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ACAUSAL REALIZATION THEORY, PART I; LINEAR DETERMINISTIC SYSTEMS*

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Abstract. We study acausal linear systems, their controllability and observability properties and the weighting patterns that they realize. A complete classification is given of all minimal real analytic realizations of a given weighting pattern and of all minimal autonomous realizations of a stationary weighting pattern.

Key words. acausal linear systems, controllable and observable on and off, minimal realization

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1. Introduction. Acausal linear systems theory is concerned with mathematical entities of the form

$$\dot{x} = Ax + Bu, \tag{1.1a}$$

$$V^0 x(t_0) + V^1 x(t_1) = v, \tag{1.1b}$$

$$y = Cx + Du, \tag{1.1c}$$

$$w = W^0 x(t_0) + W^1 x(t_1) \tag{1.1d}$$

where $x(t)$, v, $w \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$. The matrices A, B, C, D, V^0, V^1, W^0 and W^1 are dimensioned accordingly and A, B, C, D may be bounded measurable functions of t. We refer to $x(t)$ as the *state*, $u(t)$ as the *input* and $y(t)$ the *output* at time t, although t may actually represent a spatial parameter. The vector v is called the *boundary input* and the vector w the *boundary output*. We refer to (1.1) as the acausal system Σ.

We always assume that (1.1a), (1.1b) is a *well-posed problem*, i.e., for every boundary input v and square integrable $u(t)$ there exists a unique solution $x(t)$. In this case (1.1) defines a linear mapping also denoted by Σ.

$$\Sigma : \mathbb{R}^{n\times 1} \times L_2^{m\times 1}[t_0, t_1] \to \mathbb{R}^{n\times 1} \times L_2^{p\times 1}[t_0, t_1], \tag{1.2a}$$

$$\Sigma : (v, u(t)) \mapsto (w, y(t)). \tag{1.2b}$$

We say that such a mapping is the *input output* map of the acausal system (1.1) or equivalently that the acausal system (1.1) is a *realization* of the input output mapping.

In § 2 we discuss situations where such models naturally arise. Of course (1.1) is a generalization of the usual linear system where $V^0 = W^1 = I$ and $V^1 = W^0 = 0$. Such a system is *causal* because future inputs do not affect past states or outputs. Systems of the form (1.1) do not necessarily have this property, hence the term *acausal*. There are many possible generalizations of (1.1) which are of interest. We shall mention some of these in § 2, but we shall not discuss them in any great depth.

Section 3 is essentially a review of [1] where systems of the form (1.1) were first introduced under the name of *boundary value linear systems*.

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In § 4, we exhibit two other processes closely related to $x(t)$ but which are causal in some generalized sense. These processes supply the foundations for the definitions of controllability and observability given in § 5. Also in this section we begin to relate controllability and observability to minimality.

The key results of this paper are found in §§ 6 and 7. The first is a complete classification of all the minimal real analytic realizations of the mapping

$$(v=0,\ u(\cdot))\mapsto y(\cdot)$$

induced by (1.1). The second is complete classification of all the minimal autonomous realizations of such maps which are stationary.

Recently Gohberg and Kaashoek [9], [10] have made an excellent study of such systems. Their work is based on completely different concepts of controllability and observability. They do not discuss the question of minimality but treat a different question of irreducibility. At the end of § 5 we give an example that illustrates the differences between their work and ours.

2. Examples and extensions. Acausal systems naturally arise when the independent variable t is spatial rather than temporal. For example, consider a static, approximately horizontal beam, clamped at both ends, which supports a continuously distributed load. We can view this from a system theoretic point of view where the input $u(t)$ is the load density and the output $y(t)$ is the deflection of the beam. ($y(t)>0$ indicates downward deflection.) The variable t measures length along the beam. The relationship between input and output is given by

$$E(t)I(t)\frac{dy^4}{dt^4}=u(t) \tag{2.1}$$

where $E(t)$ is the modulus of elasticity and $I(t)$ is the moment of inertia of the cross section. Clamping at both ends imposes boundary conditions on $y(t_0)$, $y(t_1)$, $\dot{y}(t_0)$ and $\dot{y}(t_1)$.

This can be put in state space form (1.1) by letting $x=(x_1, x_2, x_3, x_4)=(y, \dot{y}, \ddot{y}, \dddot{y})$; then

$$A=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix},\qquad B=\begin{bmatrix}0\\0\\0\\1/EI\end{bmatrix},$$

$$C=[1\ 0\ 0\ 0],\qquad D=[0],$$

$$V^0=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\qquad V^1=\begin{bmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&1&0&0\end{bmatrix}.$$

A similar example was considered in [2] but with the bending moment as input, which resulted in a second order system.

The acausal nature of this system is apparent, the load at point s affects the deflection at every point t along the beam. One goal of the loading might be to force the beam to assume a desired shape. Such a problem can be cast as a linear quadratic optimal control problem [2].

The boundary condition (1.1) can be used to force $x(t)$ to be *cyclic*, $x(t_0)=x(t_1)$ by letting $V^0=-V^1$ and $v=0$. Similarly if $V^0=V^1$ and $v=0$ we obtain an *anticyclic* state process $x(t_0)=-x(t_1)$. Such phenomena cannot be modeled by causal systems.

If we drive (1.1) by white Gaussian noise process $u(t)$ and an independent Gaussian boundary value v then we obtain two Gaussian processes $x(t)$ and $y(t)$. If a causal system is so driven, then the state process $x(t)$ is Markov but for most acausal systems the state process is not Markov.

These stochastic systems are a convenient way of representing stochastic processes. We study the associated realization theory in the sequel [4] to this paper. Using this class of models one can formulate and solve various estimation problems for spatially distributed processes [3], [5], [6].

Even processes where the independent variable t is temporal can have an acausal character. These are systems which are anticipatory. There is an intelligent controller who modifies the evolution of the system in order to achieve a desired goal at some future time. This may be on a fixed time interval or over a moving time interval. Most messages are of this type. Before composing the message, the author usually has a fairly clear idea of what the contents should include, and particularly how it should begin and end. Another example is the tracking of an object whose ultimate destination is already known.

Causality is a property of the mapping from inputs $u(\cdot)$ to outputs $y(\cdot)$. Suppose one is studying a system where the inputs and outputs are not known a priori as is frequently the case in network theory. If it is impossible to decide a priori whether the process one wishes to model is causal or not, why restrict a priori to causal models? Besides (1.1), Luenberger's descriptor systems can be used to model acausality [12].

There are numerous extensions of the acausal linear system (1.1) which we will not go into in any depth. A straightforward one is to discretize t or we could let $t_0=-\infty$ and/or $t_1=\infty$. A more substantial generalization is to allow t to be a multidimensional variable. Such systems arise in distributed parameter control, image processing, and seismic data processing. It is somewhat surprising considering all the effort that has gone into these areas that one-dimensional acausal systems have not received more study.

Throughout this paper we consider only well-posed systems. This rules out many interesting problems. For example in the stochastic setting, we rule out a pinned Weiner process (Brownian Bridge). Generally we restrict our attention to two point boundary value processes where the solution $x(t)$ of (1.1a) is partially constrained at only two times t_0, t_1 as in (1.1b). But multipoint constrained problems will arise even in this paper. They have wide applicability in many other contexts.

3. Basic facts. Let $\Phi(t,s)$ be $n\times n$ matrix valued function satisfying

$$\frac{\partial}{\partial t}\Phi(t,s)=A(t)\Phi(t,s), \tag{3.1a}$$

$$\Phi(t,t)=I. \tag{3.1b}$$

Since $A(t)$ is assumed to be bounded and measurable, the existence, uniqueness and absolute continuity of $\Phi(t,s)$ follows from standard theorems on ODE's.

The boundary value problem (1.1a), (1.1c) is *well posed* iff the matrix

$$F=V^0+V^1\Phi(t_1,t_0) \tag{3.2}$$

is invertible. If this is satisfied then the solution to (1.1a), (1.1b) is given by

$$x(t)=\Phi(t,t_0)F^{-1}v+\int_{t_0}^{t_1}G(t,s)B(s)u(s)\,ds. \tag{3.3}$$

Green's matrix $G(t, s)$ is given by splicing together two matrix valued functions

(3.4a)
$$G(t, s) = \begin{cases} G^0(t, s) & \text{if } t > s, \\ G^1(t, s) & \text{if } s > t \end{cases}$$

along the line $t = s$, where

(3.4b)
$$G^0(t, s) = \Phi(t, t_0)F^{-1}V^0\Phi(t_0, s),$$

(3.4c)
$$G^1(t, s) = -\Phi(t, t_0)F^{-1}V^1\Phi(t_1, s) = \Phi(t, t_0)F^{-1}(V^0 - I)\Phi(t_0, s).$$

The output $y(t)$ is given by

(3.5)
$$y(t) = C(t)\Phi(t, t_0)F^{-1}v + \int_{t_0}^{t_1} W(t, s)u(s)\, ds$$

where the *weighting pattern* $W(t, s)$ is given by

(3.6)
$$W(t, s) = C(t)G(t, s)B(s) + D(t)\delta(t - s).$$

The system (1.1) defines a linear mapping Σ

$$\Sigma : \mathbb{R}^{n\times 1} \times L_2^{m\times 1}[t_0, t_1] \to \mathbb{R}^{n\times 1} \times L_2^{p\times 1}[t_0, t_1],$$

$$\Sigma : \begin{pmatrix} v \\ u(\cdot) \end{pmatrix} \mapsto \begin{pmatrix} w \\ y(\cdot) \end{pmatrix} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \begin{pmatrix} v \\ u(\cdot) \end{pmatrix}.$$

The maps Σ_{11}, Σ_{12} and Σ_{21} are of finite rank since they map from and/or to finite-dimensional vector spaces. The most important part of Σ is the infinite rank part Σ_{22} which maps between the function spaces. The usefulness of the model (1.1) depends ultimately on the fact that it describes an infinite rank mapping in a very concise and tractable fashion.

Suppose we have an integral operator Σ_{22}

$$\Sigma_{22} : L_2^{m\times 1}[t_0, t_1] \to L_2^{p\times 1}[t_0, t_1],$$

$$\Sigma_{22} : u(\cdot) \mapsto y(\cdot)$$

where

$$y(t) = \int_{t_0}^{t_1} W(t, s)u(s)\, ds.$$

The system (1.1) is said to be a *realization* of Σ_{22} (or equivalently the kernel $W(t, s)$) if $W(t, s)$ is the *weighting pattern* of (1.1) as given by (3.6). Realization theory (in the deterministic sense) is concerned with the existence and classification of the realizations of $W(t, s)$ and related questions. As an example of such a question consider the adjoint map Σ^* of Σ

(3.7a)
$$\Sigma^* : \mathbb{R}^{1\times n} \times L_2^{1\times p}[t_0, t_1] \to \mathbb{R}^{1\times n} \times L_2^{1\times m}[t_0, t_1],$$

(3.7b)
$$\Sigma^* : (\zeta, \mu(t)) \mapsto (\xi, \nu(t))$$

defined by the equation

(3.8)
$$\xi v + \int_{t_0}^{t_1} \nu(t)u(t)\, dt = \zeta w + \int_0^t \mu(t)y(t)\, dt$$

for all $(v, u(t))$ and $(\zeta, \mu(t))$. An obvious question is whether Σ^* can be realized by an acausal linear system. As was shown in [1] the following system (given in adjoint form) does the job. The functions $\lambda(t)$, $\mu(t)$ and $\nu(t)$ are the state, input and output, respectively, ζ denotes the boundary input and ξ the boundary output.

$$\dot{\lambda} = -\lambda A - \mu C, \tag{3.9a}$$

$$\lambda(t_0)M^0 + \lambda(t_1)M^1 = \zeta, \tag{3.9b}$$

$$\nu = \lambda B + \mu D, \tag{3.9c}$$

$$\xi = \lambda(t_0)N^0 + \lambda(t_1)N^1. \tag{3.9d}$$

The boundary matrices are fixed by

$$\begin{bmatrix} V^0 & V^1 \\ W^0 & W^1 \end{bmatrix} \begin{bmatrix} -M^0 & -N^0 \\ M^1 & N^T \end{bmatrix} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}. \tag{3.10}$$

It is convenient to make a change of coordinates in the space of boundary input values v so that

$$F = V^0 + V^1\Phi(t_1, t_0) = I. \tag{3.11}$$

If V^0 and V^1 satisfy this, then the boundary conditions are in *standard form* and Σ is a *standard realization* of $W(t, s)$. Causal linear systems where $V^0 = I$ and $V^1 = 0$ have boundary conditions in standard form.

