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RECIPROCAL PROCESSES AND THE STOCHASTIC REALIZATION PROBLEM FOR ACAUSAL SYSTEMS

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Reciprocal processes were introduced in 1932 by S. Bernstein as a generalization of Markov processes. Acausal linear systems were introduced in 1978 by A. Krener as a generalization of causal linear systems. We discuss the relationships between these four concepts.

1. ACAUSAL SYSTEMS

Acausal linear systems (or boundary value linear systems) were introduced in [11] and further studied in [2,8-10,12-15]. They are mathematical models of the form

(1.1a) $\dot{x} = Ax + Bu$

(1.1b) $V^0 x(t_0) + V^1 x(t_1) = v$

(1.1c) $y = Cx + Du$

(1.1d) $w = W^0 x(t_0) + W^1 x(t_1)$

where $x(t)$, v, $w \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^p$. The matrices A, B, C, D, V^0, V^1, W^0 and W^1 are dimensioned accordingly. The matrices A, B, C and D may be time varying; we are primarily interested in the case when these are either constant or real analytic functions of time. We assume throughout that (1.1a,b) is well posed, i.e., for each sufficiently regular input $u(t)$ and boundary condition v there exists a unique solution $x(t)$. This is equivalent to assuming that after a change of v coordinates we have

(1.2) $V^0 + V^1 \Phi(t_1, t_0) = I$

where $\Phi(t,s)$ is the fundamental matrix solution of the differential equation

(1.3a) $\frac{\partial \Phi}{\partial t}(t,s) = A(t)\Phi(t,s)$

(1.3b) $\Phi(t,t) = I.$

The system (2.1) induces a linear mapping from pairs $(v, u(\cdot))$ to pairs $(w, y(\cdot))$. In particular

(1.4) $$y(t) = C(t)\Phi(t,t_0)v + \int_{t_0}^{t_1} C(t)G(t,s)B(s)u(s)\,ds + D(t)u(t)$$

where $G(t,s)$ is the Green's matrix of the boundary value problem given by

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$$(1.5) \qquad G(t,s) = \begin{cases} \Phi(t,t_0)V^0\Phi(t_0,s) & \text{if } t > s \\ \Phi(t,t_0)V^1\Phi(t_1,s) & \text{if } t < s. \end{cases}$$

Models such as (1.1) are useful because they give a convenient representation of the map between function spaces (1.4). For the moment let $v = 0$ and $D = 0$. The kernel of (1.4)

$$(1.6) \qquad W(t,s) = C(t)G(t,s)B(s)$$

is called the <u>impulse response</u> or <u>weighing pattern</u> of the system (1.1).

Deterministic realization theory is concerned with the inverse question, i.e., given a weighing pattern $W(t,s)$ defining an input output map

$$(1.7) \qquad y(t) = \int_{t_0}^{t_1} W(t,s)u(s)\,ds$$

(v is assumed to be 0), describe any and all state space realizations of the form (1.1). One is particularly interested in minimal realizations, i.e., those of minimal state dimension among a given class of realization, e.g. real analytic or stationary. These two categories admit a complete realization theory given in [15]. We briefly review this work in the next section.

If we excite (1.1) with a standard white Gaussian noise $u(t)$ and take v as an independent Gaussian zero mean random variable, then both $x(t)$ and $y(t)$ are zero mean Gaussian processes with covariances $R_x(t,s)$ and $R_y(t,s)$.

Stochastic realization theory is concerned with the inverse problem. Given a covariance $R(t,s)$, describe any and all realizations of it either as the state covariance $R_x(t,s)$ or the output covariance $R_y(t,s)$ of a model such as (1.1).

For causal models, i.e., $V^0 = I$, $V^1 = 0$, these questions have received considerable attention and have been completely solved for stationary covariances $R(t,s) = R(t-s)$. Under the assumption of causality, the state process is Markov. From the work of Doob [7] and others, we know that every Gauss Markov process can be realized as the state process of a causal system.

The question of which stationary covariances can be realized as the output process of a causal system has been solved but is considerably more complicated, involving questions of spectral factorizations, splitting subspaces, etc. We will not go into this, but refer the interested reader to the recent survey article [14] and its references.

