are associated to distributions.) If D (or $\mathcal{D}$) is nonsingular, i.e., the dimension of $D(x)$ is constant over all x, then D is a *subbundle* of TM (in the usual sense of the term).

A *local frame* for $\mathcal{D}$ (or D) on an open set $\mathcal{U} \subset M$ is a family of vector fields $\{X^1, \cdots, X^d\}$ such that for each $x \in \mathcal{U}$ the vectors $\{X^1(x), \cdots, X^d(x)\}$ are a basis for $D(x)$ (clearly D is nonsingular iff around each $x \in M$ it admits a local frame). Given a singular subbundle D associated to a distribution $\mathcal{D}$, we can define a second distribution $\Gamma(D)$ as the set of all smooth vector fields $X \in \mathcal{X}(M)$ such that $X(x) \in D(x)$, $\forall x \in M$. A distribution $\mathcal{D}$ is *complete* if $\mathcal{D} = \Gamma(D)$. (After this section all distributions will be assumed to be complete, and we shall use the term distribution to mean complete distribution.)

For nonsingular D, the distinction between D and $\mathscr{D}$ (or $\Gamma(D)$) is not particularly important and a certain amount of sloppiness is tolerable. However, one must be much more careful when considering the singular case. For example, the collection of all distributions on M forms a lattice partially ordered by inclusion under the operations of submodule addition and submodule intersection. If $\mathscr{D}^1$, $\mathscr{D}^2$ are complete distributions, D^1, D^2 their associated singular subbundles and D the singular subbundle associated to $\mathscr{D}^1 \cap \mathscr{D}^2$ then

$$D(x) \subset (D^1(x) \cap D^2(x))$$

but the inclusion can be proper for some x. For example, for $M = \mathbb{R}^2$ let $\mathscr{D}^1$ be the span of $\partial/\partial x_1$ and $\mathscr{D}^2$ be the span of $\partial/\partial x_1 + x_2\,\partial/\partial x_2$ (span always means over $\mathscr{F}(M)$). Then $\mathscr{D}^1 \cap \mathscr{D}^2$ contains only the zero vector field so $D(x) = \{0\}$.

No such difficulty occurs with sums; if D is the singular subbundle associated to $\mathscr{D}^1 + \mathscr{D}^2$ then $D(x) = D^1(x) + D^2(x)$.

DEFINITION. An integral submanifold L of $\mathscr{D}$ is a connected, immersed submanifold $L \subset M$ such that for each $x \in L$, $T_xL = D(x)$. A distribution $\mathscr{D}$ is *integrable* if its maximal integral manifolds define a partition of M. This partition is called a *foliation* and the maximal integral submanifolds are its *leaves*.

DEFINITION. A distribution $\mathscr{D}$ is Ad_X *invariant* if $Y \in \mathscr{D}$ implies $\mathrm{Ad}_X^t(Y) \in \mathscr{D}$ for all t. A distribution $\mathscr{D}$ is ad_X *invariant* if $Y \in \mathscr{D}$ implies $\mathrm{ad}_X(Y) \in \mathscr{D}$.

Clearly from (2.5b), Ad_X invariance implies ad_X invariance but the converse need not hold. If everything is $\mathscr{C}^\omega$ then Lie series arguments (2.6b) imply the converse. If $\mathscr{D}$ is nonsingular then an argument of Hermann [16] also implies the converse.

DEFINITION. A distribution $\mathscr{D}$ is *involutive* if $\mathscr{D}$ is ad_X invariant for every $X \in \mathscr{D}$.

The basic integrability result is next.

THEOREM 2.1 (Sussmann [10]). *A distribution $\mathscr{D}$ is integrable iff $\mathscr{D}$ is Ad_X invariant for every $X \in \mathscr{D}$.*

This leads to the following corollaries.

COROLLARY 2.2 (Frobenius [18]). *For nonsingular distributions integrability and involutiveness are equivalent.*

COROLLARY 2.3. (Hermann [16], Nagano [17]). *For C^ω distributions integrability and involutiveness are equivalent.*

DEFINITION. A point x^0 is a *regular point* of the distribution $\mathscr{D}$ if the dimension of $D(x)$ is constant in a neighborhood of x^0 otherwise it is a *singular point*.

It is easy to see that the regular points of $\mathscr{D}$ form an open and dense submanifold of M.

COROLLARY 2.4. *An integrable distribution $\mathscr{D}$ is involutive. An involutive distribution $\mathscr{D}$ restricted to the submanifold of its regular points is integrable.*

The Ad_X and ad_X invariant distributions form lattices, while the integrable and involutive distributions form semilattices (closed under intersections but not sums). There exist minimal integrable and involutive distributions containing a given distribution $\mathscr{D}$, called the integrable and involutive closures of $\mathscr{D}$. From (2.8a,b) it follows that if $\mathscr{D}$ is Ad_X or ad_X invariant then so is its involutive closure.

DEFINITION. A *codistribution* $\mathscr{E}$ is a submodule of $\mathscr{X}^*(M)$. (Classically codistributions are called Pfaffian systems.) Associated to each codistribution $\mathscr{E}$ is a family of subspaces $E(x) \subset T_x^*M$ obtained by evaluating the one forms of $\mathscr{E}$ at x. The union $E = \bigcup E(x)$ is a *singular subbundle* of T_x^*M. Nonsingularity, local frame, completeness, etc. are all defined analogously.

There is a duality between distribution and codistributions. To each distribution $\mathscr{D}$ (codistribution $\mathscr{E}$) there is a codistribution $\mathscr{D}^\perp$ (distribution $\mathscr{E}^\perp$) called its annihilator defined by

$$\mathcal{D}^\perp = \{\omega \in \mathcal{X}^*(M) \colon \langle \omega, X\rangle = 0, \forall X \in \mathcal{D}\},$$

$$(\mathcal{E}^\perp = \{X \in \mathcal{X}(M) \colon \langle \omega, X\rangle = 0, \forall \omega \in \mathcal{E}\}).$$

One has the inclusion

$$\mathcal{D} \subset \mathcal{D}^{\perp\perp} \qquad (\mathcal{E} \subset \mathcal{E}^{\perp\perp}),$$

which may be proper unless $\mathcal{D}(\mathcal{E})$ is nonsingular and complete. Moreover

$$(\mathcal{D}^1 + \mathcal{D}^2)^\perp = \mathcal{D}^{1\perp} \cap \mathcal{D}^{2\perp} \qquad ((\mathcal{E}^1 + \mathcal{E}^2)^\perp = \mathcal{E}^{1\perp} \cup \mathcal{E}^{2\perp}),$$

$$\mathcal{D}^{1\perp} + \mathcal{D}^{2\perp} \subset (\mathcal{D}^1 \cap \mathcal{D}^2)^\perp \qquad (\mathcal{E}^{1\perp} + \mathcal{E}^{2\perp} \subset (\mathcal{E}^1 \cap \mathcal{E}^2)^\perp).$$

If $\mathcal{D}^1$, $\mathcal{D}^2$ and $\mathcal{D}^1 \cap \mathcal{D}^2$ ($\mathcal{E}^1$, $\mathcal{E}^2$ and $\mathcal{E}^1 \cap \mathcal{E}^2$) are complete and nonsingular then the last inclusion is an identity.

DEFINITION. A codistribution $\mathcal{E}$ is Ad_X (ad_X) invariant if $\omega \in \mathcal{E}$ implies $\mathrm{Ad}_X^t(\omega) \in \mathcal{E}$ $\forall t$ ($\mathrm{ad}_X(\omega) \in \mathcal{E}$).

LEMMA 2.5. *If the distribution $\mathcal{D}$ is $\mathrm{Ad}_X(\mathrm{ad}_X)$ invariant then the codistribution $\mathcal{D}^\perp$ is also. If the codistribution $\mathcal{E}$ is Ad_X (ad_X) invariant then the distribution $\mathcal{E}^\perp$ is also.*

Proof. Suppose $\mathcal{D}$ is Ad_X invariant, $\omega \in \mathcal{D}^\perp$ and $Y \in \mathcal{D}$; then $\mathrm{Ad}_X^{-t}(Y) \in \mathcal{D}$. Using (2.7a) gives

$$\langle \mathrm{Ad}_X^t(\omega), Y\rangle = \langle \mathrm{Ad}_X^t(\omega), \mathrm{Ad}_X^t(\mathrm{Ad}_X^{-t}(Y))\rangle = \mathrm{Ad}_X^t(\langle \omega, \mathrm{Ad}_X^{-t}(Y)\rangle) = 0,$$

so $\mathrm{Ad}_X^t(\omega) \in \mathcal{D}^\perp$ and $\mathcal{D}^\perp$ is Ad_X invariant. Suppose $\mathcal{D}$ is ad_X invariant, $\omega \in \mathcal{D}^\perp$ and $Y \in \mathcal{D}$; then $L_X(Y) \in \mathcal{D}$. Using (2.7b)

$$\langle L_X(\omega), Y\rangle = L_X\langle \omega, Y\rangle - \langle \omega, L_X(Y)\rangle = L_X(0) - 0 = 0$$

so $L_X(\omega) \in \mathcal{D}^\perp$ and $\mathcal{D}^\perp$ is ad_X invariant. The other assertion is proved similarly. QED

DEFINITION. A codistribution $\mathcal{E}$ is *integrable* if the distribution $\mathcal{E}^\perp$ is integrable.

Let $h\colon M \to \mathbb{R}^p$ be smooth. We denote by $\mathcal{R}(dh)$ the codistribution spanned by the one forms dh_i, $i = 1, \cdots, p$. We denote by $\mathcal{H}(dh)$ the distribution which annihilates $\mathcal{R}(dh)$, $\mathcal{H}(dh) = \mathcal{R}(dh)^\perp$.

LEMMA 2.6. *$\mathcal{R}(dh)$ is integrable.*

Proof. By definition we must show that the distribution $\mathcal{H}(dh)$ is integrable. By Sussmann's theorem this amounts to showing that $\mathcal{H}(dh)$ is Ad_X invariant for every $X \in \mathcal{H}(dh)$. By Lemma 2.6 this is equivalent to showing that $\mathcal{R}(dh)$ is Ad_X invariant for every $X \in \mathcal{H}(dh)$.

From the definition

$$\mathrm{Ad}_X^{s+t}(dh_i) = \mathrm{Ad}_X^s(\mathrm{Ad}_X^t(dh_i))$$

so

$$\frac{d}{ds}\mathrm{Ad}_X^s(dh_i) = \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_X^{s+t}(dh_i) = \mathrm{Ad}_X^s \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_X^t(dh_i).$$

By (2.5c) and (2.9b) this becomes

$$\frac{d}{ds}\mathrm{Ad}_X^s(dh_i) = \mathrm{Ad}_X^s L_X(dh_i) = \mathrm{Ad}_X^s d(L_X(h_i)).$$

But $X \in \mathcal{H}(dh)$ implies $L_X(h_i) = 0$ hence

$$\mathrm{Ad}_X^s(dh_i) = dh_i. \qquad \text{QED}$$

3. Ad_f and ad_f invariance. In the study of linear systems of the form

$$\dot{x} = Ax + Bu, \tag{3.1a}$$

$$y = Cx, \tag{3.1b}$$

$$x(0) = x^0, \tag{3.1c}$$

the invariant subspaces of the matrix A play an important role. Suppose $V \subseteq \mathbb{R}^n$ is such a subspace, i.e., $AV \subseteq V$. Then V is spanned by the real and imaginary parts of a subset of eigenvectors and generalized eigenvectors of A. The invariant subspaces are the modal subspaces of A.

The nonlinear generalizations of this are several.

DEFINITION. A distribution or codistribution is Ad_f invariant (ad_f invariant) if it is $\mathrm{Ad}_{f(\cdot,u)}$ invariant ($\mathrm{ad}_{f(\cdot,u)}$ invariant) for each constant control $u \in \mathbb{R}^n$.

Clearly Ad_f invariance implies ad_f invariance but not the converse unless the distribution or codistribution is nonsingular or $\mathscr{C}^\omega$. It is easy to see that ad_f invariance is equivalent to ad_{g^j} invariance for $j = 0, \cdots, m$. What is not so obvious, but follows from Lemmas 3.2 and 3.3, is that Ad_f invariance is equivalent to Ad_{g^j} invariance for $j = 0, \cdots, m$. As one expects from the results of § 2, the sum and intersection of Ad_f or ad_f invariant (co) distributions is also, the involutive closure of an Ad_f or ad_f invariant distribution is also and the annihilator of an Ad_f or ad_f invariant (co) distribution is also.