Frequently W^0 and W^1 are not explicitly given, but the dual systems can still be determined if one assumes that the dual boundary conditions (3.9c) satisfy the dual of (3.11), i.e.,

$$\Phi(t_1, t_0)M^0 + M^1 = I. \tag{3.12}$$

By equating the upper left blocks of (3.10) and (3.12) we see that

$$M^0 = V^1, \tag{3.13a}$$

$$M^1 = \Phi(t_1, t_0)V^0\Phi(t_0, t_1). \tag{3.13b}$$

These conditions (3.13) are called the *standard dual boundary conditions.*

Notice in the causal case when $V^0 = I$ and $V^1 = 0$, the standard dual boundary conditions are $M^0 = 0$ and $M^1 = I$. Systems with such boundary conditions are said to be *anticausal*, because under time reversal they become causal.

Having computed the standard dual boundary conditions we can in the same way compute the *standard* boundary output equation (1.1d). We assume that

$$W^0\Phi(t_0, t_1) + W^1 = I; \tag{3.14}$$

then this and the lower left block of (3.10) imply that

$$W^0 = \Phi(t_1, t_0)(V^0 - I), \tag{3.15a}$$

$$W^1 = I + \Phi(t_1, t_0)V^1. \tag{3.15b}$$

This normalization has been chosen so that for causal systems $W^0 = 0$ and $W^1 = I$, and the boundary output is $w = x(t_1)$.

It is a simple exercise to verify that for the standard boundary equations (1.1b), (1.1d) the matrix

$$\begin{bmatrix} V^0 & V^1 \\ W^0 & W^1 \end{bmatrix} \tag{3.16}$$

is nonsingular. This means that the boundary inputs and outputs are linearly independent variables. A *standard acausal system* (1.1) is one where the coefficients of the boundary equations (1.1b), (1.1d) satisfy the normalizations (3.11) and (3.14).

As we have seen, (1.1) defines a linear mapping

$$\Sigma_{22}: u(t) \mapsto y(t) = \int_{t_0}^{t_1} W(t, s)u(s)\, ds \tag{3.17}$$

where

$$W(t, s) = C(t)G(t, s)B(s) + D(t)\delta(t-s). \tag{3.18}$$

A kernel $W(t, s)$ is called *proper* if it is the sum of a bounded measurable term and a Dirac delta term as is (3.18). It is called *strictly proper* if the second term is missing, e.g., $D(t) = 0$. The next theorem is trivial but occasionally useful.

THEOREM 3.1. *A strictly proper $p \times m$ kernel $W(t, s)$ can be realized by an acausal linear system* (1.1) *iff there exists bounded measurable $p \times n$ and $n \times m$ matrices $C(t)$ and $B(s)$ and $n \times n$ matrices V^0 and V^1 such that*

$$V^0 + V^1 = I, \tag{3.19}$$

$$W(t, s) = \begin{cases} C(t)V^0B(s), & t > s, \\ -C(t)V^1B(s), & t < s. \end{cases} \tag{3.20}$$

Proof. If $C(t)$, $B(s)$, V^0 and V^1 exist then $W(t, s)$ is realized by (1.1) with $A = 0$. On the other hand, if $W(t, s)$ is realized by any acausal system (1.1) then the time dependent change of state coordinates given by $\tilde{x} = \Phi(t_0, t)x$ transforms (1.1) into an acausal system $\tilde{\Sigma}$ where $\tilde{A} = 0$, $\tilde{C}(t) = C(t)\Phi(t, t_0)$ and $\tilde{B}(s) = \Phi(t_0, s)B(s)$. The transformed boundary conditions $\tilde{V}^0 = V^0$ and $\tilde{V}^1 = V^1\Phi(t_1, t_0)$ are in standard form. It is straightforward to verify that $\tilde{\Sigma}$ also realizes $W(t, s)$. Q.E.D.

4. The inward and outward boundary value processes. In general a system of the form (1.1) defines an acausal mapping from $u(\cdot)$ to $y(\cdot)$. The output $y(t)$ at time t depends on the input $u(s)$ at all times $s \in [t_0, t_1]$, not just those $s \in [t_0, t]$ as for causal systems. A natural question to ask is whether there is any causal way of looking at (1.1).

It turns out that a related question is how to view the boundary input condition (1.1c) and the boundary output equation (1.1d). For causal systems the boundary input condition is just $x(t_0) = v$ and the boundary output equation is $w = x(t_1)$. The initial value v can be thought of as the medium which transmits the effects of past controls $u(s)$, $s \in (-\infty, t_0)$ to the state $x(t)$ and output $y(t)$ for $t \in [t_0, t_1]$. The state $x(t)$ and output $y(t)$ can also be determined if w and $u(s)$, $s \in [t, t_1]$ are known. But this is not really an anticausal representation because w depends on v and $u(s)$ for all $s \in [t_0, t_1]$.

We now present a similar interpretation of v and w in the acausal case. We need to introduce moving boundary conditions. Suppose $t_0 \leqq \tau_0 \leqq \tau_1 \leqq t_1$, then define four matrices

$$K^0 = \Phi(\tau_0, t_0)V^0\Phi(t_0, \tau_0), \tag{4.1a}$$

$$K^1 = \Phi(\tau_0, t_0)V^1\Phi(t_1, \tau_1), \tag{4.1b}$$

$$L^0 = \Phi(\tau_1, t_1)W^0\Phi(t_0, \tau_0), \tag{4.1c}$$

$$L^1 = \Phi(\tau_1, t_1)W^1\Phi(t_1, \tau_1). \tag{4.1d}$$

These matrices are functions of τ_0 and τ_1 but for notational simplicity we suppress the arguments. We shall assume that V^0, V^1, W^0 and W^1 are in standard form on the interval $[t_0, t_1]$, then it is easy to see that K^0, K^1, L^0 and L^1 are in standard form on the interval $[\tau_0, \tau_1]$.

Let $x(t)$ be the solution of (1.1) for some $u(t)$ and v. We define the inward boundary process $k(\tau_0, \tau_1)$ by

$$k(\tau_0, \tau_1) = K^0 x(\tau_0) + K^1 x(\tau_1). \tag{4.2}$$

There are two important points to be made about this process. A simple calculation shows that

$$k(\tau_0, \tau_1) = \Phi(\tau_0, t_0)v + \left(\int_{t_0}^{\tau_0} + \int_{\tau_1}^{t_1}\right) G(\tau_0, s)B(s)u(s)\, ds. \tag{4.3}$$

We can interpret this as causality in some generalized sense for the mapping $u(\cdot) \mapsto k(\cdot, \cdot)$. Think of the pair $\{\tau_0, \tau_1\}$ as being the present; the past is $[t_0, t_1] \backslash [\tau_0, \tau_1]$ and the future is (τ_0, τ_1). Then (4.3) says that the present value of $k(\tau_0, \tau_1)$ does not depend on future values of the input $u(\tau)$ for $\tau \in (\tau_0, \tau_1)$.

The second point is that given the present value $k(\tau_0, \tau_1)$ and future values $u(\tau)$, $\tau \in (\tau_0, \tau_1)$ we can compute future values $x(\tau)$ and $y(\tau)$ for $\tau \in (\tau_0, \tau_1)$. This is because the boundary value problem

$$\dot{x} = Ax + Bu, \qquad t \in [\tau_0, \tau_1], \tag{4.4a}$$

$$K^0 x(\tau_0) + K^1 x(\tau_1) = k, \tag{4.4b}$$

is well posed, and its solution for given u and v agrees with that of (1.1) on $[\tau_0, \tau_1]$ provided $k = k(\tau_0, \tau_1)$ given by (4.3).

Note that $k(t_0, t_1) = v$ and so v can be thought of as the medium that transmits the effects of $u(s)$ for $s \notin [t_0, t_1]$ to $x(t)$ and $y(t)$ for $t \in [t_0, t_1]$.

If we consider the process $l(\tau_0, \tau_1)$ defined by

$$l(\tau_0, \tau_1) = L^0 x(\tau_0) + L^1 x(\tau_1), \tag{4.5}$$

then

$$l(\tau_0, \tau_1) = \Phi(\tau_1, \tau_0)k(\tau_0, \tau_1) + \int_{\tau_0}^{\tau_1} \Phi(\tau_1, s)B(s)u(s)\, ds. \tag{4.6}$$

From knowledge of $l(\tau_0, \tau_1)$ and $u(t)$ for $t \in [t_0, t_1] \backslash (\tau_0, \tau_1)$ we can reconstruct $x(t)$ for $t \in [t_0, t_1] \backslash (\tau_0, \tau_1)$ as the solution of the well-posed four point boundary value problem

$$\dot{x} = Ax + Bu, \qquad t \in [t_0, t_1] \backslash (\tau_0, \tau_1), \tag{4.7a}$$

$$V^0 x(t_0) + V^1 x(t_1) = v, \tag{4.7b}$$

$$L^0 x(\tau_0) + L^1 x(\tau_1) = l(\tau_0, \tau_1). \tag{4.7c}$$

However the map $u(\cdot) \mapsto l(\cdot, \cdot)$ is not causal in any generalized sense because $l(\tau_0, \tau_1)$ depends on $u(s)$ for all $s \in [t_0, t_1]$.

We have just seen that the inward boundary value process of an acausal system plays a role similar to that of the forward moving state of a causal system. For acausal systems there is also an outward boundary value process which plays the same role as the future jump $x(t_1) - \Phi(t_1, \tau_0)x(\tau_0)$ of the state of a causal process caused by $u(\tau)$ differing from 0 on $[\tau_0, t_1]$.

Given $u(\tau)$ for $\tau \in [\tau_0, \tau_1]$, define $z(\tau)$ and $j(\tau_0, \tau_1)$ by

$$\dot{z} = Az + Bu, \tag{4.8a}$$

$$z(\tau_0) = 0, \tag{4.8b}$$

$$j(\tau_0, \tau_1) = z(\tau_1). \tag{4.8c}$$

The *outward boundary value process* is $j(\tau_0, \tau_1)$ and has two important properties similar to those of $k(\tau_0, \tau_1)$. The first is that

$$j(\tau_0, \tau_1) = \int_{\tau_0}^{\tau_1} \Phi(\tau_1, s)B(s)u(s)\,ds,$$

so if we think of $\{\tau_0, \tau_1\}$ as the present, $[t_0, t_1]\backslash[\tau_0, \tau_1]$ as the past and (τ_0, τ_1) as the future, then the mapping $u(\cdot)\mapsto j(\tau_0, \tau_1)$ is anticausal. In other words $j(\tau_0, \tau_1)$ does not depend on past values of $u(\tau)$.