Generally speaking the state process of an acausal model (1.1) is not Markov. In 1932, Serge Bernstein [3] introduced a generalization of the Markov property which he called reciprocal. A process $x(t)$ defined on an interval $[t_0,t_1]$ is <u>reciprocal</u> if for any subinterval $[\tau_0,\tau_1] \subseteq [t_0,t_1]$, the values of the process $x(\tau)$, $\tau \in [\tau_0,\tau_1]$ inside the subinterval are conditionally independent of the values $x(t)$, $t \notin (\tau_0,\tau_1)$ outside the subinterval given the values of the process, $x(\tau_0)$, $x(\tau_1)$, on the boundary.

In Section 5, we show that the state process of an acausal linear system (1.1) is reciprocal. However, even in the stationary Gaussian case, not every reciprocal process is the state process of an acausal linear system (1.1). We show this by describing all stationary Gaussian processes that can be realized as the state process of a one-dimensional acausal linear system. This list is a proper subset of the list of all one-dimensional stationary reciprocal processes as given by Jamison [11], Chay [5], and Carmichael, Massé and Theodorescu [4].

2. DETERMINISTIC REALIZATION THEORY

A linear mapping from a space of inputs $u(t)$ to outputs $y(t)$ is said to be causal if whenever two inputs agree up to time τ_0, the corresponding outputs also agree up to time τ_0. The key to understanding the realization theory for such mapping is the Hankel point of view that was perhaps most clearly put forth by R. Kalman.

The Hankel point of view can be loosely described as follows. Given τ_0 one considers only inputs with support to the left of τ_0 and observes the outputs only to the right of τ_0. To obtain a finite dimensional realization this map must be of finite rank. If this holds, then one factors the mapping through a finite dimensional vector space which plays the role of the state space at time τ_0. If this factorization can be done with a common state space for all τ_0 then one can construct a realization. The realization is controllable at τ_0 if the right factor is onto the state space. It is observable at τ_0 if the left factor is 1-1. A realization is minimal iff it is controllable and observable at some τ_0.

In the acausal context the situation is more complex. One chooses two times $\tau_0 < \tau_1$. First we consider how an input with support outside of $[\tau_0,\tau_1]$ affects the output on $[\tau_0,\tau_1]$. Second, we can consider how an input with support on $[\tau_0,\tau_1]$ affects the output off $[\tau_0,\tau_1]$. We assume that each of these maps factors through an n-dimensional space and the combined 2n-dimensional space can be thought of as the direct sum of state spaces at τ_0 and τ_1.

To be more concrete consider the system (1.1) and define $n \times n$ matrices

(2.1a) $K^0 = \Phi(\tau_0,t_0)V^0\Phi(t_0,\tau_0)$

(2.1b) $K^1 = \Phi(\tau_0,t_0)V^1\Phi(t_1,\tau_1)$

(2.1c) $J^0 = -\Phi(\tau_1,\tau_0)$

(2.1d) $J^1 = I$

The inward and outward boundary values $k(\tau_0,\tau_1)$ and $j(\tau_0,\tau_1)$ are the n-vectors defined by

$$(2.2)\qquad \begin{bmatrix} \kappa(\tau_0,\tau_1) \\ j(\tau_0,\tau_1) \end{bmatrix} = \begin{bmatrix} K^0 & K^1 \\ J^0 & J^1 \end{bmatrix} \begin{bmatrix} x(\tau_0) \\ x(\tau_1) \end{bmatrix}.$$

This is an invertible transformation by the well-posed assumption (1.2).

The space of κ's serves to factor the map from inputs with support off $[\tau_0,\tau_1]$ to outputs on $[\tau_0,\tau_1]$. This is because for any input $u(t)$ and $v = 0$

$$(2.3a)\qquad \kappa(\tau_0,\tau_1) = \left(\int_{t_0}^{\tau_0} + \int_{\tau_1}^{t_1} \right) G(\tau_0,s)B(s)u(s)\,ds.$$

If the support of $u(t)$ is off $[\tau_0,\tau_1]$ and $t \in [\tau_0,\tau_1]$ then

$$(2.3b)\qquad y(t) = C(t)\Phi(t,\tau_0)\kappa(\tau_0,\tau_1).$$

Similarly the space of j's serves to factor the map from inputs with support on $[\tau_0,\tau_1]$ to outputs off $[\tau_0,\tau_1]$. For any input $u(t)$

$$(2.4a)\qquad j(\tau_0,\tau_1) = \int_{\tau_0}^{\tau_1} \Phi(\tau_1,s)B(s)u(s)\,ds.$$

