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Normal Forms for Linear and Nonlinear Systems*

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1 Introduction

It is well-known that a state space description of a controllable linear system can be transformed to controllable or controller form by a linear change of state variables. A state space description of an observable linear system can be transformed to observable or observer form by a linear change of state variables. Moreover the former are closely related to right matrix fractional descriptions (RMFD) of the transfer function and the latter are closely related to left matrix fractional descriptions (LMFD). These facts can be found in many texts such as Wolovich [1] or Kailath [2]. (The reader should be warned that the controllable/controller and observable/observer terminologies are not standard, we follow that of [2]). Unfortunately there is no one treatment of this material which is suitable for our purposes so we devote Sections 2 and 3 to a review. This is by way of preparation for our discussion of the existence and uniqueness of normal forms for nonlinear systems in Sections 4 and 5. Our treatment generalizes Zeitz [22] who discussed similar forms for scalar input and scalar output nonlinear systems.

2 Linear Normal Forms

Throughout this paper we shall use the following notation. The *indices* $\ell_1, \dots, \ell_q$ are positive integers summing to n. A *prime triple* (A,B,C) with indices $\ell_1, \dots, \ell_q$ is a triple of block diagonal matrices of dimension $n \times n$, $n \times m$ and $p \times n$ of the

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form

$$A = \text{BlockDiag.} \begin{bmatrix} 0 & 1 & \cdots & 0 \\ & \ddots & 1 & \\ 0 & \cdots & 0 & \end{bmatrix}^{\ell_i \times \ell_i} \tag{1}$$

$$B = \text{BlockDiag.} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}^{\ell_i \times 1} \tag{2}$$

$$C = \text{BlockDiag.} \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^{1 \times \ell_i} \tag{3}$$

The "prime" terminology was introduced by Morse [3].

Consider the linear state space description

$$\dot{\xi} = F\xi + Gu \tag{4}$$

$$y = H\xi \tag{5}$$

where $F^{n \times n}$, $G^{n \times m}$ and $H^{p \times n}$. The system is said to be *controllable if*

$$\text{rank}\ \{F^{r-1}G^j : j = 1, \ldots, m;\ r = 1, \ldots, n\} = n \tag{6}$$

(Note: F^r denotes the r^{th} power of F, G^j denotes the j^{th} column of G and H_i is the i^{th} row of H.)

Every controllable linear system has *controllability indices* $\ell_1, \ldots, \ell_m \geq 0$ characterized by $\ell_1 + \ldots + \ell_m = n$ and

$$\text{rank}\{F^{r-1}G^j : j = 1, \ldots, m;\ r = 1, \ldots, \ell\} =$$

$$\text{rank}\{F^{r-1}G^j : j = 1, \ldots, m;\ r = 1, \ldots, \ell \wedge \ell_j\} \tag{7}$$

for $\ell = 1, \ldots, n$. The minimum of ℓ and ℓ_j is denoted by $\ell_i \wedge \ell_j$. The set of controllability indices is uniquely determined by F and H and does not change under linear state feedback. There can be some freedom, m in the ordering of the controllability indices even when the ordering of the inputs remains fixed. This is because there may be several orderings which satisfy (7). Of course a change of variables in the input space or a reordering of the inputs can change the order of the controllability indices. We could reorder the inputs so that $\ell_1 \leq \ldots \ell_m$ or the reverse but we shall not do so. To simplify notation we shall restrict our attention to systems where the controllability indices are positive, $\ell_1, \ldots, \ell_m \geq 1$. A general system can be made to satisfy this condition by deleting dependent columns of G.

An alternative characterization of property (7) of the controllability indices is that

$$F^{\ell_j} G^j = 0 \tag{8}$$

$$\bmod \{F^{r-1} G^i : i = 1, \ldots, m;\ r = 1, \ldots, (\ell_j + 1) \wedge \ell_i\}.$$

The controllabilities indices of (4), (5) are said to be *strict* if (8) holds $\bmod \{F^{r-1} G^i : i = 1, \ldots, m;\ r = 1, \ldots, \ell_j \wedge \ell_i\}$. The controllability indices are strict iff there is only one ordering of the controllability indices satisfying (7).

It is always possible to make a linear change of input coordinates $\tilde{u} = \beta u$ that makes the controllability indices strict for the new pair $(\tilde{F}, \tilde{G}) = (F, G\beta^{-1})$ without changing their order. One way of accomplishing this to define $1 \times n$ vectors $K_1, \ldots, K_m$ by

$$K_i F^{r-1} G^j = \begin{cases} 0 & 1 < r < \ell_j \\ \delta_i^j & r = \ell_j \end{cases} \tag{9}$$

and let β be the $m \times m$ non-singular matrix whose $i - j$ entry is

$$\beta_i^j = K_i F^{\ell_i - 1} G^j. \tag{10}$$

In this case β satisfies

$$\beta_i^j = \delta_i^j \quad \ell_i \leq \ell_j. \tag{11}$$

Moreover β is the only such matrix which makes the controllability indices strict and leave the order invariant. A change of input coordinates $\overline{u} = \lambda \tilde{u}$ preserves the strictness of the controllability indices while leaving the order invariant iff $\lambda_i^j = 0$ for $\ell_i > \ell_j$.

The system (4), (5) is said to be observable if

$$\operatorname{rank} \{H_i F^{r-1} : i = 1, \ldots, p;\ r = 1, \ldots, n\} = n. \tag{12}$$

Every observable linear system has *observability indices* $\ell_1, \ldots, \ell_p \geq 0$ characterized by $\ell_1 + \cdots + \ell_p = n$ and

$$\operatorname{rank} \{H_i F^{r-1} : i = 1, \ldots, p;\ r = 1, \ldots, \ell\} =$$

$$\operatorname{rank} \{H_i F^{r-1} : i = 1, \ldots, p;\ r = 1, \ldots, \ell \wedge \ell_i\} \tag{13}$$

for $\ell = 1, \ldots, n$. The set of indices is uniquely determined by H and F and does the change under linear change of coordinates in the state and output spaces and linear output injection. There can be freedom in the ordering of the observability indices even when the order of the outputs remains fixed. This is because there may be several orderings which satisfy (13). Of course a change of output variables or a reordering of the outputs can change the order of the observability indices. We could reorder the outputs so that $\ell_1 \leq \ldots \leq \ell_p$ or the reverse but we shall not do so. We shall restrict our discussion to systems where all the observability indices are positive.

Similarly an alternative characterization of property (13) of the observability indices is that

$$H_i F^{\ell_i} = 0 \tag{14}$$

$$\text{mod } \{H_j F^{r-1} : j = 1, \ldots, p;\ r = 1, \ldots, (\ell_i + 1) \wedge \ell_j\}.$$

The observability indices of (4), (5) are said to be *strict* if (14) holds mod $\{H - jF^{r-1} : j = 1, \ldots, p;\ r = 1, \ldots, \ell_i \wedge \ell_j\}$. The observability indices are strict iff there is only one ordering of them satisfying (13). It is always possible to make a linear change of output coordinates $y = \gamma \tilde{y}$ that makes the observability indices strict for the new pair $(\tilde{H}, \tilde{F}) = (\gamma^{-1} H, F)$ while not changing their order. One way of accomplishing this is to define $n \times 1$ vectors $Q^1, \ldots, Q^p$ by

$$H_i F^{r-1} Q^j = \begin{cases} 0 & 1 \leq r \leq \ell_i \\ \delta_i^j & r = \ell_i \end{cases} \tag{15}$$

and let γ be the $p \times p$ non-singular matrix whose $i - j^{th}$ entry is

$$\gamma_i^j = H_i F^{\ell_j - 1} Q^j. \tag{16}$$

In this case γ satisfies

$$\gamma_i^j = \delta_i^j \quad \ell_i \geq \ell_j. \tag{17}$$

Moreover γ is the only such matrix which makes the observability indices strict and leaves the order invariant because a change of output coordinates $\tilde{y} = \mu \bar{y}$ preserves the strictness and order of the observability indices iff $\mu_i^j = 0$ for $\ell_i < \ell_j$.

The *controllable form* of a linear system is

$$\dot{x} = Ax - \alpha C x + Bu \tag{18}$$

$$y = \gamma x \tag{19}$$

where (A, B, C) is a prime triple with indices $\ell_1, \ldots, \ell_m$ and α and γ are arbitrary matrices of dimensions $n \times m$ and $p \times n$.

The following facts are well-known and/or can be easily proved. A system in controllable form is controllable with controllability indices $\ell_1, \ldots, \ell_m$. A system (4), (5) can be transformed into controllable form (18), (19) by a linear change of state coordinates $\xi = Tx$ iff it is controllable. If (4), (5) is controllable with controllability indices $\ell_1, \ldots, \ell_m$ then the x coordinates of (18), (19) are defined by taking as a basis the columns of the matrix T

$$T = [F^{\ell_1 - 1} G^1, \ldots, G^1, \ldots, F^{\ell_m - 1} G^m, \ldots, G^m] \tag{20}$$

Let $x = T^{-1}\xi$ have components

$$x^* = (x_{11}, \ldots, s_{1\ell_1}, \ldots, x_{m1}, \ldots, x_{m\ell_m}), \tag{21}$$

where $*$ denotes transpose, then $F^{\ell_j - r}G^j$ in ξ coordinates becomes the unit vector in the x_{jr} direction in x coordinates. The j^{th} column of α and the matrix γ are given by

$$\begin{aligned} \alpha^j &= -T^{-1}F^{\ell_j}G^j & (22)\\ \gamma &= HT. & (23) \end{aligned}$$

It can be shown that $\alpha^j_{ir} = 0$ if $\ell_i - r > \ell_j$. The controllability indices are strict iff $\alpha^j_{ir} = 0$ for $\ell_i - r \geq \ell_j$. The controllable form (18), (19) of the linear system (4), (5) and the associated x coordinates (20), (21), (22), (23) are unique up to reordering of the controllability indices. The *observable form* of a linear system is

$$\begin{aligned} \dot{x} &= Ax - B\alpha x + \beta u & (24)\\ y &= Cx & (25) \end{aligned}$$

where (A, B, C) is a prime triple with indices $\ell_1, \ldots, \ell_p$ and α and β are arbitrary matrices of dimensions $p \times n$ and $n \times m$.