The second point is that given the present value $j(\tau_0, \tau_1)$ and past values $u(t)$, $t\in[t_0, t_1]\backslash[\tau_0, \tau_1]$ we can compute past values $x(t)$ and $y(t)$ for $t\in[t_0, t_1]\backslash[\tau_0, \tau_1]$. This is because the solution $x(t)$ of the well-posed four point boundary value problem (4.7a), (4.7b), and

$$\text{(4.7d)} \qquad -\Phi(\tau_1, \tau_0)x(\tau_0) + x(\tau_1) = j$$

agrees with the solution of (1.1) if $j = j(\tau_0, \tau_1)$ given by (4.8).

We have chosen the letter j for the outward boundary value process because it represents the jump that the state experiences between times τ_0 and τ_1 because of the control $u(t)\neq 0$, $t\in[\tau_0, \tau_1]$.

This suggests another viewpoint on the boundary value v. It is possible that the process $x(t)$ lies on some compactified version of the real line and v is the jump that the state experiences from time t_1 through infinity to time t_0 because of the effects of the control $u(t)$ for $t\notin[t_0, t_1]$.

5. Controllability and observability. Suppose the map $\Sigma:(v, u(\cdot))\mapsto(w, y(\cdot))$ arises from the state space model (1.1); then it factors into a mapping $(v, u(\cdot))\mapsto x(\cdot)$ followed by a mapping $x(\cdot)\mapsto(w, y(\cdot))$. It is natural that realization theory be concerned with these factor mappings. The critical issues for the minimality of a realization Σ are whether the first factor is onto and the second is one to one in some sense. In the systems literature this first property is called controllability and the second is called observability. For linear time invariant causal systems any two reasonable definitions of controllability are equivalent. The same holds for any two reasonable definitions of observability. But for time varying and/or nonlinear causal systems there are several nonequivalent definitions whose utility varies with the problem of the moment. Therefore it should come as no surprise that there are at least two useful definitions of both controllability and observability for acausal systems.

The first two definitions relate to the inward boundary value process $k(\tau_0, \tau_1)$.

DEFINITION. The system (1.1) is *controllable off* $[\tau_0, \tau_1]$ if the map

$$\text{(5.1)} \quad \{u(t): t\in[t_0, t_1]\backslash[\tau_0, \tau_1]\}\mapsto k(\tau_0, \tau_1) = \left(\int_{t_0}^{\tau_0} + \int_{\tau_1}^{t_1}\right)G(\tau_0, s)B(s)u(s)\,ds,$$

defined by (4.3) (or equivalently (4.2)) where $v = 0$, is onto.

DEFINITION. The system (1.1) is *observable on* $[\tau_0, \tau_1]$ if the map

$$\text{(5.2)} \qquad k\mapsto\{y(\tau) = C\Phi(\tau, \tau_0)k: \tau\in[\tau_0, \tau_1]\},$$

defined for $\tau\in[\tau_0, \tau_1]$ by (4.4a), (4.4b) and (1.1c), is 1-1. The control $u(t)$ is assumed to be zero on $[\tau_0, \tau_1]$.

As mentioned before, in a general sense controllability and observability are the two halves of minimality. The next result shows that we are on the right track.

THEOREM 5.1. *Let the system* (1.1) *be a realization of the weighting pattern* $W(t, s)$. *If there exists* τ_0 *and* τ_1 *where* $t_0 < \tau_0 < \tau_1 < t_1$ *such that* (1.1) *is controllable off* $[\tau_0, \tau_1]$

and observable on $[\tau_0, \tau_1]$, *then* (1.1) *is minimal, i.e., of minimal state dimension among all realizations of* $W(t, s)$. *Moreover, if* $\tilde{\Sigma}$ *is any other minimal realization of* $W(t, s)$ *then* $\tilde{\Sigma}$ *is also controllable off and observable on* $[\tau_0, \tau_1]$.

We defer the proof of this theorem for the moment. To check controllability off $[\tau_0, \tau_1]$ and observability on $[\tau_0, \tau_1]$ we need to compute the Gramians

$$\mathscr{C}]\tau_0, \tau_1[= \left(\int_{t_0}^{\tau_0} + \int_{\tau_1}^{t_1}\right) G(\tau_0, s)B(s)B^*(s)G^*(\tau_0, s)\, ds, \tag{5.3}$$

$$\mathscr{O}[\tau_0, \tau_1] = \int_{\tau_0}^{\tau_1} \Phi^*(t, \tau_0)C^*(t)C(t)\Phi(t, \tau_0)\, dt \tag{5.4}$$

where * denotes transpose. The following is a standard exercise in linear systems theory; see, for example, Desoer [11] or Brockett [17].

PROPOSITION 5.2. *The system* (1.1) *is controllable off* $[\tau_0, \tau_1]$ *iff* $\mathscr{C}]\tau_0, \tau_1[$ *is positive definite. The system* (1.1) *is observable on* $[\tau_0, \tau_1]$ *iff* $\mathscr{O}[\tau_0, \tau_1]$ *is positive definite.*

Remark. $\mathscr{O}[\tau_0, \tau_1]$ is the observability Gramian of the causal system with the same A and C matrices. Therefore observability on is the same as causal observability.

Proof of Theorem 5.1. Let Σ and $\tilde{\Sigma}$ be realizations of $W(t, s)$. (Σ is given by (1.1) and $\tilde{\Sigma}$ by a similar acausal system with tildes, i.e., $\tilde{x}$, $\tilde{A}$, etc.) Since Σ is controllable off $[\tau_0, \tau_1]$, $\mathscr{C}]\tau_0, \tau_1[$ is invertible. Given $k \in \mathbb{R}^n$ define a control $u(t; k)$ with support off (τ_0, τ_1) (i.e., with support in $[t_0, t_1]\backslash(\tau_0, \tau_1)$) by

$$u(t; k) = B^*(t)G^*(\tau_0, t)(\mathscr{C}]\tau_0, \tau_1[)^{-1}k.$$

Under (5.1)

$$u(t; k) \mapsto k(\tau^0, \tau^1) = k.$$

If we drive Σ and $\tilde{\Sigma}$ with $u(t; k)$ $(=\tilde{u}(t; k))$ and $v = 0$, $\tilde{v} = 0$, then we obtain $k(\tau_0, \tau_1)$ and $\tilde{k}(\tau_0, \tau_1)$. We have the commuting diagram (Fig. 5.1), which defines a linear mapping

$$T: k(\tau_0, \tau_1) \mapsto \tilde{k}(\tau_0, \tau_1).$$

The outputs $y(t)$ and $\tilde{y}(t)$ of Σ and $\tilde{\Sigma}$ corresponding to $u(t; k)$ must agree. Since the support of $u(t; k)$ is off (τ_0, τ_1), the output $y(\tau)$ on $[\tau_0, \tau_1]$ is given by (5.2) as a function of $k(\tau_0, \tau_1)$. A similar expression holds for $\tilde{y}(\tau)$ for $\tau \in [\tau_0, \tau_1]$. Therefore we have another commuting diagram (Fig. 5.2). From this we see that the kernel of T is contained in the kernel of the upper right mapping of Fig. 5.2 which is given by (5.2). But this latter kernel is 0 since Σ is observable on $[\tau_0, \tau_1]$.

This shows that the state dimension of $\tilde{\Sigma}$ must be greater than or equal to that of Σ. Hence Σ is minimal. If the dimensions are equal, then T is a linear isomorphism. From this it follows that $\tilde{\Sigma}$ is controllable off and observable on $[\tau_0, \tau_1]$. Q.E.D.

We note for future reference that if Σ is controllable off $[\tau_0, \tau_1]$ but not observable on $[\tau_0, \tau_1]$, the map T is well defined but not necessarily 1-1. Its kernel is contained in the kernel of $\mathscr{O}[\tau_0, \tau_1]$. In particular, for all $t \in [\tau_0, \tau_1]$

$$\tilde{C}(t)\tilde{\Phi}(t, \tau_0)T = C(t)\Phi(t, \tau_0). \tag{5.5}$$

There is an analogous development based on the outward boundary value process $j(\tau_0, \tau_1)$.

DEFINITION. The system (1.1) is *controllable on* $[\tau_0, \tau_1]$ if the map

$$\{u(\tau): \tau \in [\tau_0, \tau_1]\} \mapsto j(\tau_0, \tau_1) = \int_{\tau_0}^{\tau_1} \Phi(\tau_1, s)B(s)u(s)\, ds \tag{5.6}$$

as defined by the system (4.8) is onto.

DEFINITION. The system (1.2) is *observable off* $[\tau_0, \tau_1]$ if the map defined by (4.7a), (4.7b), (4.7d)

$$j(\tau_0, \tau_1) \mapsto \{y(t) = C(t)G(t, \tau_1)j(\tau_0, \tau_1): t \in [t_0, t_1] \backslash (\tau_0, \tau_1)\} \tag{5.7}$$

is one to one. The control is restricted to be zero off (τ_0, τ_1).

The associated Gramians are

$$\mathscr{C}[\tau_0, \tau_1] = \int_{\tau_0}^{\tau_1} \Phi(\tau_1, s)B(s)B^*(s)\Phi^*(\tau_1, s)\, ds, \tag{5.8a}$$

$$\mathscr{O}]\tau_0, \tau_1[= \left(\int_{t_0}^{\tau_0} + \int_{\tau_1}^{t_1}\right) G^*(t, \tau_1)C^*(t)C(t)G(t, \tau_1)\, dt. \tag{5.8b}$$

PROPOSITION 5.3. *The system* (1.1) *is controllable on* $[\tau_0, \tau_1]$ *iff* $\mathscr{C}[\tau_0, \tau_1]$ *is positive definite. The system* (1.1) *is observable off* $[\tau_0, \tau_1]$ *iff* $\mathscr{O}]\tau_0, \tau_1[$ *is positive definite.*

Remark. $C[\tau_0, \tau_1]$ is the controllability Gramian of the causal system with the same A and B matrices. Therefore controllability on is just causal controllability.

THEOREM 5.4. *Let the system* (1.1) *be a realization of the weighing pattern* $W(t, s)$. *If there exists a* τ_0 *and* τ_1 *where* $t_0 < \tau_0 < \tau_1 < t_1$ *such that the system is controllable on* $[\tau_0, \tau_1]$ *and observable off* $[\tau_0, \tau_1]$ *then* (1.1) *is minimal. Moreover if* $\tilde{\Sigma}$ *is any other minimal realization of* $W(t, s)$ *then* $\tilde{\Sigma}$ *is also controllable on and observable off* $[\tau_0, \tau_1]$.

Proof. It is essentially the same as the proof of Theorem 5.1. Let Σ and $\tilde{\Sigma}$ be two realizations of $W(t, s)$. Since Σ is controllable on $[\tau_0, \tau_1]$, $\mathscr{C}[\tau_0, \tau_1]$ is invertible. Given $j \in \mathbb{R}^n$, define a control $u(t; j)$ with support in $[\tau_0, \tau_1]$ by

$$u(t; j) = B^*(t)\Phi^*(\tau_1, t)(\mathscr{C}[\tau_0, \tau_1])^{-1}j;$$

then under (5.6)

$$j \mapsto u(t; j) \to j(\tau_0, \tau_1) = j.$$

Let $\tilde{j}(\tau_0, \tau_1)$ be the value of the outward boundary value process of $\tilde{\Sigma}$ corresponding to $u(t; j)$. Then we have the commuting diagram (Fig. 5.3). which defines the linear mapping

$$S: j(\tau_0, \tau_1) \mapsto \tilde{j}(\tau_0, \tau_1).$$

The outputs $y(t)$ and $\tilde{y}(t)$ of Σ and $\tilde{\Sigma}$ must agree. Since the support of $u(\tau; j)$ is on $[\tau_0, \tau_1]$, the outputs off (τ_0, τ_1) are functions (5.7) of $j(\tau_0, \tau_1)$, $\tilde{j}(\tau_0, \tau_1)$. Therefore, we have a second commuting diagram (Fig. 5.4). The kernel of S is contained in the kernel of the upper right mapping of Fig. 5.4, given by (5.6). Since Σ is observable off $[\tau_0, \tau_1]$, the latter is zero. Hence S is one to one. The state dimension of $\tilde{\Sigma}$ must be greater than or equal to that of Σ. Hence Σ is minimal. If the dimensions are equal, then S is an isomorphism. From this it follows that $\tilde{\Sigma}$ is controllable on and observable off $[\tau_0, \tau_1]$. Q.E.D.