If the support of $u(t)$ is on $[\tau_0,\tau_1]$, $v = 0$ and $t \notin [\tau_0,\tau_1]$ then

$$(2.4b)\qquad y(t) = C(t)G(t,\tau_1)j(\tau_0,\tau_1).$$

Corresponding to these two factorizations we have two definitions of controllability and observability. In each case the formulas (2.3a) and (2.4a) ((2.3b) and (2.4b)) should be viewed as a mapping from (to) a function space to (from) $\mathbb{R}^n$. The system (1.1) is <u>controllable off</u> $[\tau_0,\tau_1]$ if (2.3a) is onto and is <u>controllable on</u> $[\tau_0,\tau_1]$ if (2.4a) is onto. It is <u>observable on</u> $[\tau_0,\tau_1]$ if (2.3b) is one-to-one and <u>observable off</u> $[\tau_0,\tau_1]$ if (2.4b) is one-to-one.

We have shown that if one of the factorizations (2.3) or (2.4) is epi-mono (i.e., the (a) part is onto and the (b) part one-to-one), then the realization is minimal [15]. We have also given necessary and sufficient conditions for minimality within the category of real analytic systems and within the category of stationary systems [15]. Loosely speaking, these conditions are that the systems be both controllable and observable off every proper subinterval and that every direction of the state which is unobservable on some subinterval must be controllable on that subinterval.

We refer the reader to [15] for the exact statements of the theorems and their proofs.

Notice that $\kappa(\tau_0,\tau_1)$, $j(\tau_0,\tau_1)$ contain the same information as $x(\tau_0)$, $x(\tau_1)$. We consider the former as functions of the subinterval $[\tau_0,\tau_1]$. Relative to the inclusion, ordering the mappings from the input $u(t)$ to $\kappa(\tau_0,\tau_1)$ and $j(\tau_0,\tau_1)$ have a causal property in that $\kappa(\tau_0,\tau_1)$ does not depend on $u(\tau)$ for $\tau \in (\tau_0,\tau_1)$ and $j(\tau_0,\tau_1)$ does not depend on $u(t)$ for $t \notin [\tau_0,\tau_1]$.

3. STATIONARY GAUSS MARKOV REALIZATION THEORY

A stochastic process $x(t)$ on $[t_0,t_1]$ is Markov if for any $\tau_0 \in [\tau_0,\tau_1]$ the sigma algebras generated by the past $\{x(t): t_0 \leq t \leq \tau_0\}$ and the future $\{x(\tau): \tau_0 \leq \tau \leq t_1\}$ are conditionally independent given the present $x(\tau_0)$. The definition was formulated in 1906 by Markov for what are now called Markov chains and generalized by Kolmogorov in 1931. They were motivated by the work of the physicists of the day, people like Chapman, Fokker, Planck, Ornstein, and Uhlenbeck.

A zero mean nonsingular Gaussian process $x(t)$ is Markov iff its covariance matrix $R_x(\tau,t)$ satisfies the functional relation

$$R_x(\tau,t) = R_x(\tau,\tau_0)R_x^{-1}(\tau_0,\tau_0)R_x(\tau_0,t) \tag{3.1}$$

for all $t_0 \leq t \leq \tau_0 \leq \tau \leq t_1$. (By nonsingular, we mean $R_x(\tau_0,\tau_0)$ is nonsingular for $t_0 < \tau_0 < t_1$.) In the stationary case where $R_x(\tau,t) = R_x(\tau-t)$, this becomes

$$R_x(t+s) = R_x(t)R^{-1}(0)R_x(s) \qquad \text{for } t,\ s \geq 0. \tag{3.2}$$

It is not clear who to credit for the realization theory of stationary Gauss Markov processes which follows. Certainly we must give at least partial credit to Ornstein, Uhlenbeck, Doob, and Kalman.