A system in observable form is observable with observability indices $\ell_1, \ldots, \ell_p$. A system (4), (5) can be transformed into observable form (24), (25) by a linear change of state coordinates $\xi = Tx$ iff it is observable. If (4), (5) is observable with observability indices $\ell_1, \ldots, \ell_p$ then the x coordinates of (18), (19) are of the form

$$x^* = (x_{11}, \ldots, x_{1\ell_1}, \ldots, x_{p1}, \ldots, x_{p\ell_p}) \tag{26}$$

where T^{-1} is defined by

$$x_{ir} = H_i F^{r-1}\xi. \tag{27}$$

The i^{th} row of α and the matrix β are given by

$$\begin{aligned} \alpha_i &= -H_i F^{\ell_i} T & (28)\\ \beta &= T^{-1}G. & (29) \end{aligned}$$

It can be shown that $\alpha^{jr}_i = 0$ if $r > \ell_i + 1$. The observability indices are strict iff $\alpha^{jr}_i = 0$ for $r > \ell_i$. The observable form (24), (25) of the linear system (4), (5) and the associated x coordinates (26), (27), (28), (29) are unique up to a reordering of observability indices.

The *controller form* of linear system is

$$\begin{aligned} \dot{x} &= Ax - B\alpha x + B\beta u & (30)\\ y &= \gamma x & (31)\end{aligned}$$

where (A, B, C) is a prime triple with indices $\ell_1, \dots, \ell_m$ and α, β, γ are matrices of dimensions $m \times n, m \times m, p \times n$. These matrices are arbitrary except β must be non-singular.

A system in controller form is controllable with controllability indices $\ell_1, \dots, \ell_m$ and the controllability indices are strict relative to the input $\tilde{u} = \beta u$. A system (4),(5) can be transformed into controller form (30), (31) by a linear change of state coordinates $\xi = Tx$ iff it is controllable. If (4), (5) is controllable with controllability indices $\ell_1, \dots, \ell_m$, then let β be defined by (10). One can define a pseudo-output for (4), (5).

$$\psi = K\xi \tag{32}$$

where K is the $m \times n$ matrix defined by (9). The square system (4) and (32) is observable with strict observability indices $\ell_1, \dots, \ell_m$. The observable form realization of (4) and (32) is a controller form realization of (4), (5). The x coordinates of (30), (31) are of the form (20) and

$$x_{jr} = K_j F^{r-1} \xi. \tag{33}$$

The matrix γ is given by (23) and the i^{th} row of α is given by

$$\alpha_i = -K_i F^{\ell_i} T. \tag{34}$$

Since the observability indices of (4) and (32) are strict, we have

$$\alpha_i^{jr} = 0 \quad r > \ell_i. \tag{35}$$

In general controller form realizations are not unique. However the controller form realization which satisfies (11) and (35) is unique up to reordering of the controllability indices.

The *observer form* of a linear system is

$$\begin{aligned} \dot{x} &= Ax - \alpha Cx + \beta u & (36)\\ y &= \gamma Cx & (37)\end{aligned}$$

where (A, B, C) is a prime triple with indices $\ell_1, \dots, \ell_p$ and α, β, γ are indices of dimensions $n \times p, n \times m, p \times p$. These matrices are arbitrary except that γ must be non-singular.

A system in observer form is observable with observability indices $\ell_1, \ldots, \ell_p$ and the observability indices are strict relative to the output $\tilde{y} = \gamma^{-1}y$. A system (4), (5) can be transformed into observer form (36), (37) by a linear change of state coordinates $\xi = Tx$ iff it is observable. If (4), (5) is observable with observability indices $\ell_1, \ldots, \ell_p$, let γ be given by (16). One can define a pseudo-input μ

$$\dot{\xi} = F\xi + Q\mu \tag{38}$$

where Q is the $n \times p$ matrix defined by (15). The square system (38) and (5) is controllable with strict controllability indices $\ell_1, \ldots, \ell_p$. The controllable form realization of (38) and (5) is an observer form realization of (4), (5). The x coordinates of (36), (37) are of the form (26) and defined by $\xi = Tx$ where

$$T = [F^{\ell_1 - 1}Q^1, \ldots, Q^1, \ldots, F^{\ell_p - 1}A^p, \ldots, Q^p]. \tag{39}$$

The matrix β is given by (29) and the j^{th} column of α is given by

$$\alpha^j = -T^{-1}F^{\ell_j}Q^j.] \tag{40}$$

Since the controllability indices of (38) and (5) are strict, we have

$$\alpha^j_{ir} = 0 \quad 1 \leq r \leq \ell_i - \ell_j. \tag{41}$$

In general observer form realizations are not unique. However, the observer form realization which satisfies (17) and (41) is unique up to a reordering of the observability indices.

Remark 2.1 *The controller form (30), (31) of a system is very useful in designing a linear state variable feedback to stabilize the system. The observer form (36), (37) is very useful in designing asymptotic observers. Together they can be used to stabilize a system by dynamic output feedback (also called observer based compensation). See [1] or [2] for details.*

Remark 2.2 *Controllable and observable forms are easier to compute and are useful for finding the observer and controller forms of related systems.*

3 MFD's

The purpose of this section is to emphasize the very close relationship between the normal forms of a linear system described above and the so-called polynomial matrix fractional descriptions of its transfer function. For linear systems it is only a matter of personal preference which representation we choose to work with. This is not the case for the nonlinear systems because they don't have nice frequency domain descriptions. Our treatment is similar to that of [1] and [2].

Throughout we shall use the following notation. Given the indices $\ell_1, \ldots, \ell_q$ where $n = \ell_1 + \cdots + \ell_q$ then $\Delta(s)$, $\Phi(s)$ and $\Psi(s)$ are block diagonal polynomial matrices of dimensions $q \times q$, $q \times n$ and $n \times q$ of the form

$$\Delta(s) = \text{BlockDiag} \begin{bmatrix} s^{\ell_i} \end{bmatrix}^{1\times 1} \tag{42}$$

$$\Phi(s) = \text{BlockDiag} \begin{bmatrix} s^{\ell_i - 1} & \cdots & 1 \end{bmatrix}^{1\times \ell_i} \tag{43}$$

$$\Psi(s) = \text{BlockDiag} \begin{bmatrix} 1 \\ \vdots \\ s^{\ell_i - 1} \end{bmatrix}^{\ell_1 + 1} \tag{44}$$

The linear state space description

$$\dot{x} = Ax + v \tag{45}$$

$$z = x \tag{46}$$

with input v, state x, output z, all of dimension n, has the following polynomial matrix description in the transform domain

$$\Delta(s)\xi(s) = \Phi(s)v(s) \tag{47}$$

$$z(s) = \Psi(s)\xi(s) \tag{48}$$

where the so called "partial state" $\xi(s)$ is defined by

$$\xi(s) = Cx(s) \tag{49}$$

or equivalently

$$x(s) = \Psi(s)\xi(s) \tag{50}$$

(Here $x(s)$ denotes the Laplace transform of $x(t)$, etc.). The (A, B, C) of the above are a prime triple with indices $\ell_1, \ldots, \ell_q$ so that

$$C\Psi(s) = \Phi(s)B = I^{q\times q}. \tag{51}$$

From this we quickly obtain MFD's corresponding to the 4 normal forms of the last section. For a system in controllable form (18), (19) we use the relations

$$v(s) = -\alpha Cx(s) + Bu(s) \tag{52}$$

$$y(s) = \gamma x(s). \tag{53}$$

Let $q = m$, then from (47), (48) and (49), (50), (51) we obtain

$$u(s) = (\Delta(s) + \Phi(s)\alpha)\xi(s) \quad (54)$$
$$y(s) = \gamma\Psi(s)\xi(s) \quad (55)$$

which is a RMFD of the form

$$y(s) = N(s)D^{-1}(s)u(s) \quad (56)$$

where

$$D(s) = (\Delta(s) + \Phi(s)\alpha) \quad (57)$$
$$N(s) = \gamma\Psi(s). \quad (58)$$

Given a RMFD (56) we can always obtain a controllable form realization. Recall that a polynomial matrix is *unimodular* if it has an inverse which is a polynomial matrix. If we multiply $N(s)$ and $D(s)$ on the right by a unimodular matrix we don't change the transfer function. In this way we can insure that the matrix of highest column coefficients of D is invertible and even more equals the identity.

Let $\ell_1, \ldots, \ell_m$ be the column degrees of D, then $D(s)$ and $N(s)$ can be written as (57), (58) thus defining α and γ. This yields a controllable form realization of (56).

For a system in controller form (30), (31) we use the relations

$$v(s) = -B\alpha x(s) + B\beta u(s) \quad (59)$$
$$y(s) = \gamma z(s) \quad (60)$$

and so we obtain the RMFD (56) where

$$D(s) = \beta^{-1}(\Delta(s) + \alpha\Psi(s)) \quad (61)$$
$$N(s) = \gamma\Psi(s). \quad (62)$$

Of course we can go backwards. Given the RMFD (56) we multiply $N(s)$ and $D(s)$ on the right by a unimodular matrix so that the matrix of highest column coefficients of $D(s)$ is nonsingular. The decomposition (61), (62) defines α, β and γ of a controller realization of the transfer function.

For a system in observable form (24), (25) we use the relations

$$v(s) = -B\alpha x(s) + \beta u(s) \quad (63)$$
$$y(s) = Cz(s) \quad (64)$$

We obtain the LMFD

$$y(s) = D^{-1}(s)N(s)u(s) \tag{65}$$

where

$$\begin{aligned} D(s) &= \Delta(s) + \alpha\Psi(s) && (66) \\ N(s) &= \Phi(s)\beta. && (67) \end{aligned}$$

On the other hand given a LMFD (65) we can multiply $D(s)$ and $N(s)$ on the left by a unimodular matrix to obtain the decomposition (66), (67). This defines α and β of an observable form realization.