Before we go any further let us relate these concepts to that of an invariant subspace of a linear system (3.1). Let V be an invariant subspace, and define $\mathscr{D}$ as the set of vector fields on $\mathbb{R}^n$ which take values in V. (We are using the canonical identification of $\mathbb{R}^n$ with each of its tangent spaces $T_x\mathbb{R}^n$.) The associated subbundle D is nonsingular with $D(x) = V$ (thought of as a subspace of $T_x\mathbb{R}^n$). For each constant control $u \in \mathbb{R}^m$ we obtain the vector field $f(x, u) = Ax + Bu$ and corresponding flow $\Phi(t, x) = e^{At}(x + \int_0^t e^{-As} Bu\, ds)$.

We claim that $\mathscr{D}$ is Ad_f and ad_f invariant. By the above remarks it suffices to verify that $\mathscr{D}$ is Ad_{g^j} and ad_{g^j} invariant for $j = 0, \cdots, m$. But $g^j(x) = B^j$ (the jth column of B), a constant vector field, and any basis for V considered as constant vector fields defines a global frame for $\mathscr{D}$. Let $v \in V$ considered as a constant vector field in $\mathscr{D}$ then

$$\mathrm{ad}_{g^0}(v) = -Av, \qquad \mathrm{Ad}^t_{g^0}(v) = e^{-At}v, \tag{3.2a}$$

$$\mathrm{ad}_{g^j}(v) = 0, \qquad \mathrm{Ad}^t_{g^0}(v) = v, \quad j = 1, \cdots, m. \tag{3.2b}$$

Since a frame for $\mathscr{D}$ is invariant, it follows that all of $\mathscr{D}$ is.

We refer to such a $\mathscr{D}$ as a constant distribution on $\mathbb{R}^n$ because it has a global frame of constant vector fields but of course $\mathscr{D}$ contains nonconstant vector fields. If g^j is a constant vector field (such as B^j) and $\mathscr{D}$ is a constant distribution then $\mathscr{D}$ is always Ad_{g^j} and ad_{g^j} invariant. Therefore one need only check the Ad_{g^0} and ad_{g^0} invariance of constant distributions. This fact frequently leads to differences between the formulation of a linear result and its nonlinear generalization as we shall see throughout this paper.

We have just noted that for a linear system the constant distributions which are Ad_f or ad_f invariant are precisely the invariant subspaces of A. One might ask whether there are any nonconstant distributions which are invariant. If one restricts to nonsingular distributions the answer is essentially no.

PROPOSITION 3.1. *Suppose the linear system* (3.1) *is controllable and* $\mathscr{D}$ *is a nonsingular* Ad_f (*equivalently* ad_f) *invariant distribution for* (3.1). *Then* $\mathscr{D}$ *is a constant distribution, hence corresponds to an invariant subspace of* A.

Proof. Let the dimension of $\mathcal{D}$ be d and let $X^1, \cdots, X^d$ be a local frame. By assumption for $j=0, \cdots, m$ and $k=1, \cdots, d$

$$[g^j, X^k] \in \mathcal{D}.$$

Using the Jacobi identity (2.7b)

$$[[g^i, g^j], X^k] = [g^i[g^j, X^k]] - [g^j[g^i, X^k]] \in \mathcal{D},$$

so $\mathcal{D}$ is invariant under any bracket $[g^i, g^j]$. By repeating this argument it follows that $\mathcal{D}$ is invariant under any multiple bracket $[g^{j_1} \cdots [g^{j_{r-1}}, g^{j_r}] \cdots]$.

Now $g^0 = Ax$, $g^j = B^j$(jth column of B) and

$$\mathrm{ad}^r_{g^0}(g^j) = (-1)^r A^r B^j,$$

$$[g^i, \mathrm{ad}^r_{g^0}(g^j)] = 0,$$

where $i, j = 1, \cdots, m$ and $r \geqq 0$. The controllability assumption implies that there are n linearly independent vectors of the form $A^r B^j$. View these as constant vector fields and denote them by $Y^1, \cdots, Y^n$.

Since each Y^k is a bracket of g^j's, it leaves $\mathcal{D}$ invariant, hence there exist functions Γ_i^{kj} such that

$$[Y^k, X^j] = \sum_{i=1}^{d} X^i \Gamma_i^{kj}.$$

Let Γ^k denote the $d \times d$ matrix (Γ_i^{kj}) and X the $n \times d$ matrix $(X^1 \cdots X^d)$; we abbreviate the above as

$$[Y^k, X] = X\Gamma^k.$$

We make a change of local frame for $\mathcal{D}$ by choosing a $d \times d$ invertible matrix valued function γ, the new basis is the set of columns $\tilde{X}^1, \cdots, \tilde{X}^d$ of $\tilde{X} = X_\gamma$. We seek a basis which commutes with Y^k, i.e.

$$0 = [Y^k, \tilde{X}] = [Y^k, X\gamma] = [Y^k, X]\gamma + XL_{Y^k}(\gamma) = X(\Gamma^k\gamma + L_{Y^k}(\gamma)).$$

Hence γ should satisfy the linear partial differential equation

$$L_{Y^k}(\gamma) = -\Gamma^k\gamma.$$

There is a local solution to this equation if the integrability (mixed partial) conditions are satisfied. Since $[Y^k, Y^l] = 0$ these are

$$L_{Y^k}L_{Y^l}(\gamma) = L_{Y^l}L_{Y^k}(\gamma)$$

which reduce to

$$L_{Y^k}(\Gamma^l\gamma) = L_{Y^l}(\Gamma^k\gamma)$$

or

$$(L_{Y^k}(\Gamma^l) - \Gamma^l\Gamma^k)\gamma = (L_{Y^l}(\Gamma^k) - \Gamma^k\Gamma^l)\gamma.$$

But these follow from the Jacobi identity (2.8b) and the linear independence of the columns of X for

$$[Y^k[Y^l, X]] - [Y^l[Y^k, X]] = [[Y^k, Y^l]X] = 0,$$

$$[Y^k, X\Gamma^l] - [Y^l, X\Gamma^k] = 0,$$

$$X(\Gamma^k\Gamma^l + L_{Y^k}(\Gamma^l) - \Gamma^l\Gamma^k - L_{Y^l}(\Gamma^k)) = 0.$$

Hence we can find γ such that the vector fields $\tilde{X}^1, \cdots, \tilde{X}^d$ of the new local frame for $\mathscr{D}$ commute with the constant vector fields $Y^1, \cdots, Y^n$ which span $\mathbb{R}^n$. From this one can conclude that $\tilde{X}^1, \cdots, \tilde{X}^d$ are constant vector fields so locally $\mathscr{D}$ has a constant frame. On the common domain of definition of two such constant frames, the change of frame matrix must be constant so any such constant local frame extends to a constant global frame for $\mathscr{D}$. QED

The statement that $AV \subset V$ can be interpreted as the dynamics (3.1a) infinitesimally leaves the directions of V invariant. The statement that $e^{At}V \subset V$ can be interpreted as the flow of (3.1a) leaves the directions of V invariant. Both these statements have direct nonlinear generalizations. If $\mathscr{D}$ is ad_f invariant then the dynamics (2.1a) infinitesimally leaves the directions of $\mathscr{D}$ invariant. If $\mathscr{D}$ is Ad_f invariant then the flow of (2.1a) leaves the directions of $\mathscr{D}$ invariant.

The constant distribution $\mathscr{D}$ on $\mathbb{R}^n$ associated to any subspace V of $\mathbb{R}^n$ is integrable, the leaves of the foliation that it induces are the cosets $x+V$ for $x \in \mathbb{R}^n$. If V is an invariant subspace of A then the flow of (3.1a) for any fixed control $u(t)$ carries cosets into cosets. A concrete way of seeing this is to choose local coordinates $x = \binom{x_1}{x_2}$ such that

$$V = \{x: x_1 = 0\}.$$

In these coordinates the dynamics (3.1a) takes a block triangular form.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u. \tag{3.3}$$

The coset space $\mathbb{R}^n/V$ is coordinatized by x_1 and since x_1 evolves independently of x_2, the dynamics passes to this space.

In the nonlinear context a similar thing happens. Suppose $\mathscr{D}$ is a nonsingular, involutive Ad_f (equivalently ad_f) invariant distribution. Then locally one can choose coordinates $x = \binom{x_1}{x_2}$ so that the leaves of the foliation induced by $\mathscr{D}$ are given by $x_1 = \text{constant}$. In these coordinates the dynamics again assumes a triangular form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, u) = g_1^0(x_1) + g_1(x_1)u, \\ \dot{x}_2 &= f_2(x_1, x_2, u) = g_2^0(x_1, x_2) + g_2(x_1, x_2)u, \end{aligned} \tag{3.4}$$

and the flow for any fixed control $u(t)$ carries leaves into leaves. If the foliation induced by $\mathscr{D}$ is *regular*, i.e., the space of leaves can be given a manifold structure, then this space is locally coordinatized by x_1 and the dynamics passes to it. See [1] for details.

We close with some technical results regarding Ad_X invariance which we referred to in the beginning of this section and which will be used later on.

LEMMA 3.2. *Suppose $\mathscr{D}$ is an Ad_X invariant distribution and $c \in \mathbb{R}$; then $\mathscr{D}$ is $\mathrm{Ad}_{(cX)}$ invariant. Suppose $\mathscr{D}$ is an Ad_X invariant distribution, $X \in \mathscr{D}$ and $\varphi \in \mathscr{F}(M)$; then $\mathscr{D}$ is $\mathrm{Ad}_{(\varphi X)}$ invariant.*

Proof. The first statement follows immediately from the identity $\mathrm{Ad}^t_{(cX)} = \mathrm{Ad}^{ct}_X$. As for the second let $\tau(t, x)$ be the solution of

$$\frac{\partial}{\partial t}\tau(t, x) = \varphi(\Phi(\tau(t, x), x),$$

$$\tau(0, x) = 0,$$

where $\Phi(t, x)$ is the flow of X. Define $\Psi(t, x) = \Phi(\tau(t, x), x)$; then $\Psi(t, x)$ is the flow of φX for

$$\begin{aligned}\frac{\partial}{\partial t}\Psi(t, x) &= \frac{\partial \Phi}{\partial \tau}(\tau(t, x), x)\frac{\partial \tau}{\partial t}(\tau, x)\\ &= X(\Phi(\tau(t, x), x))\varphi(\Phi(\tau(t, x), x))\\ &= X(\Psi(t, x))\varphi(\Psi(t, x))\end{aligned}$$

and

$$\Psi(0, x) = \Phi(\tau(0, x), x) = \Phi(0, x) = x.$$

Now if $Y \in \mathscr{D}$ then by (2.3a)

$$\begin{aligned}\mathrm{Ad}^t_{(\varphi X)}(Y)(x) &= \frac{\partial \Psi}{\partial z}(-t, z)\,Y(\Psi(t, x))\\ &= \frac{\partial \Phi}{\partial z}(-\tau(t, z), z)\,Y(z)\\ &= \mathrm{Ad}^\tau_X(Y)(x) - \frac{\partial \Phi}{\partial \tau}(-\tau, z)\frac{\partial \tau}{\partial t}(t, z)\,Y(z)\end{aligned}$$

where $z = \Psi(t, x) = \Phi(\tau(t, x), x)$ and $\tau = \tau(t, x)$.

Since $\partial\Phi/\partial\tau(-\tau, z) = X(x) \in D(x)$ it follows that $\mathrm{Ad}^t_{(\varphi X)} Y \in \mathscr{D}$. QED

LEMMA 3.3. *Suppose $\mathscr{D}$ is Ad_{X^i} invariant for $i = 1, 2$. Then $\mathscr{D}$ is $\mathrm{Ad}_{(X^1+X^2)}$ invariant.*

Proof. Let $u(t) = (u_1(t), u_2(t))$ be a bounded measurable function. Let $\Phi_u(t, t_0, x^0)$ be the time dependent flow of the time dependent vector field $X_u(t, x) = X^1(x)u_1(t) + X^2(x)u_2(t)$, i.e.

$$\frac{d}{dt}\Phi_u(t, t_0, x^0) = X_u(t, \Phi_u(t, t_0, x^0)), \tag{3.5a}$$

$$\Phi_u(t_0, t_0, x^0) = x^0. \tag{3.5b}$$

By standard results from differential equations, for each t_1, the map $x_0 \to \Phi_u(t_1, t_0, x^0)$ is a local diffeomorphism. If $u^k(\cdot)$ converges to $u(\cdot)$ in the weak L^∞ topology on $[t_0, t_1]$ then $\Phi_{u^k}(t_1, t_0, \circ)$ converges to $\Phi_u(t_1, t_0, \circ)$ uniformly as small compact subsets. Moreover each of the derivatives does also.