We note for future reference that if Σ is not controllable on $[\tau_0, \tau_1]$ then the map S can be defined on the range of the map (5.6) which is the range of $\mathscr{C}[\tau_0, \tau_1]$. If Σ is observable off $[\tau_0, \tau_1]$ then on this domain S is 1-1 by the above argument. In particular, for all $s \in [\tau_0, \tau_1]$

$$S\Phi(\tau_1, s)B(s) = \tilde{\Phi}(\tau_1, s)\tilde{B}(s). \tag{5.9}$$

In general the concepts of controllability on and controllability off are independent, i.e., we can construct an example that is one but not the other. A system is *real analytic*

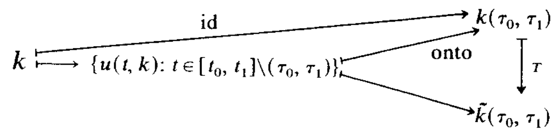

FIG. 5.1

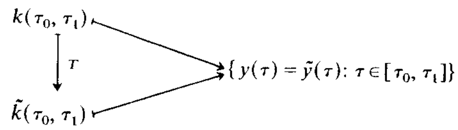

FIG. 5.2

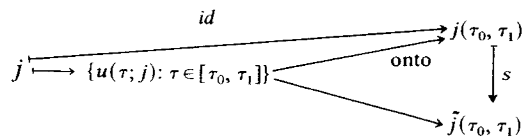

FIG. 5.3

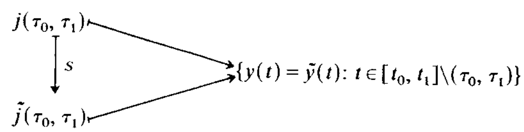

FIG. 5.4

if the matrices A, B, C and D are real analytic functions of t. Of course this includes *autonomous* systems where A, B, C, and D are constant. For real analytic systems, controllability (observability) on implies controllability (observability) off but they are not equivalent.

PROPOSITION 5.5. *Suppose* (1.1) *is a real analytic system and* $t_0 < \tau_0 < \tau_1 < t_1$.

(a) *If* (1.1) *is controllable* (*observable*) *off some* $[\tau_0, \tau_1]$ *then it is controllable* (*observable*) *off every* $[\tau_0, \tau_1]$.

(b) *If* (1.1) *is controllable* (*observable*) *on some* $[\tau_0, \tau_1]$ *then it is controllable* (*observable*) *on every* $[\tau_0, \tau_1]$.

(c) *If* (1.1) *is controllable* (*observable*) *on some* $[\tau_0, \tau_1]$ *then it is controllable* (*observable*) *off every* $[\tau_0, \tau_1]$.

Proof. (a) By definition, the Gramian $\mathscr{C}]\tau_0, \tau_1[$ is a real analytic, nonnegative definite matrix valued function of τ_0 and τ_1. Also it is monotone nonincreasing in $[\tau_0, \tau_1]$, i.e., if $[\sigma_0, \sigma_1] \supseteq [\tau_0, \tau_1]$ then $\mathscr{C}]\sigma_0, \sigma_1[\leqq \mathscr{C}]\tau_0, \tau_1[$. Suppose for some τ_0 and τ_1, $t_0 < \tau_0 < \tau_1 < t_1$, the Gramian $\mathscr{C}]\tau_0, \tau_1[$ is not positive definite then it is also not positive definite for all σ_0 and σ_1 such that $[\sigma_0, \sigma_1] \supseteq [\tau_0, \tau_1]$. For some open set of σ_0 and σ_1, the determinant of $\mathscr{C}]\sigma_0, \sigma_1[$ is zero. Real analyticity implies it is zero everywhere.

The other parts of (a) and (b) are proved in a similar fashion. (c) Suppose (1.1) is controllable on some $[\tau_0, \tau_1]$; then by (b) it is controllable on every $[\tau_0, \tau_1]$. Suppose it is not controllable off some $[\tau_0, \tau_1]$. Then we can find $0 \neq \lambda \in \mathbb{R}^{1 \times n}$ such that

$$\lambda(\mathscr{C}]\tau^0, \tau^1[)\lambda^* = 0.$$

$\mathscr{C}]\tau^0, \tau^1[$ is the sum of two nonnegative definite matrices so λ must annihilate each:

$$0 = \lambda \int_{t_0}^{\tau_0} G(\tau_0, s)B(s)B^*(s)G^*(\tau_0, s)\, ds\lambda^*$$

$$= \lambda\Phi(\tau_0, t_0)V^0\Phi(t_0, t_1)\mathscr{C}[t_0, t_1]\Phi^*(t_0, t_1)V^{0*}\Phi^*(\tau_0, t_0)\lambda^*.$$

Since $\mathscr{C}[t_0, \tau_0]$ is positive definite and $\Phi(t_0, t_1)$ is invertible we observe that

$$0 = \lambda\Phi(\tau_0, t_0)V^0.$$

From the other integral of $\mathscr{C}]\tau^0, \tau^1[$ we derive in a similar fashion that

$$0 = \lambda\Phi(\tau_0, t_0)V^1\Phi(t_1, t_0).$$

But this implies $\lambda = 0$ for $V^0 + V^1\Phi(t_1, t_0) = I$.

The other part of (c) is proved similarly. Q.E.D.

While Theorems 5.1 and 5.4 give sufficient conditions for minimality, these conditions are not necessary, as is shown by the following example, similar to one found in [10].

Example 5.6. Let $[t_0, t_1] = [0, 1]$. Consider the acausal system

$$\dot{x}_1 = 0, \qquad x_2(0) + x_1(1) - x_2(1) = v_1,$$

$$\dot{x}_2 = u, \qquad x_2(0) = v_2,$$

$$y = x_1.$$

It is a simple exercise to verify that this system is controllable and observable off every $[\tau_0, \tau_1]$ but it is not controllable nor observable on any $[\tau_0, \tau_1]$. The Gramians are

$$\mathscr{C}]\tau^0, \tau^1[= \begin{vmatrix} 1-\tau_1+\tau_0 & \tau_0 \\ \tau_0 & \tau_0 \end{vmatrix}, \qquad \mathscr{C}[\tau^0, \tau^1] = \begin{vmatrix} 0 & 0 \\ 0 & \tau_1-\tau_0 \end{vmatrix},$$

$$\mathscr{O}]\tau^0, \tau^1[= \begin{vmatrix} 1-\tau_1 & \tau_1-1 \\ \tau_1-1 & 1-\tau_1+\tau_0 \end{vmatrix}, \qquad \mathscr{O}[\tau^0, \tau^1] = \begin{vmatrix} \tau_1-\tau_0 & 0 \\ 0 & 0 \end{vmatrix}.$$

By studying this system we get an understanding of what controllability and observability off $[\tau_0, \tau_1]$ really mean. The boundary conditions allow us to control x_1 in an indirect fashion. For example, to achieve a desired $x(\tau)$, $0 < \tau < 1$, we use the control on $[0, \tau]$ to fix $x_2(\tau)$. We use the control on $[\tau_1, 1]$ to fix $x_2(1)$. The first boundary condition and dynamics imply that $x_1(\tau) = x_2(1)$.

This indirect controllability through the boundary conditions is possible in acausal systems that are controllable off every $[\tau_0, \tau_1]$. Given any $\tau \in (t_0, t_1)$ and any $x^0 \in \mathbb{R}$ there exists a $u(\cdot)$ such that $x(\tau) = x^0$. The controllability off hypothesis implies that the map $u(\cdot) \to k(\tau, \tau)$ is onto, and it follows immediately from (4.1) and (4.2) that $x(\tau) = k(\tau, \tau)$.

The boundary conditions also allow us to detect jumps or breaks in both state coordinate trajectories even though we can only observe x_1. For example, if for some unknown reason x_2 jumps at $\tau \in (0, 1)$ $(x_2(\tau^+) - x_2(\tau^-) = j_2)$, then this affects $x_1(t)$ through the first boundary condition and we can detect it through $y(t)$.

This indirect observability through the boundary conditions is possible in any system that is observable off every $[\tau_0, \tau_1]$. Given any $\tau \in (t_0, t_1)$ and jump $j = x(\tau^+) - x(\tau^-)$, the mapping $j \to y(\cdot)$ is one to one. Therefore if we know the time τ of the jump, we can detect it.

The weighting pattern of this system is $W(t, s)=1$ for all $t, s\in[t_0, t_1]$. From Theorem 3.1 it can be seen that this is a minimal realization of $W(t, s)=1$. For if Σ is a one-dimensional realization then there exists 1×1 matrices $C(t)$, $B(s)$, V^0 and V^1 such that

$$V^0+V^1=1$$

and

$$1=C(t)V^0B(s), \qquad t>s,$$

$$1=-C(t)V^1B(s), \qquad t<s.$$

The latter equations imply $C(t)$ and $B(s)$ are constant. If one is subtracted from the other we have

$$0=C(V^0+V^1)B=CB,$$

so either C or B is zero, a contradiction.

6. Minimal real analytic realizations. In the last section we gave sufficient conditions for minimality. In this one we give necessary and sufficient conditions for a real analytic realization to be minimal within the class of real analytic realizations. We describe how a real analytic realization can be reduced to a minimal real analytic realization, and how minimal real analytic realizations of the same weighting pattern can possibly differ. It may come as a bit of a surprise to readers familiar with causal systems theory that two minimal real analytic realizations can differ by more than a change of coordinates in the state space.

THEOREM 6.1. *Suppose Σ, given by* (1.1), *is a standard real analytic realization of $W(t, s)$. Σ is a minimal real analytic realization if and only if Σ is controllable and observable off every* $[\tau_0, \tau_1]$ *and*

$$\text{Kernel } \mathcal{O}[t_0, t_1]\subseteq \text{Range } \Phi(t_0, t_1)C[t_0, t_1]\Phi^*(t_0, t_1). \tag{6.1}$$

Any real analytic realization can be reduced to a minimal real analytic realization. If Σ and $\tilde{\Sigma}$ are two standard minimal real analytic realizations of the same $W(t, s)$, then there exists a real analytic invertible $n\times n$ matrix valued function $R(t)$ such that

$$\tilde{A}(t)=R(t)A(t)R^{-1}(t)+\dot{R}(t)R^{-1}(t), \tag{6.2a}$$

$$\tilde{B}(t)=R(t)B(t), \tag{6.2b}$$

$$\tilde{C}(t)=C(t)R^{-1}(t), \tag{6.2c}$$

$$\tilde{D}(t)=D(t), \tag{6.2d}$$

$$R(t)\Phi(t, s)=\tilde{\Phi}(t, s)R(s), \tag{6.2e}$$

and

$$\mathcal{O}[t_0, t_1](V^0-R^{-1}(t_0)\tilde{V}^0R(t_0))\Phi[t_0, t_1]\mathcal{C}[t_0, t_1]=0, \tag{6.3a}$$

$$\mathcal{O}[t_0, t_1](V^1-R^{-1}(t_0)\tilde{V}^1R(t_1))\mathcal{C}[t_0, t_1]=0. \tag{6.3b}$$

On the other hand, if Σ is a minimal real analytic realization of $W(t, s)$ and $\tilde{\Sigma}$ satisfies (6.2) and (6.3) for some real analytic invertible $R(t)$, then $\tilde{\Sigma}$ is also a minimal real analytic realization of $W(t, s)$.