For a system in observer form (36), (37) we use the relations

$$\begin{aligned} v(s) &= -\alpha Cx(s) + \beta u(s) && (68) \\ y(s) &= \gamma Cz(s) && (69) \end{aligned}$$

which lead to a LMFD (65) where

$$\begin{aligned} D(s) &= (\Delta(s) + \Phi(s)\alpha)\gamma^{-1} && (70) \\ N(s) &= \Phi(s)\beta. && (71) \end{aligned}$$

Given the LMFD (65) the decomposition (70), (71) defines α, β and γ of an observer form realization.

4 Nonlinear Observable and Controller Forms

Henceforth we focus our attention on the nonlinear system

$$\begin{aligned} \dot{xi} &= f(\xi) + g(\xi)u && (72) \\ y &= h(\xi) && (73) \end{aligned}$$

where $\xi \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and f, g, h are smooth (C^∞) functions. We are interested in (72), (73) in some open connected subset M of the state space containing the nominal operating point ξ^0.

We introduce some terminology and notation. The *Lie derivatives* of the function $h_i(\xi)$ by the vector fields $f(\xi)$ and $g^j(\xi)$ are functions defined by

$$\begin{aligned} L_f(h_i)(\xi) &= \frac{\partial h_i}{\partial \xi}(\xi)f(\xi) && (74) \\ L_{g^j}(h_i)(\xi) &= \frac{\partial h_i}{\partial \xi}(\xi)g^j(\xi). && (75) \end{aligned}$$

Of course these operations can be iterated,

$$L_f^r(h_i) = L_f(L_f^{r-1}(h_i)). \tag{76}$$

The *differential* dh_i of a function h_i is a one form defined by

$$dh_i(\xi) = \frac{\partial h_i}{\partial \xi}(\xi). \tag{77}$$

A *one form* ω is a row vector field or more precisely a C^∞ linear combination of differentials.

$$\omega(\xi) = (\omega^1(\xi), \ldots, \omega^n(\xi)) = \sum k_i(\xi) dh_i(\xi) \tag{78}$$

where $k_i(\xi)$ and $h_i(\xi)$ are smooth functions. A one form can be paired with a vector field (all vector fields are columns unless otherwise stated) to obtain a function

$$\langle \omega, f \rangle(\xi) = \omega(\xi) f(\xi) = \sum_{i=1}^{n} \omega^i(\xi) f_i(\xi). \tag{79}$$

A vector field can also Lie differentiate a form to obtain another one form

$$L_f(\omega) = \omega \frac{\partial f}{\partial \xi} + (\frac{\partial \omega}{\partial \xi}^* f)^* \tag{80}$$

where $*$ denotes transpose. In particular

$$L_f(dh_i) = d(L_f(h_i)). \tag{81}$$

A vector field can also Lie differentiate another vector field to yield a third vector field.

$$ad(f)g^j = [f, g^j] = \frac{\partial g^j}{\partial \xi}(\xi) f(\xi) - \frac{\partial f}{\partial \xi}(\xi) g^j(\xi). \tag{82}$$

This can be iterated,

$$ad^r(f)g^j = [f, ad^{r-1}(f)g^j]. \tag{83}$$

The operation (82) is also called the *Lie bracket* (82) of the vector fields and can be thought of both as a multiplication and as a differentiation. This is evidenced by the following Liebnitz-type formula called the Jacobi identity

$$[f, [g^i, g^j]] = [[f, g^i], g^j] + [g^i, [f, g^j]]. \tag{84}$$

Moreover the pairing (79) satisfies a Liebnitz formula with respect to Lie differentiation

$$L_f(\langle \omega, g^j \rangle) = \langle L_f(\omega), g^j \rangle + \langle \omega, [f, g^j] \rangle \tag{85}$$

For the readers unfamiliar with these concepts we suggest the calculation of the above definitions and formulas in the linear case (4), (5) where

$$f(\xi) = F\xi \tag{86}$$
$$g^j(\xi) = G^j \tag{87}$$
$$h_i(\xi) = H_i\xi. \tag{88}$$

We define

$$\mathcal{E}^\ell = C^\infty\{L_f^{r-1}(dh_i) : i = 1,\ldots,p;\ r = 1,\ldots,\ell\} \tag{89}$$

where $C^\infty\{\cdot\}$ means the linear span over C^∞ coefficients. Such a collection of one forms which is closed under addition and multiplication by C^∞ functions is called a *codistribution.* We denote by $\mathcal{E}^\ell(\xi)$ the linear space of $1 \times n$ vectors obtained by evaluating the one forms of $\mathcal{E}^\ell$ at the point ξ.

Given indices $\ell_1,\ldots,\ell_p$ we define

$$\mathcal{E}^\ell_{\ell_1,\ldots,\ell_p} = C^\infty\{L_f^{r-1}(dh_i) : i = 1,\ldots,p;\ r = 1,\ldots,\ell \wedge \ell_i\} \tag{90}$$

and $\mathcal{E}^\ell_{\ell_1,\ldots,\ell_p}(\xi)$ the vector space obtained by evaluation of these forms at ξ.

The system is (72), (73) has *observability indices* $\ell_1,\ldots,\ell_p$ around ξ^0 if $\ell_1 + \cdots + \ell_p = n$, and

$$\text{dimension } \mathcal{E}^n(\xi) = n \tag{91}$$

and

$$\mathcal{E}^\ell(\xi) = \mathcal{E}^\ell_{\ell_1,\ldots,\ell_p}(\xi) \tag{92}$$

for $\ell = 1,\ldots,n$ and all ξ in some neighborhood of ξ^0. The reader who has done the suggested calculations recognizes (91) as a generalization of (12) and (92) a generalization of (13). The *observability indices* are strict if

$$L_f^{\ell_i}(dh_i) \in \mathcal{E}^{\ell_i} \tag{93}$$

for $i = 1,\ldots,p$. This generalizes the linear definition.

The set of observability indices of (72), (73) is uniquely determined by h and f and is invariant under changes of coordinates in the state and output spaces. There can be some freedom in the ordering of the indices even when the ordering of the outputs remains fixed. The observability indices are strict iff there is only one ordering satisfying (92). To simplify notation we restrict our attention to systems where all the observability indices are positive.

Condition (91) could be called *zero input observability.* It means that the state $\xi(t)$ of (72), (73) can be distinguished from its neighbors by the output $y(t)$ and its first $n-1$ time derivatives along the trajectories near ξ^0 corresponding to

$u(t) = 0$. Unlike the linear case, (91) does not imply the existence of observability indices satisfying (92) around every ξ^0 but only for a generic, (i.e. an open and dense) set of ξ^0's. The latter condition implies that the functions

$$x_{ir} = L_f^{r-1}(h_i)(\xi) \tag{94}$$

for $i = 1, \ldots, p$; $r = 1, \ldots, \ell_i$ are valid local coordinates on the state space. When (72),(73) has observability indices around a point ξ^0 which is a critical point of f, $(f(\xi^0) = 0)$ then they agree with the observability indices of the linear approximating system to (72), (73) at ξ^0.

The *observable form* of a nonlinear system is

$$\dot{x} = Ax - B\alpha(x) + \beta(x)u \tag{95}$$

$$y = Cx \tag{96}$$

where (A, B, C) is a prime triple with indices $\ell_1, \ldots, \ell_p$ and α, β are smooth $m \times 1$, $n \times m$ matrix valued functions of x.

Proposition 4.1 . *A nonlinear system in observable form (95), (96) has observability indices $\ell_1, \ldots, \ell_p$. A nonlinear system (72), (73) can be transformed into observable form (95), (96) by a change of local coordinates around ξ^0 iff the system has observability indices around ξ^0. If (72), (73) has observability indices $\ell_1, \ldots, \ell_p$ around ξ^0 then the x coordinates of the form (26) given by (94) transform it to observable form. The observable form of a nonlinear system, the associated x coordinates and the nominal x-operating point $x^0 = T^{-1}(\xi^0)$ are unique up to a reordering of the observability indices. The functions α and β of the observable form (95), (96) are given by*

$$\alpha_i = L_f^{\ell_i}(h_i) \tag{97}$$

$$\beta_{ir}^j = L_{g^j} L_f^{r-1}(h_i). \tag{98}$$

The observability index assumption (92) implies that

$$d\alpha_i = L_f^{\ell_i}(dh_i) \in \mathcal{E}^{\ell_i+1} \tag{99}$$

which means that α_i does not depend on x_{jr} if $r > \ell_i + 1$. The observability indices are strict (93) iff α_i does not depend on a x_{jr} if $r \geq \ell_i + 1$, in other words

$$d\alpha_i = L_f^{\ell_i}(dh_i) \in \mathcal{E}^{\ell_i}. \tag{100}$$

The proof of this result is relatively straightforward, for example see [4], section 2.

We now turn to controllability properties of (72), (73). We define

$$\mathcal{D}^{\ell} = C^{\infty}\{\mathrm{ad}^{r-1}(-f)g^{j} : j = 1, \ldots, m;\ r = 1, \ldots, \ell\}. \tag{101}$$

This is a collection of vector fields closed under addition and multiplication by C^{∞} functions; such an object is called a *distribution*. Given indices $\ell_1, \ldots, \ell_m$, let

$$\mathcal{D}^{\ell}_{\ell_1,\ldots,\ell_m} = \{\mathrm{ad}^{r-1}(-f)g^{j} : j = 1, \ldots, m;\ r = 1, \ldots, \ell \wedge \ell_j\}. \tag{102}$$

The system (72), (73) has *controllability indices* $\ell_1, \ldots, \ell_m$ around ξ^0 if $\ell_1 + \cdots + \ell_m = n$ and

$$\text{dimension } \mathcal{D}^{n}(\xi) = n \tag{103}$$

and

$$\mathcal{D}^{\ell}(\xi) = \mathcal{D}^{\ell}_{\ell_1,\ldots,\ell_m}(\xi) \tag{104}$$

for $\ell = 1, \ldots, n$ and all ξ in some neighborhood of ξ^0. Of course (103) is a generalization of (6) and (104) is a generalization of (7). The *controllability indices are strict* if

$$\mathrm{ad}^{\ell_j}(-f)g^{j} \in \mathcal{D}^{\ell_j}_{\ell_i,\ldots,\ell_m} \tag{105}$$

for $i = 1, \ldots, n$. This generalizes the linear definition.