$\mathscr{D}$ is Ad_{X^i} invariant iff the flow Φ^i of X^i carries the vector field of $\mathscr{D}$ back into $\mathscr{D}$, i.e. if $Y \in \mathscr{D}$ then $\Phi^i(t)_* Y \in \mathscr{D}$. Let $u^k(\cdot)$ be equal to $(2, 0)$ and $(0, 2)$ on intervals of length $1/k$; then u^k converges weakly to $u(t) = (1, 1)$. By assumption $\Phi_{u^k}(t_1, t_0)_* Y \in \mathscr{D}$ for all t_0, t_1 and $Y \in \mathscr{D}$. By continuity $\Phi_u(t_1, t_0)_* Y \in \mathscr{D}$, hence $\mathscr{D}$ is $\mathrm{Ad}_{(X^1+X^2)}$ invariant.

Remark. One could define Ad and ad invariance with respect to time dependent vector fields such as $X_u(t, x)$. By modifying the proofs of the above lemmas one can show that $\mathscr{D}$ is Ad_{X_u} (or ad_{X_u}) invariant for any bounded measurable $u(t)$ iff $\mathscr{D}$ is Ad_{X^i} (or ad_{X^i}) invariant for all i.

LEMMA 3.4. *Suppose $\mathscr{D}$ is Ad_{X^i} invariant for $i = 1, 2$. Then $\mathscr{D}$ is Ad_Z invariant where $Z = \mathrm{Ad}^\tau_{X^1}(X^2)$ for any τ.*

Proof. Let $\Phi^i(t, x)$ denote the flow of X^i and define

$$\Psi(t, x) = \Phi^1(-\tau, \Phi^2(t, \Phi^1(\tau, x))). \tag{3.6}$$

Since

$$\frac{\partial\Psi}{\partial t}(t,x)=\Phi^1(-\tau)_*X^2(\Phi^2(t,\Phi^1(t,x)))$$
$$=\Phi^1(-\tau)_*X^2(\Phi^1(\tau,\Psi(t,x)))$$
$$=Z(\Psi(t,x)),$$
$$\Psi(0,x)=x,$$

it follows that Ψ is the flow of Z. But then

$$\Psi(t)_*Y=\Phi^1(-\tau)_*\Phi^2(\tau)_*\Phi^1(\tau)_*Y,$$

hence $Y\in\mathcal{D}$ implies $\Psi(\tau)_*Y\in\mathcal{D}$. QED

4. Nonlinear controllability and observability. In this section we review the basic concepts of nonlinear controllability and observability because they are needed in the study of disturbance decoupling and noninteracting control and they are nowhere treated in an appropriate form. The closest reference is our joint work with Hermann [12] but we must apologize for the somewhat confusing terminology that we introduced there. We hope this section rectifies the situation.

The main difficulty in passing from linear to nonlinear is that typically there are several reasonable nonlinear generalizations of a single linear concept. The appropriate choice depends on the context.

Let $\mathcal{U}$ be an open connected subset of M and T a nonnegative real number.

DEFINITION. A point x^T is *$\mathcal{U}$ accessible* from x^0 at time T if there exists a bounded measurable control $u(t)$ generating a trajectory of (2.1) $x(t)\in\mathcal{U}$ for $t\in[0,T]$ such that $x(0)=x^0$ and $x(T)=x^T$. The set of all sets x^T, $\mathcal{U}$ accessible from x^0 at time T, is denoted by $\mathcal{A}(x^0,T,\mathcal{U})$. If $\mathcal{U}$ is suppressed, M is to be understood as in $\mathcal{A}(x^0,T)=\mathcal{A}(x_0,T,\mathcal{U})$. If T is suppressed, the union over all $T\geqq 0$ is understood as in $\mathcal{A}(x^0,\mathcal{U})=\bigcup_{T>0}\mathcal{A}(x^0,T,\mathcal{U})$.

DEFINITION. The system (2.1) is *controllable* if $\mathcal{A}(x^0)=M$ for every $x^0\in M$. The system (2.1) is *locally controllable* if restricted to every open connected subset $\mathcal{U}$ of M, (2.1) is controllable, i.e., $\mathcal{A}(x_0,\mathcal{U})=\mathcal{U}$ for every $x^0\in\mathcal{U}\subset M$.

It is apparent that local controllability implies controllability but not vice versa. We shall use the modifiers local and locally to mean that a property holds for (2.1) restricted to every open connected subset of the state space and hence a local property always implies that property. These definitions capture our intuitive idea of controllability and local controllability but unfortunately they are extremely difficult to work with. Deciding when a nonlinear system is controllable or locally controllable is generally a difficult task. We are more interested in controllability as one half of what constitutes a minimal realization, therefore we introduce weaker notions. The *time reversible version* of (2.1) is

$$\dot{x}=f(x,u_0,u)=g^0(x)u_0+g(x)u, \tag{4.1a}$$

$$y=h(x), \tag{4.1b}$$

$$x(0)=x^0. \tag{4.1c}$$

DEFINITION. The system (2.1) is *reversibly controllable* if (4.1) is controllable. The system (2.1) is *locally reversibly controllable* if (4.1) locally controllable. Let $\mathcal{RA}(x_0,T,\mathcal{U})$ be the set of points accessible in $\mathcal{U}$ from x^0 along trajectories of (4.1). Equivalently the system (2.1) is (locally) reversibly controllable if for every x_0 (and $\mathcal{U}$), $\mathcal{RA}(x^0)=M(\mathcal{RA}(x^0,\mathcal{U})=\mathcal{U})$.

Clearly (local) controllability implies (local) reversible controllability but not vice versa. Throughout we use the modifiers reversible and reversibly to mean that a property holds not for (2.1) itself but for its time reversible version (4.1), hence a property generally implies the corresponding reversible property.

These definitions emphasize one aspect of what one expects in a controllable system, the ability to steer from one point to another; but there is another: namely, that there are no uncontrollable modes, no coordinates of the state space which are unaffected by the control. The following are attempts to characterize this.

DEFINITION. The system (2.1) has the *(local) accessibility property* if $\mathcal{A}(x^0)$ ($\mathcal{A}(x^0, \mathcal{U})$) has nonempty interior for every $x^0 \in M$ (and open neighborhood $\mathcal{U}$). The system (2.1) has the *(local) reversible accessibility property* if (4.1) has the (local) accessibility property.

THEOREM 4.1. *If the system* (2.1) *has the accessibility property then it is reversibly controllable. The system has the* (*local*) *reversible accessibility property iff it is* (*locally*) *reversibly controllable. The system has the local accessibility property iff it is reversibly controllable.*

Proof. Reversible accessibility is an equivalence relation which partitions M. Suppose the system has the accessibility property so that $\mathcal{A}(x^0)$ has nonempty interior. This implies that $\mathcal{R}\mathcal{A}(x^0)$ (the set of points reversibly accessible from x^0) is an open subset of the connected manifold M, hence $\mathcal{R}\mathcal{A}(x^0) = M$. The proof of the second assertion is straightforward and the proof of the third is found in Hermann-Krener [12, Thm. 2.1]. QED

In summary the logical implications between various forms of controllability are

local controllability	$\Rightarrow$	controllability
$\Downarrow$		$\Downarrow$
local accessibility property	$\Rightarrow$	accessibility property
$\Updownarrow$		$\Downarrow$
local reversible controllability	$\Rightarrow$	reversible controllability
$\Updownarrow$		$\Updownarrow$
local reversible accessibility property	$\Rightarrow$	reversible accessibility property.

One would like a simple criterion to decide when a system is controllable or not. Unfortunately none seems to exist. These are however relatively straightforward criteria for some of the others. We denote by $\mathcal{R}(f)$ the distribution spanned (over $\mathcal{F}(M)$) by $\{f(\cdot, u)\colon u \text{ constant}\}$. Let $\langle \mathrm{Ad}_f | \mathcal{R}(f) \rangle$ and $\langle \mathrm{ad}_f | \mathcal{R}(f) \rangle$ denote the smallest Ad_f and ad_f invariant distributions containing $\mathcal{R}(f)$. These are the Ad_f and ad_f *controllability distributions.*

By Lemmas 3.2 and 3.3 the former is spanned by terms of the form

$$\mathrm{Ad}_{f^k}^{t_k} \circ \cdots \circ \mathrm{Ad}_{f^1}^{t_1} f^0 \tag{4.2a}$$

where $k \geqq 0$ and $f^j(x) = f(x, u^j)$ for u^j constant. The latter is spanned by terms of the form

$$\mathrm{ad}_{f^k} \circ \cdots \circ \mathrm{ad}_{f^1} f^0 \tag{4.2b}$$

and by the Jacobi identity (2.8b) is involutive. Lemma 3.4 implies that the former is integrable and is the integral closure of the latter. The next result is related to a theorem of Chow [21].

THEOREM 4.2 (Sussmann [11]). *The system* (2.1) *is reversibly controllable iff*

$$\langle \mathrm{Ad}_f | \mathcal{R}(f) \rangle = \mathcal{X}(M). \tag{4.3}$$

While this is very elegant, the Ad_f controllability distribution is not always easy to compute so the following can be more useful.

THEOREM 4.3 (Hermann–Krener [12]). *The system* (2.1) *is locally reversibly controllable if*

$$\langle \mathrm{ad}_f | \mathcal{R}(f) \rangle = \mathcal{X}(M). \tag{4.4}$$

If (2.1) *is locally reversibly controllable and D is the subbundle of TM associated to the* ad_f *controllability distribution then on an open dense subset of* M

$$D(x) = T_x M. \tag{4.5}$$

Equation (4.5) is usually referred to as the *controllability rank condition* at x. For a linear system (3.1) the Ad_f and ad_f controllability distributions both equal

$$\mathcal{R}\{Ax, A^r B^j : r = 0, \cdots, n-1, j = 1, \cdots, m\}.$$

For $x = 0$ the controllability rank condition (4.5) reduces to the familiar

$$\mathrm{Rank}\,(B, AB, \cdots, A^{n-1}B) = n.$$

Now we turn to observability where again we follow [12] in spirit but change terminology considerably. In what follows $\mathcal{U}$ always denotes an open subset of M.

DEFINITION. Two points x^0 and x^1 are *$\mathcal{U}$ distinguishable* if there exists a bounded measurable input $u(t)$ generating solutions $x^0(t)$ and $x^1(t)$ of (2.1a) satisfying $x^i(0) = x^i$ such that $x^i(t) \in \mathcal{U}$ for all $t \in [0, T]$ and $h(x^1(t)) \neq h(x^2(t))$ for some $t \in [0, T]$. We let $\mathcal{I}(x^0, \mathcal{U})$ denote all the points $x^1 \in \mathcal{U}$ which are not $\mathcal{U}$ distinguishable from x^0. If $\mathcal{U}$ is suppressed, M is understood as in $\mathcal{I}(x^0) = \mathcal{I}(x^0, M)$.

DEFINITION. The system (2.1) is *observable* if $\mathcal{I}(x^0) = \{x^0\}$ for every x^0. The system (2.1) is *locally observable* if for every open neighborhood $\mathcal{U}$ of x^0, $\mathcal{I}(x^0, \mathcal{U}) = \{x^0\}$. The system (2.1) is (*locally*) *reversibly* observable if (4.1) is (locally) observable.

All these definitions require that x^0 be distinguishable from every other point of M. The local ones require that x^0 and x^1 be distinguishable by local experiments. Frequently it may suffice that one be able to distinguish a point from its neighbors either by local or global experiments. Therefore we introduce additional terminology which was referred to as (local) weak observability in [12].

DEFINITION. The system (2.1) has the *distinguishability property* if every x^0 has an open neighborhood $\mathcal{V}$ such that $\mathcal{I}(x^0) \cap \mathcal{V} = \{x^0\}$. The system (2.1) has the *local distinguishability property* if every x^0 has an open neighborhood $\mathcal{V}$ such that for every open $\mathcal{U}$ neighborhood of x^0, $\mathcal{I}(x^0, \mathcal{U}) \cap \mathcal{V} = \{x^0\}$. The system (2.1) has the (*local*) *reversible distinguishability property* if (4.1) has the (local) distinguishability property.

The basic implications between these definitions are as follows.

local observability		⇒	observability	
	⇓		⇓	
⇓	loc. rev. observability	⇒	⇓	rev. observability
loc. dist. prop.	⇓	⇒	dist. prop.	⇓
	⇓		⇓	
	loc. rev. dist. prop.	⇒		rev. dist. prop.

If one makes a controllability assumption more implications follow; perhaps the most interesting is

THEOREM 4.4. *If* (2.1) *is locally reversibly controllable then the local distinguishability property and the local reversible distinguishability property are equivalent.*

We defer the proof to the end of this section.