Remarks. Condition (6.1) means that any state which is unobservable on $[t_0, t_1]$ must be controllable on $[t_0, t_1]$. Equations (6.2) are the same as those that arise from a time varying change of coordinates $\tilde{x} = R(t)x$, but this is not the whole story because of (6.3). If Σ is both controllable and observable on $[t_0, t_1]$ then (6.3) becomes

$$\tilde{V}^0 = R(t_0)V^0R^{-1}(t_0), \qquad \tilde{V}^1 = R(t_0)V^1R^{-1}(t_1).$$

Then (6.2) and (6.3) represent a time varying change of state coordinates $\tilde{x} = R(t)x$ and a corresponding change of coordinates in the space of boundary inputs, $\tilde{v} = R(t_0)v$, so that $\tilde{\Sigma}$ is standard, $\tilde{V}^0 + \tilde{V}^1\tilde{\Phi}(t_1, t_0) = I$. Since $\tilde{\Sigma}$ is standard, (6.3a) and (6.3b) are equivalent.

Proof. Suppose Σ and $\tilde{\Sigma}$ are standard real and analytic realizations of $W(t, s)$. Clearly $D(t) = \tilde{D}(t)$. For simplicity henceforth we assume that $D(t) = 0$. Suppose for some τ_0 and τ_1, the system Σ is controllable and observable off $[\tau_0, \tau_1]$ and (6.1) is satisfied.

It is convenient to make a time varying change of state coordinate $x_{\text{new}}(t) = \Phi(t_0, t)x_{\text{old}}(t)$ so that in these new coordinates $A(t) = 0$. We make a similar change of coordinates on $\tilde{\Sigma}$ so that $\tilde{A}(t) = 0$. Then $\Phi(t, s) = I$, $\tilde{\Phi}(t, s) = I$ and

$$W(t, s) = C(t)V^0B(s) = \tilde{C}(t)\tilde{V}^0\tilde{B}(s) \quad \text{if } t > s, \tag{6.4a}$$

$$W(t, s) = C(t)(V^0 - I)B(s) = \tilde{C}(t)(\tilde{V}^0 - I)\tilde{B}(s) \quad \text{if } t < s. \tag{6.4b}$$

By real analyticity we conclude that these formulas must hold for all t and s so

$$C(t)B(s) = \tilde{C}(t)\tilde{B}(s). \tag{6.5}$$

Relative to the controllability on and observability on Gramians $\mathscr{C}[t_0, t_1]$ and $\mathscr{O}[t_0, t_1]$, there is a nested family of subspaces of the state space

$$\begin{aligned} \mathbb{R}^n &\supseteq \text{Range}\,(\mathscr{C}[t_0, t_1]) + \text{Kernel}\,(\mathscr{O}[t_0, t_1]) \\ &\supseteq \text{Range}\,(\mathscr{C}[t_0, t_1]) \\ &\supseteq \text{Range}\,(\mathscr{C}[t_0, t_1]) \cap \text{Kernel}\,(\mathscr{O}[t_0, t_1]) \supseteq 0. \end{aligned}$$

We can choose coordinates $x = (x_1, x_2, x_3, x_4)$, which respect this flag, i.e.,

$$\text{Range } \mathscr{C}[t_0, t_1] \cap \text{Kernel } \mathscr{O}[t_0, t_1] = \{x : x_i = 0, i = 1, 2, 3\},$$

$$\text{Range } \mathscr{C}[t_0, t_1] = \{x : x_i = 0, i = 1, 2\},$$

$$\text{Range } \mathscr{C}[t_0, t_1] + \text{Kernel } \mathscr{O}[t_0, t_1] = \{x : x_1 = 0\}.$$

This is essentially the Kalman 4 part decomposition of the state space; see Kalman [16] or Desoer [11, p. 187] for more details.

The x_1 and x_3 coordinates are observable and the x_1 and x_2 coordinates are uncontrollable on $[t_0, t_1]$. We make the same change of coordinates in the space of boundary inputs v to ensure that $V^0 + V^1 = I$.

Relative to this partition of x we have that

$$B^*(s) = (0 \quad 0 \quad B_3^*(s) \quad B_4^*(s)), \tag{6.6a}$$

$$C(t) = (C_1(t) \;\; 0 \;\; C_3(t) \;\; 0). \tag{6.6b}$$

Let d_i be the dimension of x_i; then $d_1 + d_2 + d_3 + d_4 = n$. Condition (6.1) ensures that $d_2 = 0$.

We make a similar decomposition of the state space of $\tilde{\Sigma}$, $\tilde{x}=(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4)$ of dimensions $\tilde{d}_1+\tilde{d}_2+\tilde{d}_3+\tilde{d}_4=\tilde{n}$. Of course $\tilde{d}_2$ need not be zero.

Relative to this partition of $\tilde{x}$ we have

$$\tilde{B}^*(s)=(0 \quad 0 \quad \tilde{B}_3^*(s) \quad \tilde{B}_4^*(s)), \tag{6.7a}$$

$$\tilde{C}(t)=(\tilde{C}_1(t) \quad 0 \quad \tilde{C}_3(t) \quad 0). \tag{6.7b}$$

From (6.5), (6.6) and (6.7) we see that

$$C_3(t)B_3(s)=\tilde{C}_3(t)\tilde{B}_3(s). \tag{6.8}$$

We can consider (6.8) as the weighting pattern of a causal system which can be realized on either x_3 or $\tilde{x}_3$ space. Both these realizations are minimal because the x_3 and $\tilde{x}_3$ coordinates are both controllable and observable on $[t_0, t_1]$. Hence $d_3=\tilde{d}_3$.

Now consider the map T: $k(\tau_0, \tau_1)\mapsto \tilde{k}(\tau_0, \tau_1)$ constructed in the proof of Theorem 5.1. Since Σ is controllable off $[\tau_0, \tau_1]$, T is well defined but it need not be one to one. Restricted to the range of $\mathcal{O}[\tau_0, \tau_1]$, it is one to one. The ranks of $\mathcal{O}[\tau_0, \tau_1]$ and $\tilde{\mathcal{O}}[\tau_0, \tau_1]$ are d_1+d_3 and $\tilde{d}_1+\tilde{d}_3$, respectively. The counting diagram (Fig. 5.2) implies that $d_1+d_3\leqq \tilde{d}_1+\tilde{d}_3$. Since $d_3=\tilde{d}_3$ this implies that $d_1\leqq\tilde{d}_1$.

Next consider the map $S: j(\tau_0, \tau_1)\mapsto \tilde{j}(\tau_0, \tau_1)$ of Theorem 5.4. This is not defined for all $j(\tau_0, \tau_1)$ but only those in the range of $\mathscr{C}[\tau_0, \tau_1]$. On this domain it is one to one. The ranks of $\mathscr{C}[\tau_0, \tau_1]$ and $\tilde{\mathscr{C}}[\tau_0, \tau_1]$ are d_3+d_4 and $\tilde{d}_3+\tilde{d}_4$, respectively. The commuting diagram (Fig. 5.4) implies that $d_3+d_4\leqq\tilde{d}_3+\tilde{d}_4$ and hence $d_4\leqq\tilde{d}_4$.

We have shown that $d_i\leqq\tilde{d}_i$ for $i=1, 2, 3, 4$ so $n\leqq\tilde{n}$ and hence Σ is minimal.

Suppose $\Sigma(t, s)$ is any standard real analytic realization of $W(t, s)$; we now show that Σ can be reduced to obtain a lower-dimensional realization of $W(t, s)$ which is controllable and observable off every $[\tau_0, \tau_1]$ and such that (6.1) is satisfied. In this manner we see that if a realization is minimal it is controllable and observable off every $[\tau_0, \tau_1]$, and every state unobservable on $[\tau_0, \tau_1]$ is controllable on $[\tau_0, \tau_1]$.

We assume that we have made the preliminary change of state coordinates so that $A(t)=0$. Suppose Σ is controllable and observable off every $[\tau_0, \tau_1]$. We decompose the state space relative to the Gramians $\mathscr{C}[t_0, t_1]$ and $\mathcal{O}[t_0, t_1]$ as above.

The x_2 coordinate is irrelevant to the weighting pattern of the system. From (6.6) we have that

$$W(t, s)=\begin{cases}\sum_{i=1,3}\sum_{j=3,4} C_i(t)V_{ij}^0B_j(s) & \text{if } t>s,\\ -\sum_{i=1,3}\sum_{j=3,4} C_i(t)V_{ij}^1B_j(s) & \text{if } t<s.\end{cases}$$

Therefore we can delete x_2 and the second boundary condition. The new system satisfies (6.1) so we obtain a new realization of $W(t, s)$, of lower dimension, which is minimal among real analytic realizations.

Suppose Σ is not controllable or not observable off every $[\tau_0, \tau_1]$; then we can reduce it to one that is. We decompose the state space into 4 parts relative to Gramians $\mathscr{C}]\tau_0, \tau_0[$ and $\mathcal{O}]\tau_0, \tau_0[$ for any $t_0<\tau_0<\tau_1<t_1$. We obtain a flag of subspaces

$$\begin{aligned}\mathbb{R}^n &\supseteq \text{Range}\,(\mathscr{C}]\tau_0, \tau_1[)+\text{Kernel}\,(\mathcal{O}]\tau_0, \tau_1[)\\ &\supseteq \text{Range}\,(\mathscr{C}]\tau_0, \tau_1[)\\ &\supseteq \text{Range}\,(\mathscr{C}]\tau_0, \tau_1[)\cap\text{Kernel}\,(\mathcal{O}]\tau_0, \tau_1[)\supseteq 0.\end{aligned}$$

As before we choose coordinates $x=(x_1, x_2, x_3, x_4)$ which respect this flag and we make the same change of coordinates in the space of boundary values v to keep the system in standard form.