The set of controllability indices of (72), (73) is uniquely determined by f and g and is invariant under change of coordinates in the state space and nonlinear state feedback, i.e. $u = \alpha(x) + \beta(x)v$ where $\beta(x)$ is $m \times m$ invertible. There can be some freedom in the ordering of the indices even when the output is fixed. The controllability indices are strict iff there is only one ordering satisfying (104). For notational convenience, we restrict our attention to systems where the controllability indices are positive.

Condition (103) could be called *local linear controllability* for if ξ^0 is a critical point of f, $(f(\xi^0) = 0)$ then the linear approximation to (72), (73) at ξ^0 is controllable iff (103) holds. Once again (103) does not imply the existence of controllability indices satisfying (104) around every ξ^0, only for an open, dense set of ξ^0's. When (72), (73) has controllability indices around a critical point ξ^0, they agree with the controllability indices of the linear approximating system to (72), (73) at ξ^0.

The *controller form* of a nonlinear system is

$$\begin{aligned} \dot{x} &= Ax - B\alpha(x) + B\beta(x)u && (106)\\ y &= \gamma(x) && (107) \end{aligned}$$

where (A, B, C) is a prime triple with indices $\ell_1, \ldots, \ell_m$ and $\alpha,\ \beta,\ \gamma$ are smooth $m \times 1,\ m \times m,\ p \times 1$ matrix valued functions of x which are arbitrary except that $\beta(x)$ must be nonsingular. The question of when a nonlinear control system can be transformed to controller form has been independently solved by several authors [5,6,7,8,9,22]. Some only considered special cases like $m = 1$ or $\beta(x)$ =constant. Our treatment follows Hunt and Su [8].

Recall that a distribution $\mathcal{D}$ is *involutive* if it is closed with respect to Lie bracket, i.e. $[q^1, q^2] \in \mathcal{D}$ whenever $q^1, q^2 \in \mathcal{D}$. Given a distribution we can consider the under-determined systems of partial differential equations.

$$\langle dk, q \rangle = 0 \quad \text{forall } q \in \mathcal{D} \tag{108}$$

for the unknown function $k(\xi)$. The question of existence and uniqueness of local solutions to (73) is addressed by the following.

Frobenius Theorem Suppose $\mathcal{D}$ is of constant codimension d. $\mathcal{D}$ is involutive iff locally there exists d independent solutions $k_1, \ldots, k_d$ to (73). Any other solution $k(\xi)$ is a function of $k_1(\xi), \ldots, k_d(\xi)$.

Proposition 4.2 *([8], see also [5,6,7,8,9,22]). A nonlinear system in controller form (106), (107) has controllability indices $\ell_1, \ldots, \ell_m$ which are strict relative to the input $\tilde{u} = \beta u$. A nonlinear system (72), (73) can be transformed into controller form (106), (107) by a local change of coordinates around ξ^0 iff it has controllability indices $\ell_1, \ldots, \ell_m$ and $\mathcal{D}^{\ell_j - 1}$ is involutive for $j = 1, \ldots, m$.*

Proof The proof of the first statement is a straightforward verification.

As for the second suppose (72), (73) can be transformed to controller form by $\xi = T(x)$. Using the C matrix of the prime triple we define a pseudo-output.

$$\psi = k(\xi) = CT^{-1}(\xi) \tag{109}$$

then the function k satisfies

$$L_f^{\ell_i}(k_i) = \alpha_i \tag{110}$$

$$L_{g^j} L_f^{r-1}(k_i) = \begin{cases} 0 & 1 \le r < \ell_i \\ \beta_i^j & r = \ell_i \end{cases} \tag{111}$$

Using the Liebnitz formula (85) and induction we can show that (111) is equivalent to

$$\langle L_f^s(dk_i), \mathrm{ad}^{r-s-1}(-f)g^j \rangle = \begin{cases} 0 & 1 \le r < \ell_i, \quad 0 \le s < r \\ \beta_i^j & r = \ell_i, \quad 0 \le s < r \end{cases} \tag{112}$$

From this it follows that for every $q \in \mathcal{D}^\ell$

$$\langle L_f^{r-1}(dk_i), q \rangle = 0 \tag{113}$$

for $i = 1, \ldots, n$ and $r = 1, \ldots, \ell_i - \ell$. Moreover from the invertibility of β it follows that the functions $\{L_f^{r-1}(k_i)\ i = 1, \ldots, m$ and $r = 1, \ldots, \ell_i - \ell\}$ are independent. There are as many such functions as the codimension of $\mathcal{D}^\ell$ so by the Frobenius theorem $\mathcal{D}^\ell$ is involutive for all ℓ and in particular for $\ell = \ell_j - 1;\ j = 1, \ldots, m$.

On the other hand if $\mathcal{D}^{\ell_j - 1}$ is involutive for $j = 1, \ldots, m$ then by repeated application of the Frobenius theorem one can find independent functions $k_1, \ldots, k_m$ satisfying (112) where β is some invertible $m \times m$ matrix valued function. If we define x coordinates by

$$x_{jr} = L_f^{r-1}(k_j)(\xi) \tag{114}$$

for $j = 1, \ldots, m;\ r = 1, \ldots, \ell_j$ then these coordinates transform the nonlinear system to controller form (106), (107). The functions α and β are given by (110), (111).

When it exists, the controller form of a nonlinear system is not unique. From the proof of the above we see that the controller form is completely determined by the choice of the pseudo-output $k(\xi)$ satisfying (111) for some invertible $\beta(\xi)$. If $\tilde{k}(\xi)$ is another solution of (111) then (112) and (113) imply that $\tilde{k}_i(\xi)$ is a function of $x_{jr} = L_f^{r-1}(k_j)$ for $\ell_j \geq \ell_i$ and $r = 1, \ldots, \ell_j - \ell_i$.

Notice that the nominal operating point $x^0 = T^{-1}(\xi^0)$ of the controller form is determined by the choice of $k(\xi)$. In particular there exists k such that $x^0 = 0$ iff there exists u^0 such that $f(\xi^0) + g(\xi^0)u^0 = 0$.

Another point worth mentioning is that the system (72) with pseudo-output $\psi = k(\xi)$ does not necessarily have observability indices $\ell_1, \ldots, \ell_m$. This would be the case iff in addition to (112), $k(\xi)$ satisfies

$$\langle L_f^{\ell_i}(dk_i), \mathrm{ad}^{r-1}(-f)g^j\rangle = 0$$

for $r = 1, \ldots, \ell_j - \ell_i - 1$.

We might try to obtain a unique controller form by requiring that α and β also satisfy the nonlinear generalizations of (11) and (35), namely

$$\beta_i^j(\xi) = \delta_i^j \quad \ell_i \leq \ell_j \tag{115}$$

$$\frac{\partial \alpha_i}{\partial x_{jr}} = 0 \quad r > \ell_i \tag{116}$$

But this would reduce the number of nonlinear systems that admit a controller form. The conditions (115), (116) imply that

$$\langle dk_i, \mathrm{ad}^{r-1}(-f)g^j\rangle = \begin{cases} 0 & 1 \leq r < \ell_j \\ \beta_i^j & r = \ell_j \end{cases} \tag{117}$$

This is a system of first order partial differential equations for the unknown functions $k_1, \ldots, k_m$. The solvability of such a system is addressed by the following.
Integrability Theorem Let $q^1(\xi), \ldots, q^n(\xi)$ be an n linearly independent n dimensional vector fields. There exists a solution $k = (k_1, \ldots, k_m)$ to the system

$$\langle dk_i, q^j \rangle = \begin{cases} \delta_i^j & j = 1, \ldots, m \\ 0 & j = m+1, \ldots, n \end{cases}$$

iff

$$[q^i, q^j] \in \mathcal{D} \quad i, j = 1, \ldots, n$$

where $\mathcal{D}$ is the distribution spanned by $\{q^{m+1}, \ldots, q^n\}$. The solution is unique up to a choice of $k(\xi^0)$.

From this theorem we see that there exists a solution to (117) iff

$$[\mathrm{ad}^{r-1}(-f)g^i,\ \mathrm{ad}^{s-1}(-f)g^j] \in \mathcal{D} \tag{118}$$

for $i, j = 1, \ldots, n;\ r = 1, \ldots, \ell_i,\ s = 1, \ldots, \ell_j$ where $\mathcal{D}$ is the distribution given by

$$\mathcal{D} = C^\infty\{\mathrm{ad}^{r-1}(-f)g^j :\ j = 1, \ldots, n;\ r = 1, \ldots, \ell_j - 1\}. \tag{119}$$

Condition (118),(119) is considerably more stringent then $\mathcal{D}^{\ell_j - 1}$ being involutive for $j = 1, \ldots, m$. In particular suppose we consider a generic nonlinear system (72), (73) with $n = 2$ and $m = 1$. Around a generic point ξ^0, the vector fields g^1 and $\mathrm{ad}(-f)g^1$ are linear independent hence such a system has a controllability index $\ell_1 = 2$. The distribution $\mathcal{D}^1 = C^\infty\{g^1\}$ is trivially involutive so such a system has a controller form. However condition (118), (119) which in this case is

$$[g^1,\ \mathrm{ad}(-f)g^1] \in C^\infty\{g^1\}$$

is not generically satisfied.