Consider an autonomous causal linear system

$$\dot{x} = Ax + Bu \tag{3.4a}$$

$$x(0) = x^0 \tag{3.4b}$$

where $u(t)$ and x^0 are independent and satisfy

$$u(t) \sim N(0,I\delta(t-s)) \tag{3.4c}$$

$$x^0 \sim N(0,P) \tag{3.4d}$$

The state process $x(t)$ is a zero mean Gauss Markov process defined for all $t \geq 0$ with covariance

$$R_x(t,s) = e^{At}Pe^{A^*s} + \int_0^{\min\{t,s\}} e^{A\tau}BB^*e^{A^*\tau}\, d\tau.$$

The following facts are well known.

Proposition 3.1. The process $x(t)$ defined by (3.4) is stationary iff

$$AP + PA^* = -BB^*. \tag{3.5}$$

Proposition 3.2. If the process $x(t)$ defined by (3.4) is stationary and nondegenerate then the eigenvalues of A have nonpositive real part.

Proposition 3.3. If the process $x(t)$ defined by (3.4) is stationary and (A,B) is a controllable pair then $x(t)$ in nondegenerate and the eigenvalues of A

have negative real part.

Now we turn to the inverse question and prove the following well-known result.

Theorem 3.4. Every stationary nonsingular zero mean Gauss Markov process with continuous covariance has a realization of the form (3.4) on $[0,\infty)$.

Proof. Integrate (3.2) with respect to s from 0 to $\delta > 0$ to obtain

$$\int_t^{t+\delta} R_x(s)\,ds = R_x(t)R_x^{-1}(0)\int_0^{\delta} R_x(s)\,ds. \tag{3.6}$$

By the nonsingularity of $R_x(0)$ the integral on the right is invertible for small δ, so (3.6) implies that $R_x(t)$ is C^1 and by induction C^∞. We differentiate (3.2) with respect to t at $t = 0$ to obtain

$$\dot{R}_x(s) = \dot{R}_x(0)R_x^{-1}(0)R_x(s). \tag{3.7a}$$

Let $A = \dot{R}_x(0)R_x(0)$, then

$$R_x(t) = e^{At}R_x(0). \tag{3.7b}$$

The desired realization (3.4) is obtained by setting $P = R_x(0)$ and letting B be any solution of (3.5).

4. RECIPROCAL PROCESSES

A stochastic process $x(t)$ on $[t_0,t_1]$ is reciprocal if for any subinterval $[\tau_0,\tau_1] \subseteq [t_0,t_1]$ the sigma algebras generated by the process on the inside $\sigma\{x(\tau)\colon \tau \in [\tau_0,\tau_1]\}$ and on the outside $\sigma\{x(t)\colon t \in [t_0,t_1]\setminus(\tau_0,\tau_1)\}$ are conditionally independent given the boundary values $x(\tau_0)$ and $x(\tau_1)$. This definition was formulated by Bernstein [3] in 1932 who was motivated by work of Schrödinger.

It is easy to see that a zero mean Gaussian process $x(t)$ is reciprocal iff its covariance matrix $R_x(t,\tau)$ satisfies the relation

$$R(t,\tau) = [R(t,\tau_0)R(t,\tau_1)]\begin{bmatrix} R(\tau_0,\tau_0) & R(\tau_0,\tau_1) \\ R(\tau_1,\tau_0) & R(\tau_1,\tau_1)\end{bmatrix}^{-1}\begin{bmatrix} R(\tau_0,\tau) \\ R(\tau_1,\tau)\end{bmatrix} \tag{4.1}$$

for any $\tau \in [\tau_0,\tau_1]$ and $t \notin (\tau_0,\tau_1)$. Implicit in this formulation is the assumption that the indicated inverse exists. A process for which this holds for all $t_0 < \tau_0 < \tau_1 < t_1$ is called nonsingular of order 2. Henceforth we only consider such processes.