From the choice of coordinates

$$G(t, s)B(s) = V^iB(s) = \begin{bmatrix} 0 \\ 0 \\ * \\ * \end{bmatrix} \quad \text{where } i=0 \text{ if } t>s \text{ and } i=1 \text{ if } t<s,$$

and

$$C(t)G(t, s) = C(t)V^i = [* \ 0 \ * \ 0] \quad \text{where } i=0 \text{ if } t>s \text{ and } i=1 \text{ if } t<s.$$

Since $V^0 + V^1 = I$ we obtain

$$B(s) = \begin{bmatrix} B_1(s) \\ B_2(s) \\ B_3(s) \\ B_4(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ * \\ * \end{bmatrix}, \tag{6.9a}$$

$$C(t) = [C_1(t)C_2(t)C_3(t)C_4(t)] = [* \ 0 \ * \ 0], \tag{6.9b}$$

$$V^iB(s) = \begin{bmatrix} V^i_{13}B_3(s) + V^i_{14}B_4(s) \\ V^i_{23}B_3(s) + V^i_{24}B_4(s) \\ V^i_{33}B_3(s) + V^i_{34}B_4(s) \\ V^i_{44}B_4(s) + V^i_{44}B_4(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ * \\ * \end{bmatrix}, \tag{6.9c}$$

$$\begin{aligned} C(t)V^i = [&C_1(t)V^i_{11} + C_3(t)V^i_{31} \vdots C_1(t)V^i_{12} + C_3(t)V^i_{32} \vdots \\ &C_1(t)V^i_{13} + C_3(t)V^i_{33} \vdots C_1(t)V^i_{14} + C_3(t)V^i_{34}] \\ = [&* \ 0 \ * \ 0]. \end{aligned} \tag{6.9d}$$

We can calculate $W(t, s)$ from (6.9a), (6.9d) as

$$W(t, s) = \begin{cases} C_1(t)V^0_{13}B_3(s) + C_3(t)V^0_{33}B_3(s), & t>s, \\ -C_1(t)V^1_{13}B_3(s) - C_3(t)V^1_{33}B_3(s), & t<s \end{cases} \tag{6.10a}$$

or from (6.9b), (6.9c) as

$$W(t, s) = \begin{cases} C_3(t)V^0_{33}B_3(s) + C_3(t)V^0_{34}B_4(s), & t>s, \\ -C_3(t)V^1_{33}B_3(s) - C_3(t)V^1_{34}B_4(s), & t<s. \end{cases} \tag{6.10b}$$

Equation (6.10a) shows that the x_2 and x_4 coordinates are unnecessary to realize $W(t, s)$, so we delete them and the corresponding boundary conditions to obtain a new system which is observable off every $[\tau_0, \tau_1]$. From the first component of (6.9c) we see that $V^i_{13}B_3(s) = 0$ so we can delete the x_1 coordinate and realize $W(t, s)$ by

$$\dot{x}_3 = B_3u, \tag{6.11a}$$

$$V^0_{33}x_3(t_0) + V^1_{33}x_3(t_1) = v_3, \tag{6.11b}$$

$$y = C_3x_3. \tag{6.11c}$$

This realization is controllable and observable off every $[\tau_0, \tau_1]$. As previously seen, such realizations can be further reduced so that (6.1) holds, thereby obtaining a minimal real analytic realization.

Next we study the relationship between minimal real analytic realizations. Let Σ and $\tilde{\Sigma}$ be two standard minimal real analytic realizations of $W(t, s)$. Hence they are controllable and observable off every $[\tau_0, \tau_1]$ and (6.1) is satisfied. We transform state coordinates on Σ and $\tilde{\Sigma}$ as in the first part of the proof so that $A(t)=\tilde{A}(t)=0$ and $x=(x_1, x_3, x_4)$, $\tilde{x}=(\tilde{x}_1, \tilde{x}_3, \tilde{x}_4)$. Because they are minimal $d_i=\tilde{d}_i$, $i=1,3,4$ and $d_2=\tilde{d}_2=0$.

The decomposition of the state space induces a similar decomposition of the boundary value processes, $k=(k_1, k_3, k_4)$, $\tilde{k}=(\tilde{k}_1, \tilde{k}_3, \tilde{k}_4)$, $j=(j_1, j_3, j_4)$ and $\tilde{j}=(\tilde{j}_1, \tilde{j}_3, \tilde{j}_4)$. We consider the maps T and S in more detail. If $k \in \text{kernel } \mathcal{O}[\tau_0, \tau_1]$ then $T(k)=\tilde{k} \in \text{kernel } \tilde{\mathcal{O}}[\tau_0, \tau_1]$, so $\tilde{k}=Tk$ is given by

$$\begin{pmatrix}\tilde{k}_1\\ \tilde{k}_3\\ \tilde{k}_4\end{pmatrix}=\begin{pmatrix}T_{11} & T_{13} & 0\\ T_{31} & T_{33} & 0\\ T_{41} & T_{43} & T_{44}\end{pmatrix}\begin{pmatrix}k_1\\ k_3\\ k_4\end{pmatrix}. \tag{6.12}$$

By the commuting diagram (Fig. 5.2) the upper left 2×2 block of T is invertible.

On the other hand S is only defined on the range of $C[\tau_0, \tau_1]$ (i.e. $j=(0, j_3, j_4)$), but it is one to one. So $\tilde{j}=S(0, j_3, j_4)$ is given by

$$\begin{pmatrix}\tilde{j}_1\\ \tilde{j}_3\\ \tilde{j}_4\end{pmatrix}=\begin{pmatrix}0 & S_{13} & S_{14}\\ 0 & S_{33} & S_{34}\\ 0 & S_{43} & S_{44}\end{pmatrix}\begin{pmatrix}0\\ j_3\\ j_4\end{pmatrix} \tag{6.13}$$

where the matrix is of rank d_3+d_4.

From (5.5) and (5.9) we have

$$C(t)=\tilde{C}(t)T, \tag{6.14a}$$

$$\tilde{B}(s)=SB(s), \tag{6.14b}$$

so (6.5) becomes

$$\tilde{C}(t)TB(s)=\tilde{C}(t)SB(s). \tag{6.15}$$

If we multiply by $\tilde{C}^*(t)$ and $B^*(s)$ and integrate we obtain

$$\tilde{\mathcal{O}}[t_0, t_1](T-S)\mathscr{C}[t_0, t_1]=0. \tag{6.16}$$

This implies that $T_{ij}=S_{ij}$ for $i=1,3$ and $j=3,4$. We define an $n\times n$ matrix R by

$$R=\begin{vmatrix}T_{11} & T_{13} & 0\\ T_{31} & T_{33} & 0\\ T_{41} & S_{43} & S_{44}\end{vmatrix}. \tag{6.17}$$

Since $S_{14}=T_{14}=0$ and $S_{34}=T_{34}=0$, S_{44} must be invertible for S to be of rank d_3+d_4. The upper left 2×2 block of R is from T, hence is invertible. This shows that R is invertible. Since R agrees with T in rows indexed by 1 and 3, from (6.7b) and (6.14a) we see that

$$C(t)=\tilde{C}(t)R. \tag{6.18a}$$

Since R agrees with S in columns indexed by 3 and 4, from (6.7a) and (6.14b) we see that

$$\tilde{B}(t)=RB(t). \tag{6.18b}$$

From (6.4a) and (6.18), we obtain

$$C(t)V^0B(s)=C(t)R^{-1}\tilde{V}^0RB(s).$$

We multiply on both sides and integrate to obtain

$$\mathcal{O}[t_0, t_1](V^0-R^{-1}\tilde{V}^0R)C[t_0, t_1]=0. \tag{6.19}$$

Let $\Phi(t, s)$ and $\tilde{\Phi}(t, s)$ denote the fundamental solutions for Σ and $\tilde{\Sigma}$ in their original state coordinates. Define

$$R(t) = \tilde{\Phi}(t, t_0) R \Phi(t_0, t); \tag{6.20}$$

then it is straightforward but tedious to verify (6.2) and (6.3) from (6.18), (6.19) and (6.20).

On the other hand suppose Σ and $\tilde{\Sigma}$ are two systems related by (6.2) and (6.3). Then if $t > s$ the weighting pattern of $\tilde{\Sigma}$ is given by

$$\begin{aligned}\tilde{W}(t, s) &= \tilde{C}(t)\tilde{\Phi}(t, t_0)\tilde{V}^0\tilde{\Phi}(t_0, s)\tilde{B}(s)\\ &= C(t)\Phi(t, t_0)V^0\Phi(t_0, s)B(s) + C(t)\Phi(t, t_0)(V^0 - R^{-1}(t_0)\tilde{V}^0 R(t_0))\Phi(t_0, s)B(s).\end{aligned}$$

But (6.2a) implies that this second term is zero, so $\tilde{W}(t, s) = W(t, s)$ for $t > s$. A similar calculation holds for $t < s$. Q.E.D.

Suppose Σ is a minimal realization of $W(t, s)$ and we choose state coordinates as before so that $A(t) = 0$ and $x = (x_1, x_3, x_4)$ respects the flag of subspaces associated to $\mathscr{C}[t_0, t_1]$ and $\mathcal{O}[t_0, t_1]$. Since $V^0 + V^1 = I$

$$W(t, s) = \sum_{i=1,3}\sum_{j=3,4} C_i(t)V^0_{ij}B_j(s) \quad \text{if } t > s, \tag{6.21a}$$

$$W(t, s) = \sum_{i=1,3}\sum_{j=3,4} C_i(t)(V^0 - I)_{ij}B_j(s) \quad \text{if } t < s. \tag{6.21b}$$

Then

$$W(t, s) = W_1(t, s) + W_2(t, s), \tag{6.22a}$$

$$W_1(t, s) = \begin{cases} C_3(t)V^0_{33}B_3(s), & t > s,\\ C_3(t)(V^0_{33} - I)B_3(s), & t < s\end{cases} \tag{6.22b}$$

and for all t, s

$$W_2(t, s) = C_1(t)V^0_{13}B_3(s) + C_1(t)V^0_{14}B_4(s) + C_3(t)V^0_{34}B_4(s). \tag{6.23}$$

Each kernel $W_i(t, s)$ defines a mapping

$$u(t) \mapsto y_i(t) = \int_{t_0}^{t_1} W_i(t, s)u(s)\, ds.$$

The first kernel $W_1(t, s)$ defines a mapping of infinite rank which can be realized on x_3 space. The second kernel $W_2(t, s)$ defines a mapping of finite rank

$$y_2(t) = C_1(t)V^0_{13}\int_{t_0}^{t_1} B_3(s)u(s)\, ds + (C_1(t)V^0_{14} + C_3(t)V^0_{34})\int_{t_0}^{t_1} B_4(s)u(s)\, ds.$$

There is an alternate decomposition of $W(t, s)$:

$$W(t, s) = \tilde{W}_1(t, s) + \tilde{W}_2(t, s) \tag{6.24a}$$

where

$$\tilde{W}_1(t, s) = \begin{cases} C_3(t)B_3(s), & t > s,\\ 0, & t < s,\end{cases} \tag{6.24b}$$

(6.24c)

$$\tilde{W}_2(t, s) = C_1(t)V^0_{13}B_3(s) + C_1(t)V^0_{14}B_4(s) + C_3(t)(V^0_{33} - I)B_3(s) + C_3(t)V^0_{34}B_4(s).$$

$\tilde{W}_1(t, s)$ is a causal weighing pattern which can be realized on $\tilde{x}_3$ space. As before, $\tilde{W}_1$ and $\tilde{W}_2$ are maps of infinite and finite rank, respectively.

In their excellent study of acausal systems, Gohberg and Kaashoek [9], [10] have introduced the concepts of multicontrollability and multiobservability, which are different generalizations of causal controllability and observability from those discussed above. The following example is a slight modification of one found in [10, § II.1 and illustrates some of the differences between their work and ours.

Example 6.2. Let $[t_0, t_1]=[0, 1]$.

$$\dot{x}_1=0, \qquad x_2(0)+x_1(1)-x_2(1)=v_1,$$

$$\dot{x}_2=0, \qquad x_2(0)+x_3(0)-x_3(1)=v_2,$$

$$\dot{x}_3=u, \qquad x_3(0)=v_3,$$

$$y=x_1.$$

Gohberg and Kaashoek associate with an acausal system a sequence of weighting patterns of which $W(t, s)$ given by (3.6) is the first. Relevant to these weighting patterns are the concepts of multicontrollability and multiobservability. This example is a 3 controllable and 3 observable system. They show that such systems are irreducible under similarity and reduction and are characterized up to similarity by their sequence of weighting patterns. They do not discuss minimality.

By our definition this system is not controllable nor observable off any $[\tau_0, \tau_1]$ and (6.1) is not satisfied. Therefore by Theorem 6.1 it is not minimal among real analytic realizations. If we reduce it as described above we arrive at the trivial system of state dimension 0. The weighting pattern (3.6) is $W(t, s)=0$, but the other weighting patterns of Gohberg and Kaashoek are not all zero.