Suppose (72), (73) has controllability indices $\ell_1, \ldots, \ell_m$ around ξ^0. Regardless of whether or not it admits a controller form around ξ^0, it is always possible to make the controllability indices strict by a change of input coordinates $\tilde{u} = \beta(\xi)u$ when $\beta_i^j(\xi) = \delta_i^j$ for $\ell_i \le \ell_j$ as in the linear case. We define one forms $\omega_1(\xi), \ldots, \omega_p(\xi)$ by

$$\langle \omega_i, \mathrm{ad}^{r-1}(-f)g^j \rangle = \begin{cases} 0 & 1 \le r < \ell_i \\ \delta_i^j & r = \ell_i \end{cases} \tag{120}$$

From this and the controllability index assumptions (104) it follows that

$$\langle \omega_i, \mathrm{ad}^{r-1}(-f)g^j \rangle = 0 \quad \ell_j < r < \ell_i. \tag{121}$$

Moreover by repeated use of the Liebnitz formula (85) we see that (120), (121) is equivalent to

$$\langle L_f^{r-1}(\omega_i), g^j \rangle = \begin{cases} 0 & 1 \le r < \ell_j \text{ or } \ell_j < r < \ell_i \\ \delta_i^j & r = \ell_j \end{cases} . \tag{122}$$

We define β by

$$\beta_i^j = \langle L_f^{\ell_i - 1}(\omega_i), g^j \rangle. \tag{123}$$

Immediately we see that $\beta_i^j = \delta_i^j$ for $\ell_i \le \ell_j$ so β is invertible. It is not hard to show that the system defined by $(\tilde{f}, \tilde{g}) = (f, g\beta^{-1})$ has strict controllability indices $\ell_1, \ldots, \ell_m$. Notice that if (117) is solvable then $\omega_i = dk_i$.

5 Nonlinear Controllable and Observer Forms

The *controllable form* of a nonlinear system is

$$\dot{x} = Ax + \alpha(Cx) + Bu \tag{124}$$

$$y = \gamma(x) \tag{125}$$

where (A, B, C) is a prime triple with indices ℓ_1, ℓ_m and α, γ are smooth matrix valued functions of dimensions $n \times 1$, $p \times 1$. Notice that α is a function of the pseudo-output $\psi = Cx$ while γ is a function of x.

Notice that if $\alpha(\psi)$ is a linear function of ψ then the dynamics (124) of the nonlinear controllable form agree with the dynamics (18) of the linear controllable form. Hence the question of the existence of a nonlinear controllable form is closely related to the question of linearizing the dynamics (124) by a change of state coordinates. This latter question has a long history going back to Poincare [16]. For more recent work see [17,18,19,20,21].

For the most part the controllable forms of nonlinear systems have not appeared explicitly in the literature. But as one might expect they have arisen implicitly in some of the work on observer form [4,10], and on linearization [21]. The following is a reformulation of similar results from [10], [21] and [22].

Proposition 5.1 *A nonlinear system in controllable form has controllability indices* $\ell_1, \ldots, \ell_m$. *A nonlinear system (72), (73) can be transformed into controllable form (124), (125) by a change of local coordinates around* ξ^0 *iff it has controllability indices* $\ell_1, \ldots, \ell_m$ *and*

$$[\mathrm{ad}^{r-1}(-f)g^i,\ \mathrm{ad}^{s-1}(-f)g^j] = 0 \tag{126}$$

for $i, j = 1, \ldots, m$ *and* $r = 1, \ldots, \ell_i$, $s = 1, \ldots, \ell_j$ *around* ξ^0. *Controllable form and the associated* x *coordinates are unique up to a choice of the nominal* x*-operating point* $x^0 = T^{-1}(\xi^0)$ *and up to reordering of the controllability indices.*

The dynamics (72) of a nonlinear system can be linearized, or equivalently, can be transformed to the dynamics of linear controllable form (18) by a change of state coordinates around ξ^0 iff (72) has controllability indices $\ell_1, \ldots, \ell_m$ and (126) holds for $i, j = 1, \ldots, m$ and $r = 1, \ldots, \ell_i + 1$; $s = 1, \ldots, \ell_j + 1$.

Proof Consider the nonlinear system in controllable form (124), (125). It is a straightforward calculation to show that

$$\mathrm{ad}^{r-1}(-Ax + \alpha(\psi))B^j = \begin{cases} A^{r-1}\,B^j & 1 \leq r \leq \ell_j \\ \frac{\partial \alpha}{\partial \psi_j} & r = \ell_j + 1 \end{cases} . \tag{127}$$

Hence the controllable form (124), (125) has controllability indices $\ell_1, \ldots, \ell_m$. Moreover if (72), (73) can be transformed to (124), (125) by a change of state coordinates then clearly (72), (73) must have the same controllability indices and (126) must hold.

On the other hand suppose (72), (73) has controllability indices $\ell_1, \ldots, \ell_m$ and (126) holds. By the integrability theorem of Section 4 with $m = n$, we can choose coordinate functions $x_{ir}(\xi)$, $i = 1, \ldots, m$; $r = 1, \ldots, \ell_i$ such that

$$\langle dx_{ir},\ \mathrm{ad}^{\ell_j - s}(-f)g^j \rangle = \delta_i^j \delta_r^s \tag{128}$$

for $i, j = 1, \ldots, m$ and $r = 1, \ldots, \ell_i$, $s = 1, \ldots, \ell_j$.

In the x coordinates, $\mathrm{ad}^{\ell_j - s}(-f)g^j$ becomes the unit vector in the direction x_{js} or in other words

$$\frac{\partial x}{\partial \xi}\mathrm{ad}^{\ell_j - s}(-f)g^j = A^{\ell_j - s}B^j \tag{129}$$

for $j = 1, \ldots, m$ and $s = 1, \ldots, \ell_j$, where A, B are from the prime triple with indices $\ell_1, \ldots, \ell_m$. Let $\tilde{f}(x)$ be the transform of $f(\xi)$ into x coordinates.

$$\tilde{f}(x) = \frac{\partial x}{\partial \xi}(\xi(x))f(\xi(x)) \tag{130}$$

Then from (129), (130) we have if $s > 1$

$$\begin{aligned} \frac{\partial}{\partial x_{js}}\tilde{f}(x) &= [A^{\ell_j - s}B^j,\ \tilde{f}(x)] \\ &= [\frac{\partial x}{\partial \xi}\mathrm{ad}^{\ell_j - s}(-f)g^j,\ \frac{\partial x}{\partial \xi}f] \\ &= \frac{\partial x}{\partial \xi}\mathrm{ad}^{\ell_j - s + 1}(-f)g^j \end{aligned}$$

$$= A^{\ell_j - s+1}B^j \tag{131}$$

From this we conclude that

$$\tilde{f}(x) = Ax + \alpha(Cx)$$

where $C_i x = x_{i1},\ i = 1, \ldots, m$.

We now prove the last part of the theorem. If the nonlinear dynamics (72) can be transformed to the dynamics (18) of linear controllable form then by the above it must have controllability indices $\ell_1, \ldots, \ell_m$ and (126) must hold for $r = 1, \ldots, \ell_i$ and $s = 1, \ldots, \ell_j$. Moreover we see from (127) that $\mathrm{ad}^{\ell_j}(-f)g^j$ must transform to a constant vector field in x coordinates so (126) must hold for $r = 1, \ldots, \ell_i + 1$ and $s = 1, \ldots, \ell_j + 1$.

On the other hand if (126) holds for $r = 1, \ldots, \ell_i + 1$ and $s = 1, \ldots, \ell_j + 1$ then $\mathrm{ad}^{\ell_k}(-f)g^k$ must be a constant linear combination of the frame of vector fields $\{\mathrm{ad}^{r-1}(-f)g^i,\ i = 1, \ldots, m;\ r = 1, \ldots, \ell_i\}$. To see this, suppose for some functions $\lambda_{ir}^k(\xi)$

$$\mathrm{ad}^{\ell_k}(-f)g^k = \sum_{i=1}^{m}\sum_{r=1}^{\ell_i} \mathrm{ad}^{r-1}(-f)g^i \lambda_{ir}^k.$$

Bracketing with $\mathrm{ad}^{s-1}(-f)g^j$ yields

$$0 = \sum_{i=1}^{m}\sum_{r=1}^{\ell_i} \mathrm{ad}^{r-1}(-f)g^i L_{\mathrm{ad}^{s-1}(-f)g^j}(\lambda_{ir}^k).$$

The linear independence of the vector fields of the frame implies that for $j = 1, \ldots, m;\ s = 1, \ldots, \ell_j$

$$0 = L_{\mathrm{ad}^{s-1}(-f)g^j}(\lambda_{ir}^k)$$

hence λ_{ir}^k is a constant. By (127) this implies that

$$\alpha_{ir}(\psi) = \sum_{k=1}^{m} \lambda_{ir}^k \psi_k \quad \text{QED.}$$

Notice that it is more difficult for a nonlinear system (72), (73) to have a controllable form than to have a controller form. Clearly conditions (126) implies that $\mathcal{D}^{\ell_j - 1}$ is involutive for $j = 1, \ldots, m$. This extra difficulty is partially explained by the extra freedom afforded by $\beta(x)$ in the controller form which is lacking in the controllable form. Zeitz defines controllable form with $\beta(x)$ present [22]. There is also more freedom in the α of the controller form than the α of the controllable form. The former is an $\mathbf{R}^m$ valued function of $\mathbf{R}^n$ while the latter is a $\mathbf{R}^n$ valued function of $\mathbf{R}^m$. The linear terms of the Taylor series expansion have the same number of degrees of freedom, $n\ m$, but there are more degrees of freedom in the higher order terms of the controller form that the controllable form. In particular for the terms of order 2, there are $mn(n+1)/2$ degrees of freedom in the former and $nm(m+1)/2$ in the latter.

The *observer form* of a nonlinear system is

$$\dot{x} = Ax - \alpha(Cx) + \beta(Cx)u \tag{132}$$

$$y = \gamma(Cx) \tag{133}$$

where (A, B, C) is a prime triple with indices $\ell_1, \ldots, \ell_p$ and α, β and γ are smooth matrix valued functions of dimensions $n \times 1$, $m \times m$ and $p \times 1$. They are arbitrary except that γ must be a local diffeomorphism. We let $\bar{y} = Cx$.