Let $\mathscr{R}(dh)$ denote the codistribution spanned by dh_i, $i=1,\cdots,p$ and let $\langle \mathrm{Ad}_f|\mathscr{R}(dh)\rangle$ and $\langle \mathrm{ad}_f|\mathscr{R}(dh)\rangle$ denote the smallest Ad_f and ad_f invariant codistributions containing $\mathscr{R}(dh)$. We refer to these as the Ad_f and ad_f *observability codistributions.* They are the spans (over $\mathscr{F}(M)$) of terms of the form

(4.6a) $$\mathrm{Ad}_{f^k}^{t_k}\circ\cdots\circ\mathrm{Ad}_{f^1}^{t_1}\,dh_i$$

and

(4.6b) $$\mathrm{ad}_{f^k}\circ\cdots\circ\mathrm{ad}_{f^1}dh_i$$

respectively. By (2.9a, b) the exterior differential operator d can be pulled to the front in (4.6) so that by Lemma 2.6 these codistributions are integrable.

THEOREM 4.5 (Goncalves [13]). *The system* (2.1) *has reversible distinguishability property iff*

(4.7) $$\langle \mathrm{Ad}_f|\mathscr{R}(dh)\rangle=\mathscr{X}^*(M).$$

THEOREM 4.6 (Hermann-Krener [12]). *The system* (2.1) *has the local distinguishability property if*

(4.8) $$\langle \mathrm{ad}_f|\mathscr{R}(dh)\rangle=\mathscr{X}^*(M).$$

If (2.1) *has the local distinguishability property and E is the subbundle of* T^*M *associated to the* ad_f *observability codistribution then on an open dense subset of M*

(4.9) $$E(x)=T_x^*M.$$

Equation (4.9) is usually referred to as the *observability rank condition* of x. For a linear system (3.1) the Ad_f and ad_f observability codistributions are the $\mathscr{F}(M)$ span of the rows of the familiar observability matrix,

$$\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1}\end{pmatrix}.$$

Proof of Theorem 4.4. Clearly the local distinguishability property implies the local reversible distinguishability property. To see the converse notice that the ad_f observability codistribution for (2.1) and (4.1) are the same. Hence by Theorem 4.6, the observability rank condition holds on some open dense subset $\mathscr{V}$ of M. Therefore (2.1) restricted to $\mathscr{V}$ has the local distinguishability property, every x^0 and open neighborhood $\mathscr{U}$ is such that $\mathscr{A}(x^0,\mathscr{U})$ meets $\mathscr{V}$. But this implies x^0 can be distinguished from its neighbors. QED

5. (Ad_f, g), (ad_f, g) and local (ad_f, g) invariance. In the geometric approach to linear multivariable systems, as found in Wonham [15], the concept of an (A, B) invariant subspace plays a crucial role. Recall a subspace $V\subset\mathbb{R}^n$ is (A, B) invariant if one of two equivalent conditions is satisfied,

(5.1) $$AV\subset V+\mathscr{R}(B)$$

($\mathcal{R}(B)$ denotes the subspace spanned by the columns of B) or there exists an $m \times n$ matrix F such that

$$(A + BF)V \subset V. \tag{5.2}$$

For reasons that will become apparent later we refer to these as the local and global characterizations of (A, B) invariance. The global characterization (5.2) can be interpreted as modifying the dynamics (3.1a) by linear state feedback

$$u = Fx + v \tag{5.3}$$

so as to obtain the new system

$$\dot{x} = \tilde{A}x + Bv \tag{5.4}$$

where $\tilde{A} = A + BF$. The subspace V is an invariant subspace of the new dynamics.

When working with linear systems it is convenient to restrict oneself to constant distributions and linear feedback laws (5.3). We could allow a slightly more general form, say

$$u = Fx + Gv \tag{5.5}$$

but as far as (A, B) invariance is concerned it is not needed because every constant vector field leaves every constant distribution invariant. When dealing with (A, B) controllability subspaces, feedback laws such as (5.5) naturally arise.

A nonlinear feedback (or feedback) is a pair of matrix valued functions α and β on M; $\alpha(x)$ and $\beta(x)$ are $m \times 1$ and $m \times m$ matrices smoothly varying in x. They are used to define the feedback law

$$u = \alpha(x) + \beta(x)v \tag{5.6}$$

which results in the modified system

$$x = \tilde{f}(x, v) = \tilde{g}^0(x) + \tilde{g}(x)v \tag{5.7}$$

where $\tilde{g}^0(x) = g^0(x) + g(x)\alpha(x)$, $\tilde{g}(x) = g(x)\beta(x)$ and $\tilde{g}^j(x) = g(x)\beta^j(x)$ where $\beta^j(x)$ is the jth column of $\beta(x)$. It is convenient to combine these into an $(m+1) \times (m+1)$ matrix

$$\gamma = \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix} \tag{5.8}$$

and reexpress this as

$$\tilde{f}(x) = f(x)\gamma(x) \tag{5.9}$$

where $f(x)$ and $\tilde{f}(x)$ are $n \times (m+1)$ matrices

$$f(x) = (g^0(x), g(x)), \qquad \tilde{f}(x) = (\tilde{g}^0(x), \tilde{g}(x)). \tag{5.10}$$

Hopefully this second use of the symbols f and $\tilde{f}$ will cause no confusion. If there is no drift term g^0, then the feedback γ reduces to β.

DEFINITION. A distribution $\mathcal{D}$ is (Ad_f, g) *invariant* ((ad_f, g) *invariant*) if there exists a feedback γ such that $\mathcal{D}$ is $\mathrm{Ad}\,\tilde{f}$ invariant ($\mathrm{ad}\,\tilde{f}$ invariant). If γ is invertible then the distribution is invariant *with full control* otherwise it is invariant *with partial control.*

Unless otherwise stated "invariance" means "invariance with full control". This issue does not arise in the linear theory, because for reasons mentioned above, invariance always means with full control.

DEFINITION. A distribution $\mathscr{D}$ is *locally* (ad_f, g) *invariant with full control* if and for every constant u and for every $X \in \mathscr{D}$

$$\mathrm{ad}_{f(\cdot, u)}(X) \in \mathscr{D} + \mathscr{R}(g)$$

where $\mathscr{R}(g)$ denotes the distribution spanned by the columns of g.

A distribution $\mathscr{D}$ is *locally* (ad_f, g) *invariant with partial control* if there exists a feedback γ which is not necessarily invertible such that $\mathscr{D}$ is locally $(\mathrm{ad}_{\tilde{f}}, \tilde{g})$ invariant where $\tilde{f} = f\gamma$ $\tilde{g} = g\beta$. Again unless otherwise stated "local (ad_f, g) invariance" assumes "with full control".

It is not hard to see that (Ad_f, g) invariance implies (ad_f, g) invariance which in turn implies local (ad_f, g) invariance. Before we delve further into this area we need some additional terminology.

DEFINITION. A family of distribution $\mathscr{D}^1, \cdots, \mathscr{D}^\mu$ separates the controls if there exists locally an invertible feedback γ where β has been partitioned into submatrices $\beta = (\beta^1, \cdots, \beta^{\mu+1})$ such that

$$D^\sigma(x) \cap G(x) = \tilde{G}^\sigma(x), \qquad \sigma = 1, \cdots, \mu, \tag{5.11a}$$

$$\left(\sum_{\sigma=1}^{\mu} D^\sigma(x)\right) \cap \tilde{G}^{\mu+1}(x) = \{0\}, \tag{5.11b}$$

where G, $\tilde{G}^\sigma$ and D^σ are the subbundles of TM associated to the distributions $\mathscr{R}(g)$, $\mathscr{R}(\tilde{g}^\sigma)$ and $\mathscr{D}^\sigma$ respectively. A family of distributions *completely separates* the controls if there exists an invertible feedback γ with $\beta = (\beta^1, \cdots, \beta^\mu)$ such that (5.11a) holds. Such feedbacks γ are said to be (*completely*) *separating* for the family of distributions $\mathscr{D}^1, \cdots, \mathscr{D}^\mu$. A distribution $\mathscr{D}$ *separates* the controls if considered as a one element family of distributions it separates.

Notice α does not play a role in these definitions.

LEMMA 5.1. *If $\mathscr{D}$ is nonsingular, involutive and locally separates the controls then the following are equivalent*:

(a) *$\mathscr{D}$ is locally (ad_f, g) invariant.*

(b) *There exist an open cover $\{\mathscr{U}^\rho\}$ of M and separating feedbacks γ^ρ defined on $\mathscr{U}^\rho$ such that $\mathscr{D}$ is $\mathrm{ad}\,\tilde{f}^\rho$ invariant on $\mathscr{U}^\rho$ where $f^\rho = \tilde{f}\gamma^\rho$ (in other words, locally $\mathscr{D}$ is (ad_f, g) invariant).*

(c) *There exist an open cover $\{\mathscr{U}^\rho\}$ of M and separating feedbacks γ^ρ defined on $\mathscr{U}^\rho$ such that $\mathscr{D}$ is $\mathrm{Ad}\,\tilde{f}^\rho$ invariant on $\mathscr{U}^\rho$ (in other words, locally $\mathscr{D}$ is (Ad_f, g) invariant).*

Proof. The equivalence of (b) and (c) follows from the nonsingularity of $\mathscr{D}$. It is trivial to verify that (b) implies (a). In [3] it is shown that (a) implies (b) using the stronger hypothesis that $\mathscr{D} \cap \mathscr{R}(g)$ and $\mathscr{R}(g)$ are nonsingular. But the proof only uses this to show that $\mathscr{D}$ separates the controls. Moreover the feedback so constructed is easily seen to be separating. Similar results are found in [4]. QED

This lemma explains our terminology, in particular why we refer to (5.1) and (5.2) as the local and global characterizations of (A, B) invariance. For a discussion of the topological obstructions to global invariance we refer the reader to [20].

LEMMA 5.2. *If $\mathscr{D}$ is (Ad_f, g) invariant or (ad_f, g) invariant then so is the involutive closure of $\mathscr{D}$. If $\mathscr{D}$ is locally (ad_f, g) invariant then so is the involutive closure of $\mathscr{D}$. If $\mathscr{D}^1$ and $\mathscr{D}^2$ are locally (ad_f, g) invariant then so is $\mathscr{D}^1 + \mathscr{D}^2$, hence the set of locally (ad_f, g) invariant distributions forms a semilattice under inclusion and addition.*

Proof. Since (Ad_f, g) or (ad_f, g) invariance is equivalent to $\mathrm{Ad}_{\tilde{f}}$ or $\mathrm{ad}_{\tilde{f}}$ invariance for some feedback modified dynamics $\tilde{f}$, the first statement follows from (2.7). The second statement is proved in [3] and the third follows directly from the definition of local (ad_f, g) invariance. QED

Remarks. The sum and intersection of (Ad_f, g) (or (ad_f, g)) invariant distributions and the intersection of locally (ad_f, g) invariant distributions need not be invariant in the same sense. However the sum of locally (ad_f, g) invariant distributions is again locally (ad_f, g) invariant. This semilattice structure makes them convenient to work with. In particular it implies that in any distribution $\mathcal{D}$ there exists a unique maximal locally (ad_f, g) invariant distribution which we shall denote by $\mathcal{D}^*(\mathcal{D})$. If $\mathcal{D}$ is involutive then so is $\mathcal{D}^*(\mathcal{D})$. These remarks are predicated on the assumption of invariance with full control. They must be modified when considering invariance with partial control. In particular there may be distributions contained in $\mathcal{D}$ and properly containing $\mathcal{D}^*(\mathcal{D})$ which are locally (ad_f, g) invariant with partial control.

Briefly we discuss the dual formulation of the above, for it is useful in computing maximal locally (ad_f, g) invariant distributions.

DEFINITION. A codistribution $\mathscr{E}$ is (Ad_f, g) *invariant* ((ad_f, g) invariant) if there exists a feedback γ such that $\mathscr{E}$ is $\mathrm{Ad}_{\tilde{f}}$ invariant ($\mathrm{ad}_{\tilde{f}}$ invariant). A codistribution $\mathscr{E}$ is *locally* (ad_f, g) *invariant* if for every constant u and every $\omega \in \mathscr{E} \cap \mathscr{K}(g)$

$$L_{f(\cdot,u)}(\omega) \in \mathscr{E}.$$

Recall $\mathscr{K}(g)$ is the codistribution of one forms which annihilate the columns of g.