Jamison [11] studied one-dimensional zero mean stationary Gaussian reciprocal processes. By a technique similar to that of the last section, he showed that if the covariance $R_x(t)$ is continuous then it is C^∞ and must satisfy a differential equation of the form

$$\frac{d^2}{dt^2} R_x(t) = \pm a^2 R_x(t).$$

He mistakenly concluded that the covariances had to be one of the following forms

(i) $R_x(t) = e^{at} R_x(0)$ $\quad a > 0, \quad 0 \le t \le \infty$

(ii) $R_x(t) = (1-at) R_x(0)$ $\quad a > 0, \quad 0 \le t \le 2/a$

(iii) $R_x(t) = (\cos at) R_x(0)$ $\quad a > 0, \quad 0 \le t \le \infty$

In 1972, Chay [5] partially corrected Jamison by adding to the list covariances of the form

(i') $R_x(t) = (Ae^{at} + (1-A)e^{-at}) R_x(0)$

$$a > 0, \ \frac{1}{1-e^{aT}} \le A \le \frac{1}{1+e^{aT}}, \quad 0 \le t \le T.$$

Finally in 1982, Carmichael, Massé, and Theodorescu [4] correctly completed the list by adding

(iii') $R_x(t) = (\cos at + B \sin at) R_x(0)$

$$0 < a \le \pi/R, \ -\text{cotan}\, \frac{aT}{2} \le B \le 0, \quad 0 \le t \le T.$$

This latter work utilized the concept of a conditionally Markov process introduced by Mehr and McFadden [17] in connection with the study of first passage times of Gaussian processes. A Gaussian process $x(t)$ defined on $[t_0,t_1]$ is <u>conditionally Markov at</u> τ_0 if for every x^0 the process obtained from $x(t)$ by conditioning an $x(\tau_0) = x^0$ is Markov on $[\tau_0,t_1]$. These conditional processes vary with τ_0 and x^0. They are generally not zero mean. However because of the Gaussian assumption their covariances are independent of x^0. (See Cramer [6, Sec. 24.6].) Hence we can conveniently condition on $x^0 = 0$ to obtain a zero mean process.

Let's extend the definition of Mehr and McFadden. A process $x(t)$ is <u>conditionally Markov on</u> $[t_0,t_1]$ if $x(t)$ is conditionally Markov at every $\tau_0 \in [t_0,t_1]$ and $x(-t)$ is conditionally Markov at every $-\tau_0 \in [-t_1,-t_0]$. With this definition we have a generalization of Abraham and Thomas [1].

<u>Proposition 4.1</u>. A Gaussian process is reciprocal on $[t_0,t_1]$ iff it is conditionally Markov on $[t_0,t_1]$.

Suppose $x(t)$ is a reciprocal Gaussian process on $[t_0,t_1]$. Let $\tau_0 \in [t_0,t_1]$ and $\tilde{x}(t|\tau_0)$ be the process obtained from $x(t)$ by restricting $x(\tau_0) = 0$. By the above, $\tilde{x}(t|\tau_0)$ is Markov on $[\tau_0,t_1]$ and is zero mean. Denote its covariance by

(4.3) $\quad R_{\tilde{x}(\cdot|\tau_0)}(t,\tau) = E(x(t)x^*(\tau)|x(\tau_0) = 0).$

Following [17] and [1] we obtain

$$R_x(t,\tau) = R_x(t,\tau_0)R_x^{-1}(\tau_0,\tau_0)R_x(\tau_0,\tau) + R_{\tilde{x}(\cdot|\tau_0)}(t,\tau) \tag{4.4}$$

where $t_0 \leq \tau_0 \leq \tau \leq \tau_1 \leq t \leq t_1$.

The reader should contrast (4.4) with (4.1). In particular if $x(t)$ is a stationary reciprocal Gaussian process then (4.1) becomes

$$R_x(t-\tau) = [R_x(t-\tau_0)R_x(t-\tau_1)] \begin{bmatrix} R_x(0) & R_x(\tau_0-\tau_1) \\ R_x(\tau_1-\tau_0) & R_x(0) \end{bmatrix}^{-1} \begin{bmatrix} R_x(\tau_0-\tau) \\ R_x(\tau_1-\tau) \end{bmatrix} \tag{4.5a}$$

and (4.4) becomes

$$R_x(t-\tau) = R_x(t-\tau_0)R_x^{-1}(0)R_x(\tau_0-\tau) + R_{\tilde{x}(\cdot|\tau_0)}(t,\tau_0). \tag{4.5b}$$

Note that $\tilde{x}(\cdot|\tau_0)$ is not a stationary process.