Observer form is useful in the construction of asymptotic observers

$$\dot{\hat{x}} = A\hat{x} + \alpha(\bar{y}) + \beta(\bar{y})u + M(\bar{y} - C\hat{x}) \tag{134}$$

with linear error dynamics

$$\dot{\tilde{x}} = (A - MC)\tilde{x}. \tag{135}$$

The question of when a nonlinear system can be transformed to observer form has been considered by several authors [4], [10,11,12,13,14], [22]. Most treated only special cases like $p = 1$ or γ =identity. The general solution can be found in [4]. The approach taken in [4] is similar to the approach described above for the linear case.

Suppose the nonlinear system (72), (73) can be transformed into observer form (132), (133) by a local change of coordinates around ξ^0. Using the B matrix of the prime triple we add a pseudo-input μ to (132)

$$\dot{x} = Ax - \alpha(Cx) + \beta(Cx)u + B\mu. \tag{136}$$

When u is held constant at 0, (136) can be viewed as the controllable form relative to the pseudo-input μ.

We transform (136) back to ξ-coordinates

$$\dot{\xi} = f(\xi) + g(\xi)u + \bar{q}(\xi)\mu$$

which defines the vector fields $\bar{q} = \bar{q}^1, \ldots, \bar{q}^p$. These vector fields satisfy

$$\langle L_f^{r-1}(d\bar{y}_i),\ \bar{q}^j\rangle = \begin{cases} 0 & 1 \le r < \ell_i \\ \delta_i^j & r = \ell_i \end{cases} . \tag{137}$$

If $\bar{q}$ is known then we can recover the observer form by choosing local coordinates x_{ir} to satisfy

$$\langle dx_{ir},\ \mathrm{ad}^{\ell_j - s}(-f)\bar{q}^j\rangle = \delta_i^j \delta_r^s \tag{138}$$

for $i, j = 1, \ldots, p,\ r = 1, \ldots, \ell_i$ and $s = 1, \ldots, \ell_j$. Such coordinates exist iff

$$[\mathrm{ad}^{\ell_i - r}(-f)\bar{q}^i,\ \mathrm{ad}^{\ell_j - s}(-f)\bar{q}^j] = 0 \tag{139}$$

for $i, j = 1, \ldots, p;\ r = 1, \ldots, \ell_i;\ s = 1, \ldots, \ell_j$ and

$$[\mathrm{ad}^{\ell_i - r}(-f)\bar{q}^i, g^j] = 0 \tag{140}$$

for $i = 1, \ldots, p;\ j = 1, \ldots, n$ and $r = 2, \ldots, \ell_i$.

Summarizing the discussion, an observer form (132,133) of (72), (73) exists iff there exists a change of coordinates $y = \gamma(\bar{y})$ on the output space and vector fields $\bar{q}^1, \ldots, \bar{q}^p$ determined by γ via (137) such that (139),(140) holds. In effect (139),(140) constitute an overdetermined system of partial differential equations for the change of coordinates $y = \gamma(\bar{y})$ on the output space. To analyze such equations we must introduce the geometric concept of a Koszul connection on the output space. Let $\phi^i(y), i = 1, 2, \ldots$ denote vector fields on the p dimensional output space. A *Koszul connection* on y-space is a mapping Δ from pairs of such vector fields to vector fields.

$$\Delta : (\phi^i, \phi^j) \mapsto \Delta_{\phi^i}(\phi^j) \tag{141}$$

This mapping is linear over C^∞ functions in the first argument and satisfies a Liebnitz formula in the second argument. In other words if $\lambda_i(y)$ and $\mu_j(y)$ are smooth functions then

$$\Delta_{\sum_i \lambda_i \phi^i}(\sum_j \mu_j \phi^j) = \sum_{i,j}(\lambda_i \mu_j \Delta_{\phi^i}(\phi^j) + \lambda_i L_{\phi^i}(\mu_j)\phi^j). \tag{142}$$

If $\phi^1(y), \ldots, \phi^p(y)$ is a local frame of vector fields then Δ is completely determined by its *Christoffel symbols* $\Gamma_k^{ij}(y)$ relative to this frame. These are defined by the expressions

$$\Delta_{\phi^i}(\phi^j) = \sum_k \Gamma_k^{ij}\phi^k. \tag{143}$$

Consider a second frame $\bar{\phi}^1, \ldots, \bar{\phi}^p$ related to the first by

$$\bar{\phi}^i = \sum_{p=1}^{p} \phi^\rho \mu_\rho^i \tag{144}$$

and

$$\phi^r = \sum_{k-1}^{\rho} \bar{\phi}^k \lambda_k^r \tag{145}$$

where $\Lambda = (\lambda_i^j)$ is a $p \times p$ nonsingular matrix valued function of y and $\Lambda^{-1} = (\mu_i^j)$. It follows from (141,142) (143) and (144, 145) that

$$\Gamma_k^{ij} = \sum_{\rho,\sigma,\tau} \mu_\rho^i \mu_\sigma^j \lambda_k^\tau \Gamma_\tau^{\rho\sigma} + \sum_{\rho,\tau} \mu_\rho^i \lambda_k^\tau L_{\phi^\rho}(\mu_\tau^j). \tag{146}$$

A Koszul connection Δ has *zero curvature* if there exists a frame where the Christoffel symbols are zero. From equation (146) we obtain the partial differential equation that such a change of frame must satisfy. It is more conveniently written in matrix notation where Γ^ρ denotes the $p \times p$ matrix $(\Gamma_\tau^{\rho\sigma})$ with row index τ and column index σ,

$$0 = \Gamma^\rho \Lambda^{-1} + L_{\phi^\rho}(\Lambda^{-1}) \tag{147}$$

or equivalently

$$L_{\phi^\rho}(\Lambda) = \Lambda \Gamma^\rho. \tag{148}$$

The integrability condition for this is

$$L_{\phi^\rho} L_{\phi^\sigma}(\Lambda) - L_{\phi^\sigma} L_{\phi^\rho}(\Lambda) - L_{[\phi^\sigma,\phi^\rho]}(\Lambda) = 0 \tag{149}$$

or equivalently

$$\Lambda(\Gamma^\rho\Gamma^\sigma - \Gamma^\sigma\Gamma^\rho + L_{\phi^\rho}(\Gamma^\sigma) - L_{\phi^\rho}(\Gamma^\rho) - \sum_\tau c_\tau^{\sigma\rho}\Gamma^\tau) = 0 \tag{150}$$

where $C_\tau^{\sigma\rho}$ are the *structural coefficients* of the frame

$$[\phi^\sigma, \phi^\rho] = \sum_\tau C_\tau^{\sigma\rho} \phi^\tau. \tag{151}$$

The coefficient of Λ in (150) is the *curvature* of Δ.

It is convenient to work with frames of vector fields arising from coordinates on the output space. Suppose y and $\bar{y}$ are two different coordinate systems and ϕ and $\bar{\phi}$ are the associated frames, i.e.

$$L_{\phi^i} = \frac{\partial}{\partial y_i} \quad L_{\bar{\phi}^i} = \frac{\partial}{\partial \bar{y}_i}. \tag{152}$$

These frames are related by the chain rule

$$\phi = \bar{\phi}\frac{\partial \bar{y}}{\partial y} \tag{153}$$

so

$$\Lambda = \partial \bar{y}/\partial y. \tag{154}$$

A Koszul connection Δ is *flat* if there exists a coordinate frame for which the Christoffel symbols are zero. Such coordinates are said to be *flat* relative to the connection. Suppose $\Gamma_{\tau}^{\phi\sigma}$ are the Christoffel symbols relative to coordinates y. Clearly we can find new coordinates $\bar{y}$ where Christoffel symbols are zero iff we can solve the pair of partial differential equations (147,148) and (154).

We rewrite these as

$$\frac{\partial}{\partial y_i}\Lambda = \Lambda\Gamma^i \tag{155}$$

$$\frac{\partial \bar{y}}{\partial y} = \Lambda. \tag{156}$$

The integrability condition for the first is the zero curvature condition (150) which can be rewritten as

$$\Gamma^j\Gamma^i - \Gamma^i\Gamma^j + \frac{\partial \Gamma^i}{\partial y_j} - \frac{\partial \Gamma^j}{\partial y_i} = 0 \tag{157}$$

for $i, j = 1, \ldots, p$. The integrability condition for the second is

$$\Gamma_k^{ij} - \Gamma_k^{ji} = 0. \tag{158}$$

The left side of (158) is called the torsion of the connection Δ. In summary a Koszul connection is flat (i.e. has Christoffel symbols zero relative to some coordinate frame) iff it has zero curvature (157) and zero torsion (158).

Suppose $\bar{y}$ are flat coordinates for a flat connection Δ. It follows from (155,156) that another set of coordinates y is flat iff y and $\bar{y}$ are affinely related, i.e. for some constant invertible matrix Λ

$$\bar{y} = \Lambda y + \bar{y}^0.$$

The relevance of the above for the problem of transforming a system to observer form is explained by the following lemmas.

Lemma 5.1 *Suppose the nonlinear system (72,73) has one distinct observability index $\ell = \ell_1 = \ldots = \ell_p$ of multiplicity p. Define vector fields $q^1, \ldots, q^p$ by*

$$\langle L_f^{r-1}(dy_i), q^j\rangle = \begin{cases} 0 & 1 \le r \le \ell_i \\ \delta_i^j & r = \ell_j \end{cases} \tag{159}$$

Define p^3 functions $\Gamma_k^{ij}(\xi)$ by

$$\Gamma_k^{ij} = \frac{1}{\ell}\langle L_f(dy_k), [\mathrm{ad}^{\ell-1}(-f)q^i, \mathrm{ad}^{\ell-2}(-f)q^j]\rangle. \tag{160}$$

Let $\bar{y} = \bar{y}(y)$ be a change of output coordinates and $\bar{q} = \bar{q}^1, \ldots, \bar{q}^p$ be vector fields defined by (137). Define another p^3 function $\bar{\Gamma}_k^{ij}(\xi)$ by

$$\bar{\Gamma}_k^{ij} = \frac{1}{\ell}\langle L_f(d\bar{y}_k), [\mathrm{ad}^{\ell-1}(-f)\bar{g}^i, \mathrm{ad}^{\ell-2}(-f)\bar{g}^j]\rangle. \tag{161}$$

Then Γ_k^{ij} and $\bar{\Gamma}_k^{ij}$ are related by

$$\bar{\Gamma}_k^{ij} = \sum_{\rho,\sigma,\tau} \frac{\partial y_\rho}{\partial \bar{y}_i}\frac{\partial y_\sigma}{\partial \bar{y}_j}\frac{\partial \bar{y}_k}{\partial y_\tau}\Gamma_\tau^{\rho\sigma} + \sum_{\rho,\tau} \frac{\partial y_\rho}{\partial \bar{y}_i}\frac{\partial \bar{y}_k}{\partial y_\tau}\frac{\partial}{\partial y_\rho}(\frac{\partial y_\tau}{\partial \bar{y}_j}). \tag{162}$$

The proof of this lemma can be found in [15]. Notice that the lemma asserts that Γ_k^{ij} transform like the Christoffel symbols of a connection on the output space, not that they are Christoffel symbols. If $\Gamma_k^{ij}(\xi)$ are actually only functions of y then they define a connection on the output space and this connection is independent of the choice of output coordinates.