LEMMA 5.3. *If the distribution $\mathcal{D}$ is (Ad_f, g) invariant ((ad_f, g) invariant) then the codistribution $\mathcal{D}^\perp$ is (Ad_f, g) invariant ((ad_f, g) invariant). If the codistribution $\mathscr{E}$ is (Ad_f, g) invariant ((ad_f, g) invariant) then the distribution $\mathscr{E}^\perp$ is (Ad_f, g) invariant ((ad_f, g) invariant). If the distribution $\mathcal{D}$ is locally (ad_f, g) invariant then the codistribution $\mathcal{D}^\perp$ is locally (ad_f, g) invariant. If the codistribution $\mathscr{E}$ is locally (ad_f, g) invariant and $\mathscr{E}$ and $\mathscr{E} \cap \mathscr{K}(g)$ are nonsingular then the codistribution $\mathscr{E}^\perp$ is locally (ad_f, g) invariant.*

Proof. The first two assertions are almost immediate. As for the third let $\omega \in \mathcal{D}^\perp \cap \mathscr{K}(g) = (\mathcal{D} + \mathscr{R}(g))^\perp$ and $X \in \mathcal{D}'$, then

$$0 = L_{f(\cdot,u)}\langle \omega, X\rangle = \langle L_{f(\cdot,u)}(\omega), X\rangle + \langle \omega, \mathrm{ad}_{f(\cdot,u)}(X)\rangle. \tag{5.12}$$

Since $\mathcal{D}$ is locally (ad_f, g) invariant the second term on the right is zero hence $L_{f(\cdot,u)}(\omega) \in \mathcal{D}^\perp$.

The last assertion follows in a similar fashion. Let $\omega \in (\mathscr{E}^\perp + \mathscr{R}(g))^\perp = \mathscr{E} \cap \mathscr{K}(g)$ (by the nonsingularity of $\mathscr{E}$) and $X \in \mathscr{E}^\perp$. Since $\mathscr{E}$ is locally (ad_f, g) invariant the first term on the right of (5.12) is zero hence $\mathrm{ad}\, f(\cdot, u)(X) \in (\mathscr{E}^\perp + \mathscr{R}(g))^{\perp\perp} = \mathscr{E}^\perp + \mathscr{R}(g)$ by the nonsingularity of $(\mathscr{E}^\perp + \mathscr{R}(g))^\perp = \mathscr{E} \cap \mathscr{K}(g)$. QED

In disturbance decoupling and other problems one wishes to find $\mathcal{D}^*(\mathscr{K}(dh))$, the maximal locally (ad_f, g) invariant distribution in $\mathscr{K}(dh)$. We now present an algorithm from [1, p. 342] for the computation of $\mathcal{D}^*(\mathcal{D})$ for an arbitrary distribution $\mathcal{D}$ which works when all the distributions and codistributions involved in the calculations are nonsingular. We then specialize to compute $\mathcal{D}^*(\mathscr{K}(dh))$. When there is no possibility of confusion we shall abbreviate, $\mathcal{D}^* = \mathcal{D}^*(\mathscr{K}(dh))$.

Let $\mathcal{D}$ be an arbitrary distribution and $\mathscr{E}_*(\mathcal{D})$ be the minimal locally (ad_f, g) codistribution containing $\mathcal{D}^\perp$.

Define an increasing sequence of codistributions by

$$\mathscr{E}_0 = \mathcal{D}^\perp \quad \text{and} \quad \mathscr{E}_{k+1} = \mathscr{E}_k + L_f(\mathscr{E}_k \cap \mathscr{K}(g))$$

where the second term on the right denotes the $\mathscr{F}(M)$ span of all one forms like $L_{g^j}(\omega)$ for $j = 0, \cdots, m$ and $\omega \in \mathscr{E}_k \cap \mathscr{K}(g)$.

THEOREM 5.4 (invariant subdistribution algorithm (ISA)). *If there exists a* k_* *such that* $\mathscr{E}_{k_*} = \mathscr{E}_{k_*+1}$ *then* $\mathscr{E}(\mathscr{D}) = \mathscr{E}_{k_*}$. *If in addition* $\mathscr{E}_{k_*}$ *and* $\mathscr{E}_{k_*} \cap \mathscr{K}(g)$ *are nonsingular then* $\mathscr{D}^*(\mathscr{D}) = \mathscr{E}_{k_*}^{\perp}$.

Proof. By definition $\mathscr{E}_*(\mathscr{D})$ contains $\mathscr{E}_0 = \mathscr{D}^{\perp}$ and is locally (ad_f, g) invariant. A simple induction shows that $\mathscr{E}_*(\mathscr{D})$ contains $\mathscr{E}_k$ for all k. If $\mathscr{E}_{k_*} = \mathscr{E}_{k_*+1}$ then $\mathscr{E}_{k_*}$ is locally (ad_f, g) invariant and clearly minimal.

If $\mathscr{E}_{k_*}$ and $\mathscr{E}_{k_*} \cap \mathscr{K}(g)$ are nonsingular then $\mathscr{E}_{k_*}^{\perp}$ is a locally (ad_f, g) invariant distribution by Lemma 5.3. By duality it is the maximal such distribution contained in $\mathscr{D}$. QED

Computation of $\mathscr{D}^* = \mathscr{D}^*(\mathscr{K}(dh))$ *by* ISA.

$$\mathscr{E}_0 = \mathscr{K}(dh)^{\perp} = \mathscr{R}(dh) = \mathscr{R}(dh_1, \cdots, dh_{p_0})$$

where $p_0 = p$. Let $A_0(x)$ be the $p_0 \times m$ matrix whose ith, jth element is $\langle dh_i, g^j\rangle(x)$. Let $B_0(x)$ be the $p_0 \times 1$ vector whose ith element is $\langle dh_i, g^0\rangle(x)$. Assume the rank of $A_0(x)$ is constant and equal to r_0. By rearranging $h_1, \cdots, h_{p_0}$ if necessary we assume that the first r_0 rows of $A_0(x)$ are linearly independent at each x. Choose $m \times 1\, \alpha_0(x)$ and invertible $m \times m$ $\beta_0(x)$ such that

$$A_0(x)\alpha_0(x) + B_0(x) = \begin{pmatrix} 0 \\ \varphi_0 \end{pmatrix}, \tag{5.13a}$$

$$A_0(x)\beta_0(x) = \begin{pmatrix} I^{r_0 \times r_0} & 0 \\ \psi_0 & 0 \end{pmatrix}, \tag{5.13b}$$

where φ_0 and ψ_0 are arbitrary $1 \times (p_0 - r_0)$ and $(p_0 - r_0) \times r_0$ matrix valued functions.

Define $\tilde{g}_0^0 = g^0 + g\alpha_0$, $\tilde{g}_0 = g\beta_0 = (\tilde{g}_0^1, \tilde{g}_0^2)$ where $\tilde{g}_0^1$ is the first r_0 vector fields of $\tilde{g}_0$ and $\tilde{g}_0^2$ the last $m - r_0$. From the functions which are the entries of φ and ψ_0 if (5.13a, b), choose a maximal set whose differentials are linearly independent at each x mod $\mathscr{E}_0$. Call these $h_{p_0+1}, \cdots, h_{p_1}$. We claim that $\mathscr{E}_1 = \mathscr{R}\{dh_1, \cdots, dh_{p_1}\}$.

By definition $\mathscr{E}_1 = \mathscr{E}_0 + L_f(\mathscr{E}_0 \cap \mathscr{K}(g))$, but a straightforward calculation shows that this is the same as $\mathscr{E}_0 + L_{\tilde{f}_0}(\mathscr{E}_0 \cap \mathscr{K}(g))$ where

$$\tilde{f}_0 = f_{\gamma_0}, \qquad \gamma_0 = \begin{pmatrix} 1 & 0 \\ \alpha_0 & \beta_0 \end{pmatrix}$$

because γ_0 is invertible. From (5.13b) we see that

$$dh_i \notin \mathscr{K}(g), \qquad i = 1, \cdots, r_0, \tag{5.14a}$$

$$dh_i - \sum_{k=1}^{r_0} \psi_{0_i}^k\, dh_k \in \mathscr{K}(g), \qquad i = r_0 + 1, \cdots, p_0, \tag{5.14b}$$

so $\mathscr{E}_1$ is the sum of $\mathscr{E}_0$ and the Lie derivatives of (5.14b) by $L_{\tilde{f}_0}$, i.e.

$$L_{\tilde{f}_0}(dh_i - \Sigma \psi_{0i}^k dh_k) = L_{\tilde{f}_0}(dh_i) - \sum (L_{\tilde{f}_0}(\psi_{0i}^k) dh_k + \psi_{0i}^k L_{\tilde{f}_0}(dh_k).$$

But $dh_k \in \mathscr{E}_0$ and

$$L_{\tilde{f}_0}(dh_k) = dL_{\tilde{f}_0}(h_k) = d(0 \text{ or } 1) = 0$$

for $k = 1, \cdots, r_0$. Therefore $\mathscr{E}_1$ is spanned by $\mathscr{E}_0$ and the entries of $L_{\tilde{f}_0}(dh_i)$ for $i = r_0 + 1, \cdots, p_0$. But the latter are either zero or the differentials of the components of φ_0 and ψ_0.

$\mathscr{E}_2$ is constructed in a similar fashion. Let $A_1(x)$ be the $p_1 \times m$ matrix $\langle dh_i, g^j\rangle(x) B_1(x)$ be the $p_1 \times m$ vector $\langle dh_i, g^j\rangle(x)$. Assume $A_1(x)$ is of rank r_1 and rearrange $h_{r_0+1}, \cdots, h_{p_0}$ so that the first r_1 rows of $A_1(x)$ are linearly independent at each x. Choose α_1, β_1, etc.

Notice that at each stage of this algorithm we obtain codistributions $\mathscr{E}_k$ spanned by exact one-forms, hence they are integrable. The new feedbacks $\alpha_{k+1}, \beta_{k+1}$ can be obtained by suitably updating α_k and β_k. Moreover α_{k_*} and β_{k_*} are feedbacks which leave $\mathscr{D}^*$ invariant. If $r_{k_*} < m$ then we can partition $\tilde{g}_{k_*} = g\beta_{k_*} = (\tilde{g}^1_{k_*}, \tilde{g}^2_{k_*})$ where $R(\tilde{g}^2_{k_*}) = \mathscr{D}^* \phi R(g)$. We shall make use of this later on.

6. Disturbance decoupling. Consider the nonlinear system

$$\dot{x} = f(x, u) + p(x)w = g^0(x) + g(x)u + p(x)w, \tag{6.1a}$$

$$y = h(x), \tag{6.1b}$$

$$x(0) = x^0. \tag{6.1c}$$

The additional input $w(t)$ represents a disturbance which can be neither controlled nor predicted. We assume it is a bounded measurable function taking values in $\mathbb{R}^l$. The way it affects the dynamics is described by the l vector fields which in local coordinates are the l columns of $p(x)$.

DEFINITION. In the system (6.1) the disturbance is *decoupled* from the output if for each bounded measurable $u(t)$, the output $y(t)$ does not depend on the disturbance $w(t)$. The *disturbance decoupling problem* (DDP) is solvable if there exists a feedback γ such that the disturbance is decoupled from the output for the feedback modified system

$$\begin{aligned} \dot{x} = \tilde{f}(x, v) + p(x)w &= \tilde{g}^0(x) + \tilde{g}(x)v + p(x)w \\ &= g^0(x) + g(x)(\alpha(x) + \beta(x)v) + p(x)w. \end{aligned} \tag{6.2}$$

The *reversible disturbance decoupling problem* (RDDP) is solvable if there exists a feedback γ such that the disturbance is decoupled from the output for the time reversible version of the feedback modified system

$$\begin{aligned} \dot{x} = \tilde{f}(x, v_0, v) + p(x)w &= \tilde{g}^0(x)v_0 + \tilde{g}(x)v + p(x)w \\ &= g^0(x)v_0 + g(x)(\alpha(x)v_0 + \beta(x)v) + p(x)w. \end{aligned} \tag{6.3}$$

Notice that in contrast with controllability and observability, reversible decoupling implies decoupling rather than vice versa. Notice also that the solvability of the RDDP for the original system implies the solvability of the DDP for the time reversible version of the original system but is not equivalent to it. This is because in the former the invertible feedback γ must be of the form $\left(\begin{smallmatrix} 1 & 0 \\ \alpha & \beta \end{smallmatrix}\right)$ while in the latter any invertible feedback is allowed.

The solvability of the DDP and generalizations involving dynamic output feedback are treated at considerable length in [1], see also [2]. We would like to review some of this work using the terminology introduced in this paper and also discuss the solvability of the RDDP. We consider only the solvability of the DDP and RDDP with full control. If a partial control solution is acceptable it can be thought of as full control solution for the system with the unneeded controls deleted.