7. Autonomous and stationary systems. An acausal linear system is *autonomous* if A, B, C, D are constant with respect to t. The transition matrix and Green's matrix are given by

$$\Phi(t, s)=e^{(t-s)A}, \tag{7.1}$$

$$G(t, s)=e^{(t-t_0)A}V^0 e^{(t_0-s)A} \quad \text{if } t>s, \tag{7.2a}$$

$$G(t, s)=-e^{(t-t_0)A}V^1 e^{(t_1-s)A} \quad \text{if } t<s. \tag{7.2b}$$

From this it is easy to give alternate tests for controllability and observability.

PROPOSITION 7.1. *Let Σ be an autonomous acausal system and $t_0<\tau_0<\tau_1<t_1$:*

(a) *Σ is controllable on $[\tau_0, \tau_1]$ iff the matrix*

$$\mathscr{C}=[B, \cdots, A^{n-1}B] \tag{7.3a}$$

is of rank n.

(b) *Σ is controllable off $[\tau_0, \tau_1]$ iff the matrix*

$$\mathscr{C}^b=[V^0B, \quad V^1B, \quad \cdots, \quad V^0A^{n-1}B, \quad V^1A^{n-1}B] \tag{7.3b}$$

is of rank n.

(c) *Σ is observable on $[\tau_0, \tau_1]$ iff the matrix*

$$\mathscr{O}=\begin{bmatrix} C \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{7.3c}$$

is of rank n.

(d) Σ *is observable off* $[\tau_0, \tau_1]$ *iff the matrix*

$$\mathcal{O}^b = \begin{bmatrix} CV^0 \\ CV^1 \\ \vdots \\ CA^{n-1}V^0 \\ CA^{n-1}V^1 \end{bmatrix} \tag{7.3d}$$

is of rank n.

(e) *Condition* (6.1) *is satisfied iff*

$$\text{Kernel } \mathcal{O} \text{ Range } \mathcal{C}. \tag{7.3e}$$

Proof. Assertion (a) and (c) are well known from the causal theory, and (e) is straightforward. We only prove (b) since the proof of (d) is essentially the same. Suppose (7.3b) fails; then there exists an $1 \times n$ vector $\lambda \neq 0$ such that

$$\lambda V^0 A^k B = \lambda V^1 A^k B = 0 \tag{7.4}$$

for any $k \geqq 0$ (by Cayley-Hamilton). Hence for any t

$$\lambda V^0 e^{At} B = \lambda V^1 e^{At} B = 0. \tag{7.5}$$

Now for any $t_0 < \tau_0 < \tau_1 < t_1$,

$$\begin{aligned} 0 &= \lambda \Phi(t_0, \tau_0)(\mathcal{C}]\tau_0, \tau_1[)\Phi^*(t_0, \tau_0)\lambda^* \\ &= \int_{t_0}^{\tau_0} \lambda V^0 e^{A(t_0 - s)} BB^* e^{A^*(t_0-s)} V^{0*}\lambda^* \, ds \\ &\quad + \int_{\tau_1}^{t_1} \lambda V^1 e^{A(t_1-s)} BB^* e^{A^*(t_1-s)} V^{1*}\lambda^* \, ds. \end{aligned} \tag{7.6}$$

Therefore $\mathcal{C}]\tau_0, \tau_1[$ is not positive definite and the system is not controllable off $[\tau_0, \tau_1]$.

On the other hand, if there exists λ such that (7.6) holds for all $t_0 < \tau_0 < \tau_1 < t_1$, then (7.5) must hold. Differentiating (7.5) yields (7.4). Q.E.D.

Gohberg and Kaashoek [13] have demonstrated that a weighting pattern can have a minimal autonomous realization and a minimal real analytic realization of lower dimension. They showed that the weighting pattern $W(t, s) = 1 - s$ has this property. Their realizations are as follows.

Example 7.2. Let $[t_0, t_1] = [0, 1]$.

$$\begin{aligned} \dot{x}_1 &= 0, & x_2(0) + x_3(0) + x_1(1) - x_2(1) &= v_1, \\ \dot{x}_2 &= x_3, & x_2(0) &= v_2, \\ \dot{x}_3 &= u, & x_3(0) &= v_3, \\ y &= x_1. \end{aligned}$$

The system is controllable off any $[\tau_0, \tau_1]$ but not observable off any $[\tau_0, \tau_1]$, where $0 < \tau_0 < \tau_1 < 1$. Therefore by Theorem 6.1 it is not a minimal realization within the class of real analytic realizations. In fact $W(t, s) = 1 - s$ can also be realized by the following.

Example 7.3. Let $[t_0, t_1] = [0, 1]$.

$$\begin{aligned} \dot{x}_1 &= (1-t)u, & x_1(0) &= v_1, \\ \dot{x}_2 &= (1-t)u, & x_2(1) &= v_2, \\ y &= x_1 - x_2. \end{aligned}$$

This is controllable and observable off every $[\tau_0, \tau_1]$ and (6.1) is satisfied. Hence by Theorem 6.1 this is a minimal real analytic realization of $W(t, s) = 1 - s$. Moreover the other minimal real analytic realizations of $W(t, s)$ are described by this theorem. If one of them were autonomous there would exist an invertible 2×2 matrix $R(t)$ such that for some constant matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$

$$\tilde{A} = \dot{R}(t)R^{-1}(t), \tag{7.7a}$$

$$\tilde{B} = R(t)\begin{pmatrix} 1-t \\ 1-t \end{pmatrix}, \tag{7.7b}$$

$$\tilde{C} = (1 \ \ -1)R^{-1}(t). \tag{7.7c}$$

Without loss of generality we can assume that $R(0) = I$ so $\tilde{B} = (1 \ \ 1)^*$ and $\tilde{C} = (1 \ \ -1)$. Equation (7.7b) implies that $\tilde{B}$ is an eigenvector of $R(t)$ with eigenvalue $1/(1-t)$. On the other hand, equation (7.7a) implies that $R(t) = e^{\tilde{A}t}$ and hence $1/(1-t)$ cannot be an eigenvalue of $R(t)$. We conclude that there are no two-dimensional autonomous realizations of $W(t, s) = 1 - s$ but there are two-dimensional real analytic realizations.

We were a bit surprised by these examples for we had conjectured the opposite, namely that the class of autonomous models admitted a self-contained minimal realization theory. Being autonomous is a property of the system and not of the weighting pattern. Therefore we should have expected that we need a "nice" class of weighting patterns to obtain a self-contained minimal realization theory. The stationary weighting patterns are such a class.

A weighting pattern $W(t, s)$ is *stationary* if it is only a function of $t - s$, in abuse of notation $W(t, s) = W(t - s)$. An acausal linear system is *stationary* if it is autonomous and its weighting pattern is stationary. Every autonomous causal system is stationary, and hence the stationary acausal systems generalize the autonomous causal systems. The corollary to the following theorem was first stated in [3]; see also [9], [10].

PROPOSITION 7.4. *A standard autonomous acausal linear system is stationary iff for all* $k, l = 0, \cdots, n-1$.

$$CA^k[A, V^0]A^lB = 0, \tag{7.8a}$$

$$CA^k[A, V^1]A^lB = 0 \tag{7.8b}$$

where $[A, V^i] = AV^i - V^iA$, $i = 1, 2$.

Proof. Suppose the system is stationary; then

$$Ce^{A(t-t_0)}V^0 e^{A(t_0-s)}B = C e^{A(t+r-t_0)}V^0 e^{A(t_0-r-s)}B. \tag{7.9}$$

If we differentiate this with respect to r at $r = 0$ we obtain

$$0 = C e^{A(t-t_0)}[A, V^0] e^{A(t_0-s)}B. \tag{7.10}$$

Differentiation of this with respect to t and s one or more times yields (7.8). The steps are reversible, so the converse holds. Q.E.D.

COROLLARY 7.5. *Suppose Σ is a standard autonomous acausal system which is controllable and observable on* $[t_0, t_1]$. *Σ is stationary iff*

$$[A, V^0] = 0, \tag{7.11a}$$

$$[A, V^1] = 0. \tag{7.11b}$$

The Gohberg-Kaashoek phenomenon cannot happen for stationary weighting patterns; in other words, a minimal autonomous realization is also a minimal real analytic realization.

THEOREM 7.6. (i) *Suppose* Σ (1.1) *is a standard stationary realization of a stationary weighting pattern* $W(t-s)$. Σ *is a minimal stationary (equivalently, autonomous) realization iff*

(7.12a) $$\text{rank } \mathscr{C}^b = n,$$

(7.12b) $$\text{rank } \mathscr{O}^b = n,$$

(7.12c) $$\text{Kernel } \mathscr{O} \subseteq \text{Range } \mathscr{C}.$$

(ii) *A minimal stationary realization is also a minimal real analytic realization.*

(iii) *Any stationary realization can be modified and reduced to a minimal stationary realization.*

(iv) *Suppose* Σ *and* $\tilde{\Sigma}$ *are stationary minimal realizations of* $W(t-s)$ *then there exists an invertible constant matrix R such that*

(7.13a) $$(A - R^{-1}\tilde{A}R)\mathscr{C} = 0,$$

(7.13b) $$\mathscr{O}(A - R^{-1}\tilde{A}R) = 0,$$

(7.13c) $$\tilde{B} = RB,$$

(7.13d) $$\tilde{C} = CR^{-1},$$

(7.13e) $$\tilde{D} = D$$

and

(7.14a) $$\mathscr{O}(V^0 - R^{-1}\tilde{V}^0 R)\mathscr{C} = 0,$$

(7.14b) $$\mathscr{O}(V^1 - R^{-1}\tilde{V}^1 R)\mathscr{C} = 0.$$

On the other hand, if Σ *is a minimal stationary realization of* $W(t-s)$ *and* $\tilde{\Sigma}$ *is an autonomous system related to* Σ *by* (7.13) *and* (7.14) *for some invertible R then* $\tilde{\Sigma}$ *is also a minimal stationary realization of* $W(t-s)$.

Proof. (i) Suppose Σ is an autonomous realization of a stationary weighting pattern which satisfies (7.12); then by Proposition 7.1, Σ is controllable and observable off every $[\tau_0, \tau_1]$ and (6.1) is satisfied. By Theorem 6.1, Σ is a minimal real analytic realization and hence a minimal stationary realization.

Suppose Σ is an autonomous realization of a stationary weighting pattern $W(t-s)$ which does not satisfy (7.12). To show that Σ is not minimal we shall construct a new autonomous realization $\tilde{\Sigma}$ of smaller state dimension which does satisfy (7.12) and hence is minimal. As the reader has seen, the way one obtains a lower-dimensional realization of a weighting pattern is to find an appropriate subspace of the state space which is left invariant by the dynamics. One either restricts to this subspace or quotients by this subspace to reduce the dimension of the state space. In the context of real-analytic systems we have the luxury of making a time varying change of coordinates so that the invariant subspaces are time invariant. In the context of autonomous systems we do not have this option.

The natural subspaces associated with an autonomous system (1.1) are formed from the matrices found in Proposition 7.1, e.g.,

(7.15a) $$\text{Range } \mathscr{C},$$

(7.15b) $$\text{Kernel } \mathscr{O},$$

(7.15c) $$\text{Range } \mathscr{C}^b,$$

(7.15d) $$\text{Kernel } \mathscr{O}^b.$$

The first and second are clearly invariant by definition.

(7.16a) $$A(\text{Range } \mathscr{C}) \subseteq \text{Range } \mathscr{C},$$

(7.16b) $$A(\text{Kernel } \mathscr{O}) \subseteq \text{Kernel } \mathscr{O}.$$

The third and fourth are generally not. It is this latter fact which seems to cause the Gohberg-Kaashoek phenomenon.