Lemma 5.2 *Suppose the nonlinear system (72,73) has one distinct observability index $\ell = \ell_1 = \ldots = \ell_p$ and can be transformed to observer form (132,133) then $\Gamma_k^{ij}(\xi)$ defined by (160) are functions only of y and define a flat connection on the output space.*

Proof We compute the symbols $\bar{\Gamma}_k^{ij}$ given by (161) where $\bar{y} = Cx$ are the transformed output coordinates of the observer form. The vector fields $\bar{q}(\xi)$ defined by (143) transform to B in x coordinates. By induction we obtain

$$\mathrm{ad}^{r-1}(-Ax + \alpha(\bar{y}))B^j = \begin{cases} A^{r-1}B^j & 1 \le r \le \ell \\ \frac{-\partial\alpha}{\partial \bar{y}^j} & r = \ell + 1 \end{cases} \tag{163}$$

From

$$\begin{aligned}&[\mathrm{ad}^{\ell-1}(-Ax + \alpha(\bar{y}))B^i, \mathrm{ad}^{\ell-2}(-Ax + \alpha(\bar{y}))B^j] = \\ &\qquad [A^{\ell-1}B^i, A^{\ell-2}B^j] = 0.\end{aligned} \tag{164}$$

It follows that

$$\bar{\Gamma}_k^{ij} = 0. \tag{165}$$

If Γ_k^{ij} are defined by (160) then (162) and (165) shows that they are functions of y alone and can be transformed to zero by a change of output coordinates. Hence they define a flat connection on the output space. **QED**.

From these lemmas we immediately obtain the following theorem [15].

Theorem 1 *Suppose the nonlinear system (72,73) has one distinct observability index $\ell = \ell_1 = \ldots = \ell_p$ around ξ^0. It can be transformed to observer form around ξ^0 iff* *the $\Gamma_k^{ij}(\xi)$ defined by (160) are functions only of y, hence define a Koszul connection on the output space*

this connection is flat

for any flat coordinates $\bar{y}$ on the output space the vector fields defined by (137) satisfy the commutative conditions (139,140).

Consider a system with one distinct observability index $\ell = \ell_1 = \ldots = \ell_p$ which is in nonlinear observable form, i.e.

$$\dot{\rho}_{ir} = \begin{cases} \xi_{ir+1} & 1 \leq r < \ell \\ f_i(\xi) & r = \ell \end{cases} \tag{166}$$

for $i = 1, \ldots, p$ and $r = 1, \ldots, \ell$.

The vector yields $q^1, \ldots, q^p$ defined by (159) are just the unit vectors in the directions $\xi_{1\ell}, \ldots, \xi_{p\ell}$. The Γ_k^{ij} defined by (160) are given by

$$\Gamma_k^{ij} = \frac{-1}{\ell} \frac{\partial}{\partial \xi_{j2}} \frac{\partial}{\partial \xi_{i\ell}} f_k(\xi). \tag{167}$$

The change of output coordinates $\bar{y} = \gamma^{-1}(y)$ must satisfy the partial differential equations (155,156) or

$$\frac{\partial}{\partial y_i}\left(\frac{\partial \bar{y}_\rho}{\partial y_j}\right) = \frac{-1}{\ell} \sum_k \frac{\partial \bar{y}_\rho}{\partial y_k} \frac{\partial^2 f_k}{\partial \xi_{j2} \partial \xi_{i\ell}}. \tag{168}$$

The integrability conditions for this are the zero curvature condition (157) or

$$\sum_\rho \frac{\partial^2 f_\sigma}{\partial \xi_{\rho 2} \partial \xi_{i\ell}} \frac{\partial^2 f_\rho}{\partial \xi_{r2} \partial \xi_{j\ell}} - \frac{\partial^2 f_\sigma}{\partial \xi_{\rho 2} \partial \xi_{j\ell}} \frac{\partial^2 f_\rho}{\partial \xi_{r2} \partial \xi_{i\ell}} =$$

$$\ell\left(\frac{\partial^3 f_\sigma}{\partial \xi_{j1} \partial \xi_{r2} \partial \xi_{i\ell}} - \frac{\partial^3 f_\sigma}{\partial \xi_{i1} \partial \xi_{r2} \partial \xi_{j\ell}}\right) \tag{169}$$

and the zero torsion condition (158) or

$$\frac{\partial^2 f_k}{\partial \xi_{j2} \partial \xi_{i\ell}} = \frac{\partial^2 f_k}{\partial \xi_{i2} \partial \xi_{j\ell}}. \tag{170}$$

If these are satisfied then we can solve (168). If (139,140) are satisfied then we can solve (138) to find the x coordinates of observer form.

Needless to say this is a very tedious process. There is a necessary condition that a system in nonlinear observable form (166) must satisfy to be transformable to observer form. We define the *degree* of the variable ξ_{ir} to $r-1$ and the degree of a product of such variables to be the sum of the degrees of its factors. If (166) can be transformed into observer form then $f_i(\xi)$ must be a polynomial of degree at most ℓ. We refer the reader to [4] for a proof of this. In particular if $\ell = 2$ then this degree condition, the zero curvature condition (169) and (139, 140) are necessary and sufficient. The torsion free condition (170) is trivially satisfied. It follows from (169) that (139) need only be checked for $r = s = 1$ and $i \neq j$.

If $p = 1$ then trivially the curvature and torsion are zero and (168) reduces to a first order linear ordinary differential equation for the quantity $d\bar{y}/dy$. It is solvable if the degree condition on f_1 is satisfied. In particular when $p = 1$ and $\ell = 2$ the degree condition and (140) are necessary and sufficient for the existence of observer form.

We now discuss the case where there are several distinct observability indices. The general approach is as before. To find the observer form of (72,73) if it exists we seek an appropriate change of output coordinates $\bar{y} = \gamma^{-1}(y)$ which allows us to define vector fields $\bar{q}$ via (137). If (139,140) are satisfied then we can solve (138) for the x coordinates of the observer form.

The presence of several distinct observability indices complicates the search for $\bar{y}$ and forces us to proceed in stages. Notice that for a system in observer form the observability indices are strict for the output $\bar{y} = Cx$. This is because

$$L^{r-1}_{(Ax-\alpha(y))}(d\bar{y}) = CA^{r-1} \bmod \mathcal{E}^{r-1}$$

and the output indices are strict for the pair C, A. So any nonlinear system that admits an observer form must admit a change of output coordinates which make the output indices strict. Moreover, the problem of transforming a nonlinear system with strict observability indices into observer form is greatly simplified by the following fact.

A change of output coordinates $\bar{y} = \gamma^{-1}(y)$ preserves the order and strictness of the observability indices iff

$$\frac{\partial \bar{y}_i}{\partial y_j} = 0 \text{ for } \ell_i < \ell_j. \tag{171}$$

To find a change of output coordinates which make the observability indices of (72,73) strict we start by defining vector fields $q^1, \ldots, q^p$ via (159).

It follows by the standard induction argument using the Liebnitz formula (85) that (143) implies

$$\langle dy_i, \mathrm{ad}^{r-1}(-f)q^j \rangle = \begin{cases} 0 & 1 \le r < \ell_i \\ \beta_i^j & r = \ell_j \\ 0 & \ell_i < r < \ell_j \end{cases} \tag{172}$$

Moreover the vector fields $\mathrm{ad}^{r-1}(-f)q^j$ $j=1,\ldots,p;$ $r=1,\ldots,\ell_j$ form a frame of n independent vector fields. These characterize $\mathcal{E}^\ell$ as

$$\mathcal{E}^\ell = \{\mathrm{ad}^{r-1}(-f)q^j : \ j=1,\ldots,p; \ r=1,\ldots,\ell_j-\ell\} \perp$$
$$\mathcal{E}^\ell = \{\text{oneforms } \omega : \ \langle \omega, \mathrm{ad}^{r-1}(-f)q^j\rangle = 0 \ j=1,\ldots,p; \ r=1,\ldots,\ell_j-\ell_i\}. \tag{173}$$

Suppose $\bar{y} = \gamma^{-1}(y)$ is a change of output coordinates which preserves the ordering of the observability indices. The observability indices are strict relative to the $\bar{y}$ output iff

$$L_f^{\ell_i}(d\bar{y}_i) \in \mathcal{E}^{\ell_i} \tag{174}$$

or equivalently by (173)

$$\langle L_f^{\ell_i}(d\bar{y}_i)\mathrm{ad}^{r-1}(-f)q^j\rangle = 0 \tag{175}$$

for $r=1,\ldots,\ell_j-\ell_i$. By induction and the Liebnitz formula this is equivalent to

$$\langle d\bar{y}_i, \mathrm{ad}^{r-1}(-f)q^j\rangle = 0 \tag{176}$$

for $r=\ell_{i+1},\ldots,\ell_j$. Since $d\bar{y} = \partial\bar{y}/\partial y \ dy$, (172) implies that (176) must hold for $r=1,\ldots,\ell_j-1$ also. We have shown that the observability indices are strict relative to the output $\bar{y}$ iff (176) holds for $r=1,\ldots,\ell_j-1$ when $\ell_j \le \ell_i$ and for $r=1,\ldots,\ell_j$ when $\ell_j > \ell_i$.