We state the basic results and defer the proofs to the end of the section.

THEOREM 6.1. *The* RDDP *is solvable iff there exists an* (Ad_f, g) *invariant distribution* $\mathscr{D}$ *such that* $\mathscr{R}(p) \subset \mathscr{D} \subset \mathscr{K}(dh)$.

Every (Ad_f, g) invariant distribution is also (ad_f, g) invariant so the above theorem implies that if the RDDP is solvable then there must exist an (ad_f, g) invariant distribution $\mathscr{D}$ such that $\mathscr{R}(p)\subset\mathscr{D}\subset\mathscr{H}(dh)$. Since $\mathscr{H}(dh)$ is involutive we can conclude that there must exist such a $\mathscr{D}$ which is involutive. But one can make a stronger statement.

THEOREM 6.2. *If the* DDP *is solvable then there exists an involutive* (ad_f, g) *invariant distribution* $\mathscr{D}$ *such that* $\mathscr{R}(p)\subset\mathscr{D}\subset\mathscr{H}(dh)$.

The converse of this theorem is not true as is shown by Example 6.6. Recall that the (Ad_f, g) or (ad_f, g) invariant distributions do not form a semilattice while the locally (ad_f, g) invariant ones do. We denote by $\mathscr{D}^*$ the maximal locally (ad_f, g) invariant distribution in $\mathscr{H}(dh)$, $\mathscr{D}^*=\mathscr{D}^*(\mathscr{H}(dh))$. In § 5 an algorithm for the computation of $\mathscr{D}^*$ was presented.

DEFINITION. The DDP (RDDP) is *locally solvable* if every $x^0\in M$ has an open neighborhood $\mathscr{U}$ and a feedback γ defined on $\mathscr{U}$ which solves the DDP(RDDP) restricted to $\mathscr{U}$.

THEOREM 6.3. *If the* DDP *is locally solvable then* $\mathscr{R}(p)\subset\mathscr{D}^*$. *If* $\mathscr{R}(p)\subset\mathscr{D}^*$ *and* $\mathscr{D}^*$ *is nonsingular and separates the controls then the* RDDP *is locally solvable.*

The proofs of the above depend heavily upon the following lemmas. (These lemmas describe basic properties of (Ad_f, g) and (ad_f, g) controllability distributions, concepts which will be introduced in the next section.) Let $\langle \mathrm{Ad}_{f,p})|\mathscr{R}(p)\rangle$ denote the minimal Ad_f and Ad_p invariant distribution which contains $\mathscr{R}(p)$. By Lemmas 3.2, 3.3, 3.4 and Sussmann's Theorem 2.1, this distribution is integrable. Let $(u_0(t), u(t))$ be a bounded measurable input defined on $[0, T]$ for the time reversible version (6.4) of (6.1),

$$\dot{x}=f(x, u_0, u)+p(x)w=g^0(x)u_0+g(x)u=p(x)w. \tag{6.4}$$

Let $\mathscr{R}\mathscr{A}(x^0, T, u_0(t), u(t))$ denote the set of points accessible from x^0 at time T along trajectories of (6.4) with $(u_0(t), u(t))$ fixed and $w(t)$ varying over all bounded measurable disturbances. Let x^T be the endpoint of the trajectory for $w(t)=0$.

LEMMA 6.4. *Let L be the leaf through* x^T *of the foliation induced by* $\langle \mathrm{Ad}_{(f,p)}|\mathscr{R}(p)\rangle$; *then* $\mathscr{R}\mathscr{A}(x^0, T, u_0(t), u(t))\subset L$. *Moreover for some piecewise constant control* $(u_0(t), u(t))$, $x^0=x^T$ *and* $\mathscr{R}\mathscr{A}(x^0, T, u_0(t), u(t))$ *is a neighborhood of* x^0 *in the topology of the leaf containing* x^0.

Let $\langle \mathrm{ad}_{(f,p)}|\mathscr{R}(p)\rangle$ denote the minimal ad_f and ad_p invariant distribution containing $\mathscr{R}(p)$. By the Jacobi identity (2.8b) this distribution is involutive. Let $\mathscr{U}$ be an open neighborhood of x^0 and $u(t)$ be a bounded measurable control defined on $[0, T]$ which generates a trajectory $x(t)$ of (2.1) from x^0 which lies in $\mathscr{U}$ for all $t\in[0, T]$. Let $\mathscr{A}(x^0, T, \mathscr{U}, u(t))$ be set of points accessible under (6.1) from x^0 in $\mathscr{U}$ at time T with $u(t)$ *fixed and* $w(t)$ varying over all bounded measurable disturbances.

LEMMA 6.5. *Let* $\mathscr{U}$ *be an open neighborhood of* x^0 *on which* $\langle \mathrm{ad}_{(f,p)}|\mathscr{R}(p)\rangle$ *is nonsingular, hence integrable. Let L be the leaf through* x^T; *then* $\mathscr{A}(x^0, T, \mathscr{U}, u(t))\subset L$. *Moreover there exists a piecewise constant control* $u(t)$ *such that* $\mathscr{A}(x^0, T, \mathscr{U}, u(t))$ *has nonempty interior in the topology of this leaf.*

Next we give the proofs of these results and a counterexample to the converse of Theorem 6.2.

Proof of Lemma 6.4. Without loss of generality we can assume that $\langle \mathrm{Ad}_{(f,p)}|\mathscr{R}(f, p)\rangle=\mathscr{X}(M)$ or in other words, with $u(t)$ and $w(t)$ as controls, (6.1) is reversibly controllable. For if not, $\langle \mathrm{Ad}_{(f,p)}|\mathscr{R}(f, p)\rangle$ is an integrable distribution and by replacing the state space by the leaf of this distribution through x^0 we obtain a reversibly controllable system. The assumption of reversible controllability insures that $\mathscr{D}=\langle \mathrm{Ad}_{(f,p)}|\mathscr{R}(p)\rangle$ is nonsingular, for any x^0 and x^T can be joined by a trajectory constructed from the flows of g^j, $j=0,\cdots,m$ and p^k, $k=1,\cdots,l$. But if D is the subbundle of TM corresponding to $\mathscr{D}$ then the Jacobian of these composed flows is an isomorphism between $D(x^0)$ and $D(x^T)$.

Now suppose $(u_0(t), u(t))$ and $w(t)$ are bounded measurable functions on $[0, T]$. Let $x(t)$ be the solution (6.4) and $\Phi(t, s, x)$ the time dependent flow of (6.4) with $w(t) = 0$, i.e.,

$$\frac{\partial\Phi}{\partial t}(t, s, x) = g^0(\Phi(t, s, x))u_0(t) + g(\Phi(t, s, x))u(t),$$

$$\Phi(t, t, x) = x.$$

The mapping $x \to \Phi(t, s, x)$ is smooth and its Jacobian carries $D(x)$ onto $D(\Phi(t, s, x))$. Consider the trajectory $\tilde{x}(s)$ defined by

$$\tilde{x}(s) = \Phi(T, s, x(s)).$$

Clearly $\tilde{x}(0) = x^T$ (the endpoint of the solution of (6.4) with $w(t) = 0$) and $\tilde{x}(T) = x(T)$ (the endpoint of the solution of (6.4) with $w(t)$ as above). Moreover

$$\frac{d}{ds}\tilde{x}(s) = \frac{\partial\Phi}{\partial x}(T, s, x(s))(p(x(s))w(s))$$

hence is an element of $D(\tilde{x}(s))$. The nonsingularity of $\mathcal{D}$ implies $\tilde{x}(s)$ lies in the leaf L of $\mathcal{D}$ through x^T. Therefore $x(T) = \tilde{x}(T) \in L$ and $\mathcal{R}\mathcal{A}(x^0, T, u_0(t), u(t)) \subset L$.

To prove the second assertion first we note that $\mathcal{D}$ is spanned by expressions of the form

$$\text{(6.5)} \qquad \mathrm{Ad}^{s_k}_{(f^k+p^k)} \circ \cdots \circ \mathrm{Ad}^{s_1}_{(f^1+p^1)} p^0$$

where $f^j(x) = g^0(x)u_0^j + g(x)u^j$ and $p^j(x) = p(x)w^j$ for some constants u_0^j, u^j, w^j and s_j. By rescaling u_0^j, u^j, w^j we can assume $s_j < 0$ and fix the sum $\sum_{j=1}^k s_j$ arbitrarily. Choose an expression of the form (6.5) which is not zero at x^0, and define piecewise constant functions

$$\text{(6.6)} \qquad u_0(t) = u_0^j, \quad u(t) = u^j, \quad w(t) = w^j \quad \text{for } t \in [t_{j-1}, t_j)$$

where $t_k = T$ and $t_{j-1} - t_j = s_j$. Assume that $\sum_{j=1}^k s_j = -T/2$. Let $x(t)$ be the solution of (6.4) satisfying the terminal condition $x(T) = x^0$. If we modify $w(t)$ to $w(t; \varepsilon)$

$$\text{(6.7)} \qquad w(t; \varepsilon) = \begin{cases} w^1 + w^0 & \text{if } t \in [t_0, t_0 + |\varepsilon|) \text{ and } \varepsilon > 0, \\ w^1 - w^0 & \text{if } t \in [t_0, t_0 + |\varepsilon|) \text{ and } \varepsilon < 0, \\ w(t) & \text{otherwise,} \end{cases}$$

and let $x(t, \varepsilon)$ be the solution of (6.4) satisfying the initial condition $x(t_0, \varepsilon) = x(t_0)$ then $\partial/\partial\varepsilon\ (x(T; 0))$ is precisely (6.5) evaluated at x^0.

By reversing the order and the signs of the inputs (6.6) we can get from x^0 to $x(t_0)$ in time $T/2$ and use the original sequence of inputs (6.6) to go back to x^0. Suppose we vary ε only on the second half according to (6.7), $x(t; \varepsilon)$ is now the endpoint of the total trajectory and it sweeps out a one-dimensional C^1 submanifold containing x^0 in its interior which is contained in the integral manifold L of $\mathcal{D}$ through x^0.

If the dimension of $\mathcal{D}$ is greater than one we repeat the process, this time at $x(t_0)$ instead of x^0. We also choose a new expression (6.5) which is linearly independent of the tangent to our one-dimensional submanifold pulled back to $x(t_0)$. This is always possible since expressions of the form (6.5) span $\mathcal{D}$ at x^0.

In this way we generate a one parameter family of controls which generates a one-dimensional manifold with $x(t_0)$ in its interior. When this is pulled on to x^0 along the original variation we get a two-dimensional submanifold of L with x^0 in its interior. We repeat the construction until the dimension of $\mathcal{D}$ is achieved. QED

Proof of Theorem 6.1. If the RDDP is solvable by a feedback, γ, let $\tilde{f}=f\gamma$ and let $\mathcal{D}=\langle \mathrm{Ad}_{(\tilde{f},p)}|\mathcal{R}(p)\rangle$. Clearly $\mathcal{D}$ is $\mathrm{Ad}_{\tilde{f}}$ invariant, hence (Ad_f, g) invariant and contains $\mathcal{R}(p)$. Suppose $\mathcal{D}$ is not contained in $\mathcal{H}(dh)$. Then at some x^0, $D(x^0)\not\subset dh(x^0)$. By Lemma 6.4 there exist T and a piecewise constant $(u_0(t), u(t))$ such that $\mathcal{R}\mathcal{A}(x^0, T, u_0(t), u(t))$ is a neighborhood of x^0 in the leaf N of $\mathcal{D}$ through x^0. This implies that $\mathcal{R}\mathcal{A}(x^0, T, u_0(t), u(t))$ is not contained in a level set of h, contradicting the solvability of the RDDP.

On the other suppose such a $\mathcal{D}$ exists. Let γ be the feedback such that $\mathcal{D}$ is $\mathrm{Ad}_{\tilde{f}}$ invariant for $\tilde{f}=f\gamma$. The integrable closure of $\mathcal{D}$ must contain $\langle \mathrm{Ad}_{(\tilde{f},p)}|\mathcal{R}(p)\rangle$ and since $\mathcal{H}(dh)$ is integrable it must contain the integrable closure of $\mathcal{D}$. Hence

$$\mathcal{R}(p)\subset\langle \mathrm{Ad}_{(\tilde{f},p)}|\mathcal{R}(p)\rangle\subset\mathcal{H}(dh).$$

Lemma 6.4 implies that for any fixed $(u_0(t), u(t))$ and T, $\mathcal{R}\mathcal{A}(x^0, T, u^0(t), u(t))$ is contained in a leaf of $\langle \mathrm{Ad}_{(\tilde{f},p)}|\mathcal{R}(p)\rangle$ which in turn is contained in a level set of h. Therefore the RDDP is strongly solvable. QED

Proof of Lemma 6.5. The first assertion follows from an application of Lemma 6.4 to the system restricted to $\mathcal{U}$, for the nonsingularity of $\langle \mathrm{ad}_{(f,p)}|\mathcal{R}(p)\rangle$ implies it equals $\langle \mathrm{Ad}_{(f,p)}|\mathcal{R}(p)\rangle$.