The close relationship between conditionally Markov and reciprocal processes allowed Carmichael, Massé and Theodorescu [4] to exhibit all one-dimensional stationary reciprocal processes as

$$x(t) = g(t)W(f(t)/g(t)) + h(t)Z \tag{4.6a}$$

where $W(t)$ is a one-dimensional Wiener process and Z an independent Gaussian variable. For the cases listed above $h(t) = R_x(t)$ and

(i') $f(t) = e^{-at} - e^{at}$, $g(t) = A^2e^{at} - (1-A)^2e^{-at}$

(ii') $f(t) = t$, $g(t) = 2a - a^2t$

(iii') $f(t) = \sin at$, $g(t) = (1-B^2)\sin at - 2B\cos at$.

In each case $f(t)$, $g(t)$ and $h(t)$ satisfy

$$h(t-\tau) = h(t)h(\tau) + g(t)f(\tau) \tag{4.6b}$$

which is just a particular case of (4.5b) with $\tau_0 = 0$.

Now suppose $R_x(t-\tau)$ is continuous and we integrate (4.5) with respect to t from τ_1 to $\tau_1+\delta$ for small $\delta > 0$. Recall that (4.5a) is valid for $\tau \in [\tau_0,\tau_1]$ and $t \notin (\tau_0,\tau_1)$.

$$\int_{\tau_1-\tau}^{\tau_1-\tau+\delta} R_x(s)\,ds = [I\delta + o(\delta) \quad o(\delta)]\begin{bmatrix} R_x(\tau_0-\tau) \\ R_x(\tau_1-\tau) \end{bmatrix} \tag{4.7a}$$

where $o(\delta)/\delta \to 0$ as $\delta \to 0$. Similarly if we integrate (4.5) with respect to t from $\tau_0 - \delta$ to τ_0 we obtain

$$\int_{\tau_0-\tau-\delta}^{\tau_0-\tau} R_x(s)\,ds = [o(\delta) \quad I\delta + o(\delta)]\begin{bmatrix} R_x(\tau_0-\tau) \\ R_x(\tau_1-\tau) \end{bmatrix}. \tag{4.7b}$$

By differentiating (4.7) with respect to τ we see that $R_x(\tau_0-\tau)$ and $R_x(\tau_1-\tau)$ are C^1 (and by induction C^∞) for $\tau \in (\tau_0,\tau_1)$. But $[\tau_0,\tau_1]$ is arbitrary so we have shown the following

Proposition 4.2. Let $x(t)$ be a stationary reciprocal zero mean Gaussian process on $[t_0,t_1]$ with continuous covariance $R_x(t)$. Then the covariance is C^∞ for $t \in (t_0-t_1,0) \cup (0,t_1-t_0)$.

This proposition allows us to derive a differential equation for such covariances generalizing Jamison [11]. Let $s = t - \tau$ and $\sigma = \tau - \tau_0 = \tau_1 - \tau$. Then (4.5) becomes

$$R_x(s) = [R_x(s+\sigma) \quad R_x(s-\sigma)]\begin{bmatrix} H_1(\sigma) \\ H_2(\sigma) \end{bmatrix} \tag{4.8a}$$

where

$$\begin{bmatrix} H_1(\sigma) \\ H_2(\sigma) \end{bmatrix} = \begin{bmatrix} R(0) & R(-2\sigma) \\ R(2\sigma) & R(0) \end{bmatrix}^{-1} \begin{bmatrix} R(-\sigma) \\ R(\sigma) \end{bmatrix}. \tag{4.8b}$$

If we assume $x(t)$ is nonsingular of order 2, then it is not hard to see that $H_1(\sigma)$ and $H_2(\sigma)$ are continuous for all small σ and C^∞ except possibly at $\sigma = 0$. Moreover $H_1(\sigma) = H_2(-\sigma)$ so $H_1(0) = H_2(0) = \frac{1}{2}I$. We differentiate (4.8a) twice with respect to σ and evaluate at $\sigma = 0$ to obtain

$$\ddot{R}_x(s) = 2\dot{R}_x(s)(\dot{H}_2(0)-\dot{H}_1(0)) - R_x(s)(\ddot{H}_1(0)+\ddot{H}_2(0)). \tag{4.9}$$

We formalize this as follows.

Proposition 4.3. Let $x(t)$ be a stationary reciprocal zero mean Gaussian process on $[t_0,t_1]$ with continuous covariance. Then the covariance satisfies a second order n-dimensional linear autonomous differential equation on $[t_0,t_1]$.