We define p distributions

$$\mathcal{Y}^i = C^\infty\{\mathrm{ad}^{r-1}(-f)q^j : \ r=1,\ldots,\ell_j-1 \text{ if } \ell_j \le \ell_i \text{ and}$$
$$r=1,\ldots,\ell_j \text{ if } \ell_j > \ell_i. \tag{177}$$

As we have just seen a change of coordinates $\bar{y} = \gamma^{-1}(y)$ preserves the ordering of the observability indices and makes them strict iff

$$d\bar{y}_i \perp \mathcal{Y}^i \ i=1,\ldots,p. \tag{178}$$

This is an underdetermined system of first order PDE's for $\bar{y}$. By employing the Frobenius Theorem, we obtain the following reformulation of Theorem 4.2 in [4].

Proposition 5.2 *Suppose the nonlinear system (72,73) has observability indices $\ell_1,\ldots,\ell_p$ around ξ^0. There exists a local change of output coordinates $\bar{y} = \gamma^{-1}(y)$ which preserves the ordering of the observability indices and makes them strict iff the distributions $\mathcal{Y}^1,\ldots,\mathcal{Y}^p$ are involutive.*

Lemma 5.3 *Suppose the nonlinear system has strict observability indices $\ell_1,\ldots,\ell_p$ and $\ell = \min\{\ell_1,\ldots,\ell_p\}$. Define vector fields q by (159) and symbols Γ_k^{ij} by (160) then*

$$\Gamma_k^{ij} = 0 \text{ if } \ell_i > \ell \text{ or } \ell_k > \ell.$$

Proof Equation (160) can be rewritten as

$$\begin{aligned}\ell\Gamma_k^{ij} &= L_{\mathrm{ad}^{\ell-1}(-f)g^i} L_{\mathrm{ad}^{\ell-2}(-f)g^j} L_f(y_k) \\ &\quad - L_{\mathrm{ad}^{\ell-2}(-f)q^j} L_{\mathrm{ad}^{\ell-1}(-f)q^i} L_f(y_k).\end{aligned} \tag{179}$$

By (159)

$$L_{\mathrm{ad}^{\ell-2}(-f)q^j} L_f(y_k) = L_{q^j} L_f^{\ell-1}(y_k) = \begin{cases} 0 & \ell_k > \ell \\ \delta_k^j & \ell_k = \ell \end{cases}$$

so the first term on the right of (179) is always zero. If $\ell_k > \ell$ then

$$L_{\mathrm{ad}^{\ell-1}(-f)g^j} L_f(Y_k) = L_{q^j} L_f^{\ell}(y_k) = \begin{cases} 0 & \ell_k > \ell+1 \\ \delta_k^j & \ell_k = \ell+1 \end{cases}$$

so $\Gamma_k^{ij} = 0$.

Suppose $\ell_k = \ell$ and $\ell_i > \ell$. Then $q^i \in \mathcal{E}^{\ell\perp}$ so by the strictness assumption (174,175) it follows

$$L_{\mathrm{ad}^{\ell-1}(-f)q^i} L_f(y_k) = L_{q^i} L_f^{\ell}(y_k) = 0$$

so $\Gamma_k^{ij} = 0$.

Lemma 5.4 *Suppose the nonlinear system has strict observability indices $\ell_1, \ldots, \ell_p$ and $\ell = \min\{\ell_1, \ldots, \ell_p\}$. Define vector fields q by (159) and symbols Γ_k^{ij} by (160). Let $\bar{y} = \bar{y}(y)$ be a change of coordinates among those outputs of lowest observability index, i.e.*

$$\frac{\partial \bar{y}_i}{\partial y_j} = \delta_i^j \quad \text{if } \ell_i \text{ or } \ell_j > \ell. \tag{180}$$

Define $\bar{q}$ by (137) and $\bar{\Gamma}_k^{ij}$ by (161) then Γ_k^{ij} and $\bar{\Gamma}_k^{ij}$ are related as Christoffel symbols (162).

The proof of this is similar to that of Lemma 5.2, see [15].

Lemma 5.5 *Suppose the nonlinear system (72,73) has strict observability indices $\ell_1, \ldots, \ell_p$ and $\ell = \min\{\ell_1, \ldots, \ell_p\}$. If (72,73) admits an observer form (132,133) then the $\Gamma_k^{ij}(\xi)$ defined by (160) are functions only of y and define a flat connection on the output space.*

Proof By Lemma 5.6 we know that $\Gamma_k^{ij} = 0$ if $\ell_i > \ell$ or $\ell_k > \ell$. So all we need to show is the existence of an observer form for (72,73) implies the existence of a change of coordinates among those outputs of lowest observability index (180) which to transform the Γ_k^{ij} to zero for $\ell_i = \ell_j = \ell_k = \ell$. But it is clear from the proof of Lemma 5.3 that if we were to compute the Γ_k^{ij} defined by (160) for a system in observer form then they are zero.

By Lemma 5.7 the Γ_k^{ij} for $\ell_i = \ell_j = \ell_k = \ell$ transform like Christoffel symbols under a change of coordinates among those outputs of lowest observability index. By (180) the change of output coordinates to observer form $\bar{y} = \gamma^{-1}(y)$ transform the outputs of lowest observability index among themselves and can be used to take Γ_k^{ij} to zero for $\ell_i = \ell_j = \ell_k = \ell$.

If $\Gamma_k^{ij}(\xi)$ defined by (160) are the Christoffel symbols of a flat connection on the output space then we can solve the partial differential equations (155,156) to find flat coordinates $\bar{y}$. These coordinates are not necessarily the $\bar{y}$ of the observer form if it exists. But at least those of lowest observability index are because of (171). We change notation and denote the flat coordinates by y.

The next stage is to find the next smallest distinct observability index $\ell' = \min\{\ell_i > \ell\}$. We define new symbols

$$\Gamma_k^{ij} = \frac{1}{\ell'}\langle L(dy_k), [\mathrm{ad}^{\ell'_i - 1}(-f)q^i, \mathrm{ad}^{\ell'_j - 1}(-f)q^j]\rangle \tag{181}$$

where $\ell'_i = \ell' \wedge \ell'$.

It is not hard to see by an argument similar to Lemma 5.7 that $\Gamma_k^{ij} = 0$ if ℓ_i or $\ell_j > \ell'$. Moreover if $\ell_i = \ell_j = \ell_k = \ell$ then the Γ_k^{ij} of (181) are just ℓ/ℓ' times the Γ_k^{ij} of (160). The latter are zero by our choice of flat output coordinates.

For reasons explained below, if the system admits an observer form then Γ_k^{ij} defined by (181) define a flat connection on the output space. If this is so then we solve (155,156) for new flat output coordinates $\bar{y}$. Because of the above remarks the change of coordinates will satisfy

$$\frac{\partial \bar{y}_i}{\partial y_j} = \delta_i^j \text{ if } \ell_i = \ell' \text{ or } \ell_j > \ell'.$$

We continue on in this fashion until we have exhausted the list of observability indices or found symbols which do not define a flat connection. If the latter does not happen then the last flat coordinates $\bar{y}$ are the desired output coordinates of the observer form. The observer form will exist if (139) is satisfied for $r = 1, \ldots, \ell_i$; $s = 1, \ldots, \ell_j$ and (140) holds for $r = 2, \ldots, \ell_i$; $j = 1, \ldots, m$.

To see why this approach is valid consider a system (72), (73) which can be transformed into observer form. Using Lemmas 5.7 and 5.8 we can assume that $\bar{y}_i = y_i$ for those outputs of lowest observability index $\ell_i = \ell$. Assuming that

$u = 0$ we have

$$
\begin{array}{ll}
y_i = \xi_{i1} & y_i = x_{i1} \\
\dot{\xi}_{i1} = \xi_{i2} & \dot{x}_{i1} = x_{i2} - \alpha_{i\ell} \\
\vdots & \vdots \\
\dot{\xi}_{i\ell} = f_i(\xi) & \dot{x}_{i\ell} = -\alpha_{i\ell}
\end{array}
\tag{182}
$$

By comparing these we arrive at

$$
\xi_{ir} = x_{ir} - \sum_{s=1}^{r-1} (\frac{d}{dt})^{r-s-1} \alpha_{is} \tag{183}
$$

and

$$
f_i(\xi) = -\sum_{s=1}^{\ell} (\frac{d}{dt})^{\ell-s} \alpha_s. \tag{184}
$$

We add dummy state variables ξ_{ir}, x_{ir} for $r = \ell + 1, \ldots, \ell'$ to (182) as follows

$$
\begin{array}{ll}
y_i = \xi_{i1} & y_i = x_{i1} \\
\dot{\xi}_{i1} = \xi_{i2} & x_{i1} = x_{i2} - \alpha_{i\ell} \\
\vdots & \vdots \\
\dot{\xi}_{i\ell} = \xi_{i\ \ell+1} + f_i(\xi) & \dot{x}_{i\ell} = x_{i\ \ell+1} - \alpha_{i\ell} \\
\dot{\xi}_{i\ \ell+1} = \xi_{i\ \ell+2} & \dot{x}_{i\ \ell+1} = x_{i\ \ell+2} \\
\vdots & \vdots \\
\dot{\xi}_{i\ell'} = 0 & \dot{x}_{i\ell'} = 0
\end{array}
\tag{185}
$$

It is not hard to see using (184) that these are transforms of each other under (183) and

$$
\xi_{ir} = x_{ir} \quad \ell < r \leq \ell'. \tag{186}
$$

Hence if the original system (182) can be transformed to observer form and $y_i = \bar{y}_i$ then so can the modified system (185). Moreover for the modified system the smallest observability index is now ℓ' rather that ℓ so we can apply Lemmas 5.7 and 5.8. It is a straightforward calculation to show that the symbols of the modified system defined by (160) with ℓ replaced by ℓ' are the same as those given by (181).

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