The proof of the second is similar to that of [1, Lemma 3.5]. Suppose $\mathcal{D}=\langle \mathrm{ad}_{(f,p)}|\mathcal{R}(p)\rangle$ is of dimension d on $\mathcal{U}$. $\mathcal{D}$ is spanned by terms of the form

$$\mathrm{ad}_{(f^k+p^k)}\circ\cdots\circ\mathrm{ad}_{(f^1+p^1)}p^0. \tag{6.8}$$

$\mathcal{D}$ is the involutive closure of $\langle \mathrm{ad}_f|\mathcal{R}(p)\rangle$ which is spanned by terms of the form

$$\mathrm{ad}_{f^k}\circ\cdots\circ\mathrm{ad}_{f^1}p^0. \tag{6.9}$$

Choose an expression (6.9) which is not 0 at x^0 define $u(t)$, $w(t,\varepsilon)$ for small $\varepsilon>0$ by

$$u(t)=u^j,\qquad t\in[t_{j-1}, t_j],$$
$$w(t,\varepsilon)=\begin{cases} w^0 & \text{if } t\in[t_0, t_0+\varepsilon),\\ 0 & \text{otherwise,}\end{cases}$$

where $t_0=0$ and $t_1,\cdots,t_n$ are to be determined. We have to take care choosing u^j, w^0, and t_j sufficiently small so that the trajectories $x(t;\varepsilon)$ of (6.1) from x^0 remain in $\mathcal{U}$ and $t_k<T$. Henceforth we shall not mention this point.

Since (6.9) is not zero for some choice of $t_1,\cdots,t_n$, as we vary ε, $x(t_k;\varepsilon)$ sweeps out a one-dimensional submanifold. Suppose that at some point on this submanifold there is an expression of the form (6.9) which is not tangent to the submanifold. Then we can repeat this process and construct a two-dimensional submanifold of points accessible at some later time. We repeat the process until we obtain a d-dimensional submanifold of accessible points such that every expression of the form (6.9) is tangent to it. Since the vector fields tangent to a manifold are trivially involutive and $\mathcal{D}$ is the involutive closure of (6.9), this manifold must be an integral manifold of $\mathcal{D}$. This shows that $\mathcal{A}(x^0, T, \mathcal{U}, u(t))$ has nonempty interior in the leaf topology. QED

Proof of Theorem 6.2. Suppose the DDP is solvable using feedback γ, let $\tilde{f}=f\gamma$ and $\mathcal{D}=\langle \mathrm{ad}_{(\tilde{f},p)}|\mathcal{R}(p)\rangle$. Clearly $\mathcal{D}$ is involutive and contains $\mathcal{R}(p)$, so all we need to show is that $\mathcal{D}\subset\mathcal{H}(dh)$.

Recall that x^0 is a regular point of $\mathcal{D}$ if $\mathcal{D}$ is nonsingular in a neighborhood of x^0. The regular points of $\mathcal{D}$ are open and dense in M hence by continuity it suffices to verify that at each regular point x^0 the subbundle D associated to $\mathscr{D}$ satisfies $D(x^0)\perp dh(x^0)$.

Let x^0 be a regular point and $\mathscr{U}$ a neighborhood on which $\mathscr{D}$ is nonsingular. By Lemma 6.5, there exists $x^T \in \mathscr{U}$ such that $\tilde{\mathscr{A}}(x^0, T, \mathscr{U}, v(t))$ is a neighborhood of x^T in the leaf of $\mathscr{D}$ containing x^T. (We use $\tilde{\mathscr{A}}$ and $v(t)$ instead of $\mathscr{A}$ and $u(t)$ to indicate this is the $\mathscr{U}$ accessible set for fixed $v(t)$ and variable $w(t)$ of the feedback modified dynamics (6.3).) Since the feedback decouples the output from the disturbance we conclude that $D(x^T)\perp dh(x^T)$. But x^T is arbitrarily close to x^0 so $D(x^0)\perp dh(x^0)$. QED

The following example shows that the converse to Theorem 5.3 is not true. We present it as a time varying linear system

$$\dot{x} = A(t)x + B(t)u + E(t)w, \tag{6.10a}$$

$$y = C(t)x, \tag{6.10b}$$

$$x(0) = x^0, \tag{6.10c}$$

which can easily be made into an autonomous nonlinear system (6.1) by letting time be an extra state coordinate, say $x_0 = t$.

Example 6.6. Let $\rho(t)$ be a $\mathscr{C}^\infty$ function such that $\rho(t)=0$ for $t \leqq 0$, $\rho(t)=\pi/2$ for $t \geqq 1$ and $\dot{\rho}(t)>0$ for $t\in(0,1)$. Define

$$A(t) = \dot{\rho}(t)\begin{pmatrix} -\sin\rho(t) & -\cos\rho(t) & 0 \\ \cos\rho(t) & -\sin\rho(t) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for $t \leqq 1.5$ and for $t \geqq 1.5$

$$A(t) = \dot{\rho}(t-2)\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin\rho(t-2) & -\cos\rho(t-2) \\ 0 & \cos\rho(t-2) & -\sin\rho(t-2) \end{pmatrix}.$$

The free dynamics (6.10a) for $u=0$ is constant except for $t\in(0,1)\cap(2,3)$. On the time interval $(0,1)$ the x_1-x_2 plane is rotated through an angle of $\pi/2$ and on $(2,3)$ the x_2-x_3 plane is similarly rotated. Let

$$B(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad E(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad C(t) = (0 \quad 0 \quad 1).$$

Viewed as a nonlinear system (6.1) on the extended four-dimensional space $(x_0 = t, x_1, x_2, x_3)$ the system is not disturbance decoupled. Disturbances at small positive times affect the x_1 coordinate and are rotated to affect the x_2 coordinate. Later after $t=2$ these disturbances are rotated to affect x_3 and hence the output. Since $B(t)=0$ the system cannot be disturbance decoupled.

However there is an (ad_f, g) invariant distribution $\mathscr{D}$ such that $\mathscr{R}(p)\subset\mathscr{D}\subset\mathscr{K}(dh)$. Of course $\mathscr{D}$ must be singular else it would be (Ad_f, g) invariant and Theorem 6.1 would apply. Let $\sigma(t)$ be a $\mathscr{C}^\infty$ function such that $\sigma(t)=1$ for $t\leqq 1$ and $\sigma(t)=0$ for $t \geqq 2$. Let $\mathscr{D}$ be spanned by the vector fields

$$X^1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad X^2 = \begin{pmatrix} 0 \\ 0 \\ \sigma(x_0) \\ 0 \end{pmatrix}.$$

We leave it to the reader to verify that $\mathscr{D}$ is (ad_f, g) (in fact ad_f) invariant, as a start note that

$$g^0 = \begin{pmatrix} 1 \\ A(x_0)x \end{pmatrix}, \qquad g = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Proof of Theorem 6.3. For given x^0 and $\mathscr{U}$, let γ be the feedback which solves the DDP on $\mathscr{U}$. By Theorem 6.2 on $\mathscr{U}$ there exists an (ad_f, g) distribution $\mathscr{D}$ such that $\mathscr{R}(p) \subset \mathscr{D} \subset \mathscr{K}(dh)$. Let $\mathscr{V}$ be an open neighborhood of x^0 whose closure is contained in $\mathscr{U}$ and let φ be a $\mathscr{C}^\infty$ function 1 on $\mathscr{V}$ and 0 off $\mathscr{U}$. The distribution $\varphi\mathscr{D} = \{\varphi X : X \in \mathscr{D}\}$ is globally defined and satisfies $\varphi\mathscr{D} \subset \mathscr{K}(dh)$. Moreover $\varphi\mathscr{D}$ is (ad_f, g) invariant hence locally (ad_f, g) invariant so $\varphi\mathscr{D} \subset \mathscr{D}^*$. Therefore for each x^0 there exists neighborhood $\mathscr{V}$ such that on $\mathscr{V}$, $\mathscr{R}(p) \subset \mathscr{D} = \varphi\mathscr{D} \subset \mathscr{D}^*$, hence $\mathscr{R}(p) \subset \mathscr{D}^*$.

As for the second assertion, give x^0 Lemma 5.1 allows us to conclude that in some neighborhood of $\mathscr{U}$ and x^0, $\mathscr{D}^*$ is (Ad_f, g) invariant. Since $\mathscr{R}(p) \subset \mathscr{D}^* \subset \mathscr{K}(dh)$, Theorem 6.1 implies the RDDP is solvable on $\mathscr{U}$. QED

7. Controllability distributions. We now define the nonlinear generalizations of the concept of an (A, B) controllability subspace. These were introduced by Krener and Isidori [6], see also [14].

DEFINITION. A distribution $\mathscr{C}$ (with associated subbundle $C \subset TM$) is an (Ad_f, g) *controllability distribution* ((ad_f, g) *controllability distribution*) if there exists an invertible feedback γ with β partitioned as $(\beta^1\beta^2)$ such that $\mathscr{C}$ separates the controls (see (5.11)), i.e., if $\tilde{g}^\sigma = g\beta^\sigma$ then for every x

(7.1a) $$C(x) \cap G(x) = \tilde{G}(x)^1, \qquad C(x) \cap \tilde{G}(x)^2 = \{0\}$$

and

(7.1b) $$\mathscr{C} = \langle \mathrm{Ad}_{\tilde{f}} | \mathscr{R}(\tilde{g}^1)\rangle \qquad (\mathscr{C} = \langle \mathrm{ad}_{\tilde{f}} | \mathscr{R}(\tilde{g}^1)\rangle).$$

It follows immediately from (7.1b) that any such $\mathscr{C}$ is (Ad_f, g) invariant ((ad_f, g) invariant), that (Ad_f, g) controllability distributions are integrable and that (ad_f, g) controllability distributions are involutive. Notice that (Ad_f, g) controllability does not necessarily imply (ad_f, g) controllability because the inclusion

$$\langle \mathrm{ad}_{\tilde{f}} | \mathscr{R}(\tilde{g}^1)\rangle \subset \langle \mathrm{Ad}_{\tilde{f}} | \mathscr{R}(\tilde{g}^1)\rangle$$

could be proper. However if $\mathscr{C}$ is (ad_f, g) controllable and nonsingular then the inclusion is an identity, hence $\mathscr{C}$ is (Ad_f, g) controllable.

We have already encountered several examples of such distributions. The Ad_f and ad_f controllability distributions of § 4 are (Ad_f, g) and (ad_f, g) controllability distributions for the time reversible system (4.1). (Here u_0 is an additional control and $\gamma = \beta^1 = I$.) Other important examples are the Ad_f and ad_f *exact time controllability distributions* $\langle \mathrm{Ad}_f | \mathscr{R}(g)\rangle$ and $\langle \mathrm{ad}_f | \mathscr{R}(g)\rangle$. These first appeared in the work of Sussmann and Jurdjevic [18], who considered only analytic systems, so there was no need to distinguish between the two. The first is integrable and for each x^0 and T there exists a leaf which contains $\mathscr{A}(x^0, T)$. The second is involutive; if $\mathscr{U}$ is a neighborhood of x^0 on which it is nonsingular then for each T sufficiently small, $\mathscr{A}(x^0, T, \mathscr{U})$ is contained in a leaf of $\langle \mathrm{ad}_f | \mathscr{R}(g)\rangle$ and has nonempty interior in the leaf topology. These statements follow from Lemmas 6.4 and 6.5.

These lemmas can be applied to arbitrary (Ad_f, g) or (ad_f, g) controllability distributions. The vector fields $\tilde{g}^0$, $\tilde{g}^1$ and $\tilde{g}^2$ of the feedback modified dynamics for the controllability distribution become g^0, p and g respectively in the context of these lemmas.

As one might expect there is a local version of the above concepts.

DEFINITION. Let $\mathscr{D}$ be an arbitrary distribution; we denote by $\mathscr{C}^*(\mathscr{D})$ the minimal distribution $\hat{\mathscr{D}}$ which satisfies

$$\hat{\mathscr{D}} = \mathscr{D} \cap (\mathrm{ad}_f(\hat{\mathscr{D}}) + \mathscr{R}(g)). \tag{7.2}$$

The notation $\mathrm{ad}_f(\hat{\mathscr{D}})$ denotes the $\mathscr{F}(M)$ span of all vector fields $[g^j, X]$ when $j = 0, \cdots, m$ and $X \in \hat{\mathscr{D}}$. It is not apparent that the set of distributions satisfying (7.2) is closed under intersection, hence we do not know that $\mathscr{C}^*(\mathscr{D})$ always exists. This will be shown by the controllability subdistribution algorithm.