5. ACAUSAL STATE PROCESSES

We return to the study of state space models (1.1) excited by standard white Gaussian noise $u(t)$ and independent zero mean Gaussian boundary value

$$E(u(t)u^*(\tau)) = I\delta(t-\tau) \tag{5.1a}$$

$$E(vv^*) = Q \tag{5.1b}$$

$$E(u(t)v^*) = 0 \tag{5.1c}$$

The state process $x(t)$ is a zero mean Gaussian process given by

$$x(t) = \Phi(t,t_0)v + \int_{t_o}^{t_1} G(t,s)B(s)u(s)\, ds. \tag{5.2}$$

The covariance is

$$R_x(t,\tau) = \Phi(t,t_0)Q\Phi^*(\tau,t_0) + \int_{t_0}^{t_1} G(t,s)B(s)B^*(s)G^*(\tau,s)\, ds. \tag{5.3}$$

The inward and outward boundary value processes $\kappa(\tau_0,\tau_1)$ and $j(\tau_0,\tau_1)$ are defined by (2.2) and given by

$$\kappa(\tau_0,\tau_1) = \Phi(\tau_0,t_0)v + \left(\int_{t_0}^{\tau_0} + \int_{\tau_1}^{t_1} \right) G(\tau_0,s)B(s)u(s)\, ds \tag{5.4a}$$

$$j(\tau_0,\tau_1) = \int_{\tau_0}^{\tau_1} \Phi(\tau_1,s)B(s)u(s)\, ds. \tag{5.4b}$$

We consider $\kappa(\tau_0,\tau_1)$ and $j(\tau_0,\tau_1)$ as stochastic processes where the parameter is the interval $[\tau_0,\tau_1]$. In particular, they have the following three properties which can be easily verified.

I. The Gaussian vectors $\kappa(\tau_0,\tau_1)$ and $j(\tau_0,\tau_1)$ are a linear transform of $x(\tau_0)$ and $x(\tau_1)$ and $\kappa(\tau_0,\tau_1)$ is orthogonal to $j(\tau_0,\tau_1)$.

II. $\kappa(\tau_0,\tau_1)$ and $j(\tau_0,\tau_1)$ are Markov processes relative to the inclusion ordering on intervals.

Suppose $\tau \in [\tau_0,\tau_1]$ and $t \notin (\tau_0,\tau_1)$. Let $\hat{x}(\tau|\kappa(\tau_0,\tau_1))$ be the optimal estimate of $x(\tau)$ given $\kappa(\tau_0,\tau_1)$ and $\tilde{x}(\tau|\kappa(\tau_0,\tau_1))$ be the error of this estimate. Let $\hat{x}(t|j(\tau_0,\tau_1))$ be the optimal estimate of $x(t)$ given $j(\tau_0,\tau_1)$ and $\tilde{x}(t|j(\tau_0,\tau_1))$ be the error of this estimate. It is easy to see that

$$\hat{x}(\tau|\kappa(\tau_0,\tau_1)) = \Phi(\tau,\tau_0)\kappa(\tau_0,\tau_1) \tag{5.5a}$$

$$\hat{x}(t|j(\tau_0,\tau_1)) = G(t,\tau_1)j(\tau_0,\tau_1). \tag{5.5b}$$

The third property then can be stated as follows.

III. $\tilde{x}(\tau|\kappa(\tau_0,\tau_1))$ is orthogonal to $\tilde{x}(t|j(\tau_0,\tau_1))$.

Theorem 5.1. The state process of an acausal system (1.1) excited by white Gaussian noise and independent Gaussian boundary values is a reciprocal Gaussian process.

Proof. For any $\tau \in [\tau_0,\tau_1]$, $t \notin (\tau_0,\tau_1)$

$$x(\tau) = \hat{x}(\tau|\kappa(\tau_0,\tau_1)) + \tilde{x}(\tau|\kappa(\tau_0,\tau_1))$$

$$x(t) = \hat{x}(t|j(\tau_0,\tau_1)) + \tilde{x}(t|j(\tau_0,\tau_1)).$$

Using properties I and III above, it is easy to see that $R_x(t,\tau)$ satisfies (4.1).