Even if the system (1.1) is stationary it is not always true that (7.15b), (7.15d) are A invariant. However, they are nearly so, for (7.8) implies that

(7.17a) $$A(\text{Range } \mathscr{C}^b) \subseteq \text{Range } \mathscr{C}^b + \text{Kernel } \mathscr{O},$$

(7.17b) $$A(\text{Kernel } \mathscr{O}^b \cap \text{Range } \mathscr{C}) \subseteq \text{Kernel } \mathscr{O}^b.$$

The reader with a background in geometric linear control theory recognizes (7.17a) as a form of (A, B) invariance (or controlled invariance) and (7.17b) as a form of (C, A) (or conditioned invariance). For details see [14] and [15].

Let D be a matrix such that

(7.18) $$\text{Range D} = \text{Kernel } \mathscr{O};$$

then by a standard lemma there exists a matrix F such that

(7.19) $$(A+DF) \text{ Range } \mathscr{C}^b \subseteq \text{Range } \mathscr{C}^b.$$

Moreover, since (7.16a) holds we can choose F so that

(7.20) $$\text{Kernel } F \supset \text{Range } \mathscr{C}.$$

Let $\tilde{A} = A + DF$; then (7.18) and (7.20) imply that

(7.21a) $$C\tilde{A}^k = CA^k,$$

(7.21b) $$\tilde{A}^k B = A^k B$$

for all k, so for all t, s

(7.22a) $$C\, e^{\tilde{A}t} = C\, e^{At},$$

(7.22b) $$e^{\tilde{A}s} B = e^{As} B.$$

Therefore we can modify Σ by replacing A by $\tilde{A}$ and not change the weighting pattern $W(t-s)$.

In this way we obtain another autonomous realization of $W(t-s)$ such that (7.15c) is $\tilde{A}$ invariant. By restricting this system to the subspace of the state space given by (7.15c) we obtain a smaller autonomous realization of $W(t-s)$ which satisfies (7.12a).

The property described by equation (7.17b) is called (C, A) invariance (or conditioned invariance). It is the dual of the property described by (7.17a). If we choose a matrix E such that

(7.23) $$\text{Range } \mathscr{C} = \text{Kernel } E,$$

then it is a standard exercise to show that there exists a matrix G such that

(7.24) $$(A+GE) \text{ Kernel } \mathscr{O}^b \subseteq \text{Kernel } \mathscr{O}^b.$$

Moreover, because of (7.16b) we can choose G such that

(7.25) $$\text{Range } G \subset \text{Kernel } \mathscr{O}.$$

If we define $\tilde{A} = A + GE$ then because of (7.23) and (7.25), (7.21) and (7.22) hold. We can replace A by $\tilde{A}$ without changing the weighting pattern. For this new realization (7.15d) is $\tilde{A}$ invariant and we can project it out. The resulting realization satisfies (7.12b).

In this way we obtain a realization satisfying (7.12a), (7.12b); in other words, one that is controllable and observable off every $[\tau_0, \tau_1]$. To reduce this to a realization satisfying (7.12c) we choose coordinates that respect the flag of subspaces

$$\begin{aligned}\mathbb{R}^n &\supseteq \text{Range } \mathscr{C} + \text{Kernel } \mathscr{O}\\ &\supseteq \text{Range } \mathscr{C}\\ &\supseteq \text{Range } \mathscr{C} \cap \text{Kernel } \mathscr{O} \supseteq 0.\end{aligned}$$

In other words, $x = (x_1, x_2, x_3, x_4)^*$ and

$$\begin{aligned}\text{Range } \mathscr{C} + \text{Kernel } \mathscr{O} &= \{x\colon x_1 = 0\},\\ \text{Range } \mathscr{C} &= \{x\colon x_1 = 0, x_2 = 0\},\\ \text{Range } \mathscr{C} \cap \text{Kernel } \mathscr{O} &= \{x\colon x_1 = 0, x_2 = 0, x_3 = 0\}.\end{aligned}$$

If we define $B(s) = e^{As}B$ and $C(t) = C\,e^{At}$ then in these coordinates

$$e^{As}B = B(s) = \begin{bmatrix} 0 \\ 0 \\ B_3(s) \\ B_4(s) \end{bmatrix}, \tag{7.26a}$$

$$C\,e^{At} = C(t) = [C_1(t) \;\; 0 \;\; C_3(t) \;\; 0], \tag{7.26b}$$

and

$$W(t-s) = \begin{cases} \sum_{i=1,3} \sum_{j=3,4} C_i(t-t_0)\,V^0_{ij}B_j(t_0 - s), & t > s, \\ -\sum_{i=1,3} \sum_{j=3,4} C_i(t-t_0)\,V^1_{ij}B_j(t_1 - s), & t < s. \end{cases} \tag{7.26c}$$

Hence we can delete the x_2 coordinate and the corresponding boundary input condition without changing $W(t-s)$. This completes the proof of statement (i).

(ii) By (i) a minimal stationary realization must satisfy (7.12). Hence by Theorem 6.1 and Proposition 7.1 it must also be a minimal real analytic realization.

(iii) In the proof of (i) we showed how a stationary realization can be modified and reduced to a realization satisfying (7.12). By Theorem 6.1 such a system is a minimal real analytic realization, hence a minimal stationary realization.

(iv) If Σ is a minimal stationary realization of $W(t-s)$ and $\tilde{\Sigma}$ is an autonomous realization related to Σ by (7.13) and (7.14), then it is easy to verify that $\tilde{\Sigma}$ realizes $W(t-s)$, hence is a minimal stationary realization.

If Σ and $\tilde{\Sigma}$ are two minimal stationary realizations of $W(t-s)$ then by Theorem 6.1 there exists a real analytic matrix valued function $R(t)$ satisfying (6.2) and (6.3). In particular (6.2b) implies that

$$\tilde{B} = R(t)B,$$

so $R(t)$ is constant on Range B. If we differentiate this expression using (6.2a) we obtain

$$\tilde{A}\tilde{B} = R(t)AB.$$

Further differentiations yield

$$\tilde{\mathscr{C}} = R(t)\mathscr{C}, \tag{7.27a}$$

$$0 = \dot{R}(t)\mathscr{C}. \tag{7.27b}$$

In a similar fashion repeated differentiations of (6.2c) yield

(7.28a) $$\tilde{\mathcal{O}} = \mathcal{O}R^{-1}(t),$$

(7.28b) $$0 = \mathcal{O}R^{-1}(t) = -\mathcal{O}\dot{R}^{-1}(t)R(t)R^{-1}(t).$$

Rewrite (6.2a) as

(7.29) $$A - R^{-1}(t)\tilde{A}R(t)R(t)\dot{R}(t)$$

and multiply by $\mathscr{C}$ on the right using (7.27b) to obain

(7.30a) $$(A - R^{-1}(t)\tilde{A}R(t))\mathscr{C} = 0.$$

We multiply (7.29) by $\mathcal{O}$ on the left and use (7.28b) to obtain

(7.30b) $$\mathcal{O}(A - R^{-1}(t)\tilde{A}R(t)) = 0.$$

If we let $R = R(t_0)$, a constant matrix, then (7.13) follows immediately. Moreover, (6.3a) implies (7.14a) and (7.27a) and (6.3b) imply (7.14b). Q.E.D.

Remark. While minimal stationary realizations are related by (7.13) and (7.14) for some constant matrix R, they can also be related by nonconstant matrices.

Example 7.7. Consider the time varying change of state coordinates $\tilde{x} = R(t)x$ for Example 5.6, where

$$R(t) = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}.$$

The new system is given by

$$\begin{aligned} &\dot{\tilde{x}}_1 = 0, \qquad \tilde{x}_2(0) + 2\tilde{x}_1(1) - \tilde{x}_2(1) = v_1, \\ &\dot{\tilde{x}}_2 = \tilde{x}_1 + u, \qquad \tilde{x}_2(0) = v_2, \\ &y = \tilde{x}_1. \end{aligned}$$

These two systems are also related by (7.13) and (7.14), where $R = I$.

Recall that an acausal system is *causal* if $V^0 = I$ and $V^1 = 0$ and *anticausal* if $V^0 = 0$ and $V^1 = I$. It is *strictly acausal* if both V^0 and V^1 are invertible.

PROPOSITION 7.8. *Suppose Σ is a standard stationary acausal system which is controllable and observable on $[t_0, t_1]$. The state and boundary space of Σ can be decomposed $x = (x_1, x_2, x_3)$, $v = (v_1, v_2, v_3)$ into a causal part x_1, anticausal part x_2, and an acausal part x_3.*

(7.31a) $$\dot{x}_1 = A_{11}x_1 + B_1u,$$

(7.31b) $$x_1(t_0) = v_1,$$

(7.32a) $$\mathring{x}_2 = A_{22}x_2 + B_2u,$$

(7.32b) $$x_2(t_1) = e^{A_{22}(t_1 - t_0)}v_2,$$

(7.33a) $$\dot{x}_3 = A_{31}x_1 + A_{32}x_2 + A_{33}x_3 + B_3u,$$

(7.33b) $$V^0_{33}x_3(t_0) + V^1_{33}x_3(t_1) = v_3 - V^0_{31}x_1(t_0) - V^0_{32}x_2(t_0) - V^1_{31}x_1(t_1) - V^1_{32}x_2(t_1).$$

Proof. Consider the flag of subspaces

$$\mathbb{R}^m \subseteq \text{Range } V^1 \subseteq \text{Range } V^0 \cap \text{Range } V^0 \subseteq 0$$

and choose state coordinates that respect this flag $x = (x_1, x_2, x_3)$. In other words,

$$\begin{aligned} &\text{Range } V^1 = \{x \colon x_1 = 0\}, \\ &\text{Range } V^0 \cap \text{Range } V^1 = \{x \colon x_1 = 0, x_2 = 0\}, \\ &\text{Range } V^0 = \{x \colon x_2 = 0\}. \end{aligned}$$

We choose the same coordinates on the space of boundary input values v. Since A commutes with V^0 and V^1 it leaves their ranges invariant. Because $V^0 + V^1 e^{A(t_1-t_0)} = I$, V^0 and V^1 also commute hence leave their ranges invariant. This implies that

$$A = \begin{vmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{vmatrix},$$

$$V^0 = \begin{vmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ V^0_{31} & V^0_{32} & V^0_{33} \end{vmatrix},$$

$$V^1 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & e^{A_{22}(t_0-t_1)} & 0 \\ V^1_{31} & V^1_{32} & V^1_{33} \end{vmatrix}$$

where $V^0_{33} + V^1_{33} e^{A_{33}(t_1-t_0)} = I$. Q.E.D.

Consider the acausal part of the above system, assuming that the causal state coordinates $x_1(t)$ and anticausal state coordinates $x_2(t)$ are identically zero. This yields an acausal system

(7.34a) $$\dot{x}_3 = A_{33}x_3 + B_3 u,$$

(7.34b) $$V^0_{33}x_3(t_0) + V^1_{33}x_3(t_1) = v_3,$$

which can also be decomposed into causal, anticausal and acausal parts. The decomposition process can be repeated until the acausal part is strictly acausal. In this way the original system can be decomposed into causal and anticausal parts which feed into causal and anticausal parts through the state differential equations and boundary conditions. The pattern may be repeated several times until it terminates in a strictly acausal system.

The boundary condition (1.1b) of a stationary system is said to be *separable* if Range $V^0 \cap$ Range $V^1 = 0$.

COROLLARY 7.9. *Suppose Σ is a standard stationary acausal system which is controllable and observable on $[t_0, t_1[$. If the boundary condition is separable then Σ separates into independent causal and anticausal systems.*

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