DEFINITION. A distribution $\mathscr{C}$ is *locally* (ad_f, g) *controllable* if $\mathscr{C}$ is locally (ad_f, g) invariant and $\mathscr{C} = \mathscr{C}^*(\mathscr{C})$.

CONTROLLABILITY SUBDISTRIBUTION ALGORITHM (CSA, compare with [15, p. 110]).

Let $\mathscr{D}$ be an arbitrary distribution, $\mathscr{C}^0 = \{0\}$ and

$$\mathscr{C}^k = \mathscr{D} \cap (\mathrm{ad}_f(\mathscr{C}^{k-1}) + \mathscr{R}(g)). \tag{7.3}$$

Clearly $\mathscr{C}^0 \subset \mathscr{C}^1$, by induction $\mathscr{C}^{k-1} \subset \mathscr{C}^k$. For if $\mathscr{C}^{k-2} \subset \mathscr{C}^{k-1}$ then

$$\mathscr{C}^{k-1} = \mathscr{D} \cap (\mathrm{ad}_f(\mathscr{C}^{k-2}) + \mathscr{R}(g)) \subset \mathscr{D} \cap (\mathrm{ad}_f(\mathscr{C}^{k-1}) + \mathscr{R}(g)) = \mathscr{C}^k. \tag{7.4}$$

We claim that $\bigcup \mathscr{C}^k$ is the minimal distribution satisfying (7.2), i.e.

$$\mathscr{C}^*(\mathscr{D}) = \bigcup_{k \geqq 0} \mathscr{C}^k. \tag{7.5}$$

Clearly $\bigcup \mathscr{C}^k$ satisfies (7.2). If $\hat{\mathscr{D}}$ is any distribution satisfying (7.2) then $\mathscr{C}^0 \subset \hat{\mathscr{D}}$ and an induction similar to (7.4) shows that $\mathscr{C}^k \subset \hat{\mathscr{D}}$ for all k. Hence $\bigcup \mathscr{C}^k$ is the minimal distribution satisfying (7.2).

It is very important to note that for an arbitrary distribution $\mathscr{D}$, $\mathscr{C}^*(\mathscr{D})$ is *not* invariant and hence not controllable in any of the above senses.

LEMMA 7.1. *Let $\mathscr{D}$ be locally (ad_f, g) invariant; then $\mathscr{C}^*(\mathscr{D})$ is locally (ad_f, g) invariant and in fact is the unique maximal locally (ad_f, g) controllability distribution contained in $\mathscr{D}$.*

Proof. Suppose $X \in \mathscr{C}^*(\mathscr{D})$ as defined by (7.5). Since $\mathscr{D}$ is locally (ad_f, g) invariant there exists $Y \in \mathscr{R}(g)$ such that

$$\mathrm{ad}_f(X) + Y \in \mathscr{D}. \tag{7.6}$$

(A word of caution regarding notation is in order. By the above we mean that for $i = 0, \cdots, m$ there exists $Y^i \in \mathscr{R}(g)$ such that

$$\mathrm{ad}_{g^i}(X) + Y^i \in \mathscr{D}.$$

In (7.6) Y is a matrix whose columns are $Y^0, \cdots, Y^m$. Without mentioning it again we will continue to abuse notation in this fashion.) Since $\mathscr{C}^*(\mathscr{D})$ satisfies (7.2) it follows that

$$\mathrm{ad}_f(X) + Y \in \mathscr{C}^*(\mathscr{D})$$

so $\mathscr{C}^*(\mathscr{D})$ is locally (ad_f, g) invariant.

Let $\mathscr{C}^k$ be defined (7.3). Then $\mathscr{C}^k \subset \mathscr{C}^*(\mathscr{D})$ so

$$\mathscr{C}^k = \mathscr{C}^*(\mathscr{D}) \cap \mathscr{C}^k = \mathscr{C}^*(\mathscr{D}) \cap (\mathrm{ad}_f(\mathscr{C}^{k-1}) + \mathscr{R}(g)),$$

$$\mathscr{C}^k = \mathscr{C}^*(\mathscr{D}) \cap (\mathrm{ad}_f(\mathscr{C}^{k-1}) + \mathscr{R}(g)),$$

hence $\mathscr{C}^*(\mathscr{D})$ is locally (ad_f, g) controllable.

Suppose $\mathscr{C}$ is any other local (ad_f, g) controllability distribution in $\mathscr{D}$, define $\hat{\mathscr{C}}^0 = \{0\}$ and

$$\hat{\mathscr{C}}^k = \hat{\mathscr{C}} \cap (\mathrm{ad}_f(\hat{\mathscr{C}}^{k-1}) + \mathscr{R}(g)).$$

Since $\hat{\mathscr{C}}^0 = \mathscr{C}^0$ and $\hat{\mathscr{C}} \subset \mathscr{C}$, a simple induction similar to (7.4) shows that $\hat{\mathscr{C}}^k \subset \mathscr{C}^k$ and hence $\hat{\mathscr{C}} = \bigcup \hat{\mathscr{C}}^k \subset \bigcup \mathscr{C}^k = \mathscr{C}^*(\mathscr{D})$. Therefore $\mathscr{C}^*(\mathscr{D})$ is maximal. QED

From this lemma we see that every distribution $\mathscr{D}$ contains a unique maximal locally (ad_f, g) controllable distribution. The argument proceeds in two steps. Since the locally (ad_f, g) invariant distributions form a semilattice under addition, every distribution contains a unique maximal locally (ad_f, g) invariant distribution $\mathscr{D}^*(\mathscr{D})$. By the above lemma this distribution contains an unique maximal locally (ad_f, g) controllability distribution $\mathscr{C}^*(\mathscr{D}^*(\mathscr{D}))$. Note that $\mathscr{C}^*(\mathscr{D}^*(\mathscr{D})) \subset \mathscr{C}^*(\mathscr{D})$ but generally this is a proper inclusion. Frequently we shall wish to compute $\mathscr{C}^*(\mathscr{D}^*(\mathscr{K}(dh)))$ which we shall abbreviate $\mathscr{C}^*$ when there is no possibility of confusion. At the end of this section we discuss the computation of $\mathscr{C}^*$ by extending the algorithm for $\mathscr{D}^*$ of § 5.

The above remarks are predicted on the assumption of full control. There may exist distributions $\hat{\mathscr{D}}$ which are locally (ad_f, g) invariant with partial control such that $\mathscr{D}^*(\mathscr{D}) \subseteq \hat{\mathscr{D}} \subset \mathscr{D}$. On the other hand from the CSA we see that if $\hat{\mathscr{D}}$ is any locally (ad_f, g) controllability distribution with partial control that is contained in $\mathscr{D}$ then $\hat{\mathscr{D}} \subset \mathscr{C}^*(\mathscr{D})$.

The set of locally (ad_f, g) controllability distributions is a semilattice under addition.

LEMMA 7.2. *Suppose $\mathscr{C}_1$ and $\mathscr{C}_2$ are locally (ad_f, g) controllable. Then so is $\mathscr{C} = \mathscr{C}_1 + \mathscr{C}_2$.*

Proof. Of course $\mathscr{C}$ is locally (ad_f, g) invariant. Let $\mathscr{C}_i^k$ and $\mathscr{C}^k$ be defined by the controllability subdistribution algorithm (7.3) applied to $\mathscr{C}_i$ and $\mathscr{C}$. Clearly $\mathscr{C}_i^0 = \mathscr{C}^0$ and $\mathscr{C}_i \subset \mathscr{C}$ so by induction $\mathscr{C}_i^k \subset \mathscr{C}^k$. But

$$\mathscr{C} = \mathscr{C}_1 + \mathscr{C}_2 = \left(\bigcup_{k \geqq 0} \mathscr{C}_1^k \right) + \left(\bigcup_{k \geqq 0} \mathscr{C}_2^k \right) = \bigcup_{k \geqq 0} (\mathscr{C}_1^k + \mathscr{C}_2^k) \subset \bigcup_{k \geqq 0} \mathscr{C}^k \subset \mathscr{C}. \qquad \text{QED}$$

The next lemma is important for it shows that if $\mathscr{D}$ is (ad_f, g) invariant then $\langle \mathrm{ad}_{\tilde{f}} | \mathscr{D} \cap \mathscr{R}(g) \rangle$ is independent of the choice of feedback so long as it leaves $\mathscr{D}$ invariant.

LEMMA 7.3. *Suppose $\mathscr{D}$ is (ad_f, g) invariant under γ. Let $\mathscr{C}^k$ and $\mathscr{C}^*(\mathscr{D})$ be defined by the* CSA (7.3), (7.5) *applied to $\mathscr{D}$. Then for $k \geqq 1$*

$$\mathscr{C}^k = \sum_{j=0}^{k} \mathrm{ad}_{\tilde{f}}^{j-1} (\mathscr{D} \cap \mathscr{R}(g))$$

and

$$\mathscr{C}^*(\mathscr{D}) = \langle \mathrm{ad}_{\tilde{f}} | \mathscr{D} \cap \mathscr{R}(g) \rangle.$$

Proof. The second assertion follows from the first which follows by induction. For $k = 1$ it is clearly true. Suppose it holds for $k-1$. Let $X \in \mathscr{C}^{k-1}$. Then

$$\mathrm{ad}_{\tilde{f}}(X) = \mathrm{ad}_f(X)\gamma - f L_X(\gamma).$$

Since $\gamma = \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix}$ it follows that $L_X(\gamma) = \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix}$ and $fL_X(\gamma)$. Moreover γ is invertible so

$$\mathrm{ad}_{\tilde f}(X) + \mathscr{R}(g) = \mathrm{ad}_f(X) + \mathscr{R}(g).$$

This allows us to express $\mathscr{C}^k$ as

$$\mathscr{C}^k = \mathrm{ad}_{\tilde f}(\mathscr{C}^{k-1}) + (\mathscr{D} \cap \mathscr{R}(g)).$$

$$= \sum_{j=0}^{k} \mathrm{ad}_{\tilde f}^{j-1}(\mathscr{D} \cap \mathscr{R}(g)). \qquad \text{QED}$$

COROLLARY 7.4. *If $\mathscr{C}$ is (ad_f, g) controllable then $\mathscr{C}$ is locally (ad_f, g) controllable.*

Proof. If $\mathscr{C}$ is (ad_f, g) controllable then it is (ad_f, g) invariant, hence locally (ad_f, g) invariant. Let γ be a feedback which leaves $\mathscr{C}$ invariant and separates the controls, so that $C_x \cap G_x = \tilde G_x^1$ then $\mathscr{C}^*(\mathscr{C}) = \langle \mathrm{ad}_{\tilde f} | \mathscr{C} \cap \mathscr{R}(g) \rangle = \langle \mathrm{ad}_{\tilde f} | \mathscr{R}(\tilde g^1) \rangle = \mathscr{C}$ so $\mathscr{C}$ is locally (ad_f, g) controllable. QED

COROLLARY 7.5. *If $\mathscr{C}$ is nonsingular, involutive and separates the controls then the following are equivalent.*

(a) *$\mathscr{C}$ is locally (ad_f, g) controllable.*

(b) *There exist an open cover $\{\mathscr{U}^\rho\}$ of M and separating feedbacks γ^ρ such that $\mathscr{C}$ is (ad_f, g) controllable on $\mathscr{U}^\rho$ under γ^ρ (in other words, locally $\mathscr{C}$ is (ad_f, g) controllable).*

(c) *There exist an open cover $\{\mathscr{U}^\rho\}$ of M and separating feedbacks γ^ρ such that $\mathscr{C}$ is (Ad_f, g) controllable on γ^ρ under γ^ρ (locally $\mathscr{C}$ is (Ad_f, g) controllable).*

Proof. This follows directly from Lemmas 5.1 and 7.3. QED

Computation of $\mathscr{C}^* = \mathscr{C}^*(\mathscr{D}^*(\mathscr{K}(dh))$. One could apply the CSA to $\mathscr{D}^* = \mathscr{D}^*(dh)$ computed by the ISA of § 5. A more convenient approach is to apply Lemma 7.3 so that in the notation of the end of § 5,

$$\mathscr{C}^* = \langle \mathrm{ad}_{\tilde f_{k_*}} | \mathscr{R}(\tilde g^2_{k_*}) \rangle.$$

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