At this point one might be tempted to conjecture that every stationary reciprocal process can be realized by the state process of an autonomous acausal model. But this is not true as can be seen by listing all the one-dimensional stationary processes arising in such a fashion.

Suppose $x(t)$ is a one-dimensional stationary process arising from (1.1) and (5.1) where $t_0 = 0$ and $t_1 = T$. From (5.3) it follows that

(5.6a) $$2AR_x(0) = B^2(1-2V^0)$$

(5.6b) $$R_x(0) = Q + (V^1B)^2 \int_0^T e^{2At}\,dt.$$

We enumerate the cases.

<u>Case 1</u>. $B = 0$. Then (5.6a) implies $A = 0$ and $x(t)$ is a constant random process.

(5.7) $$R_x(t) = R_x(0).$$

<u>Case 2</u>. $B \neq 0$ and $A = 0$. Then (5.6a) implies $V^0 = \frac{1}{2}$ and (1.2) implies $V^1 = \frac{1}{2}$. Using (5.3) we obtain

(5.8) $$R_x(t) = Q + \frac{B^2}{4}(T-2t)$$

which we recognize as a reformulation of (ii) of the list of stationary reciprocal processes of Section 4.

<u>Case 3</u>. $B \neq 0$ and $A \neq 0$. Without loss of generality we can assume $A < 0$. From (5.6a) we obtain

(5.9a) $$R_x(0) = \frac{B^2(1-2V^0)}{2A}.$$

From (5.6b) and (1.2) we obtain

(5.9b) $$0 \le Q = R_x(0) - \frac{B^2}{2A}(1-V^0)^2(1-e^{-2AT}).$$

From (5.9) we obtain a quadratic inequality for V^0,

(5.10) $$0 \le \frac{B^2}{2A}(e^{-2AT} - 2e^{-2AT}V^0 + (1-e^{-2AT})(V^0)^2).$$

This constrains V^0 to satisfy

(5.11) $$\frac{1}{1+e^{AT}} \le V^0 \le \frac{1}{1-e^{AT}}.$$

Next we solve (5.3) to obtain

(5.12) $$R_x(t) = \frac{B^2}{2A}((1-V^0)e^{-At} - V^0e^{At})$$

which we recognize as (i') of Section 4.

At this point we have exhausted all the processes obtainable from the one-dimensional acausal models (1.1), (5.1), yet we have not obtained (iii'). Hence we conclude that the acausal state processes are a proper subset of the reciprocal processes.

We conjecture that every stationary Gaussian reciprocal process of dimension n can be realized as the output process of an acausal model of dimension less than or equal to $2n$. We also conjecture that those stationary reciprocal Gaussian processes which can be realized as an acausal state process are characterized by Properties I, II and III above.

We close with two further observations. Suppose $x(t)$ is a stationary Gaussian process realized on $[0,T]$ as an acausal autonomous state process (1.1), (5.1). By differentiating (5.3) we obtain a differential equation for $R_x(t)$, i.e.,

$$\dot{R}_x(t) = AR_x(t) - BB^* e^{A^*(T-t)} V^{1*} \tag{5.13}$$

Clearly $R_x(t)$ satisfies the boundary conditions

$$V^0 R_x(0) + V^1 R_x(T) = Q. \tag{5.14}$$

If we differentiate a second time we obtain

$$\ddot{R}_x(t) = A\dot{R}_x(t) - \dot{R}_x(t)A^* + AR_x(t)A^* \tag{5.15}$$

This should be compared with (4.9).

Second, the unrealized one-dimensional reciprocal processes (iii') do have realizations as the output process of nonautonomous and nonstationary causal state processes.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a^2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b \\ \frac{\dot{g}(t)}{g(t)} b \end{pmatrix} u \tag{5.16a}$$

$$y(t) = x_1(t) \tag{5.16b}$$

where

$$g(t) = (1-B^2)\sin at - 2B\cos at$$

and

$$b = \sqrt{-2aB}.$$

The initial condition is

$$x(0) = x^0 \tag{5.16c}$$

where

$$R_x(0) = R_y(0)\begin{vmatrix} 1 & aB \\ aB & a^2B^2 \end{vmatrix} \text{ if } B \neq 0 \quad\text{and}\quad R_x(0) = R_y(0)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \text{ if } B = 0.$$

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