Proceed. 26th CDC '87
LOS ANGELES
pp 519-523

HIGHER ORDER LINEAR APPROXIMATIONS TO NONLINEAR CONTROL SYSTEMS*

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Abstract: One traditional approach in the analysis and design of nonlinear control systems is a first order approximation by a linear system. A new approach is to use nonlinear change of coordinates and feedback to construct linear approximations that are accurate to second and higher orders. However, the algebraic calculations required to obtain these aproximations are somewhat lengthy. In this paper, the theoretical framework for finding such change of coordinates for a nonlinear system are described. A software package that symbolically solves these transformations is currently being prepared.

1. Introduction

There is no general method for dealing with all nonlinear systems because nonlinear differential equations are virtually devoid of a general method of attack. A well-known and straightforward way to analyze nonlinear control systems is to obtain a linear approximation of the plant dynamics around a nominal operating point and design a feedback law for the resulting linear system. If the nonlinearities are strong, this approximation is valid for only a limited range of the operating regime, and performance degradation or loss of stability of the control system may occur as the system moves away from the nominal point. Then it may be necessary to repeat the linearization and design a new controller for the updated linear representation. This process is repeated as often as necessary, as dictated by the nonlinearities in the plant.

Another approach is to feed some nonlinear correction terms into the linearized plant model to compensate for the inaccuracies involved in the approximations. However, it is usually not straightforward to find such correction terms. Poincaré's theory of normal forms produces a fruitful technique for transforming a nonlinear vector field to a simpler form in the neighborhood of an equilibrium point. Another method employed in robotics is the cancellation of all the nonlinear terms by feedback. Alternatively, with the method of linearizing transformations one seeks a change of coordinates and state feedback to transform the nonlinear system into a linear one. Various forms of this question have been addressed by Brockett [8], Hunt and Su [3], Jakubczyk and Respondek[4], Sommer [9], and Krener [1,5]. The concepts on which the present paper is based, and the necessary and sufficient conditions for the existence of a solution, have been treated by Krener in [2].

The method proposed is to find a nonlinear change of coordinates for a nonlinear system to construct a linear approximation of the plant dynamics accurate to second or higher order. Based on these more accurate approximations one should be able to design controllers that give improved performance over a wider range of operating conditions. The computations required to calculate these transformations are somewhat complicated. As suggested in [2], this difficulty may be overcome with the aid of a symbolic algebraic computation package. The goal of this paper is to describe the theoretical framework for finding the required transformations.

2. Linearizing Transformations

Let us consider a nonlinear system in which the control u enters the dynamics in a linear fashion:

$$\dot{x} = f(x) + g(x)u \tag{1a}$$

$$x(0) = x^{\circ}. \tag{1b}$$

where $x \in \Re^n$ and $u \in \Re^m$. The system is assumed to be at rest at the nominal operating point (x°; $u^{\circ} = 0$). For brevity of the expressions we will assume $x^{\circ} = 0$. The calculations can be easily extended to the case $x^{\circ} \neq 0$. First, consider the linearization of (1) at x°:

$$\dot{x} = Ax + Bu \tag{2a}$$

$$A = \frac{\partial f}{\partial x}(0), \;\; B = g(0). \tag{2b}$$

We will seek a coordinate change for (1) of the form identity plus

* Research supported by the Air Force Office of Scientific Research under Grant No. 85-0267.

higher order terms, such that the resulting linear plant will agree with (1) up to an error of order $O(x,u)^{p+1}$ (i.e. terms of $O(x)^{p+1}$ and $O(x)^p u$) where p is the degree of approximation. Obviously, Eqn. (2) results when p = 1. In the following, the case for p = 2 will be derived and the results will be generalized to any arbitrary order p by induction.

We assume a transformation of the form:

$$x = z + \phi^{(2)}(z) \tag{3}$$

where z denotes the new set of coordinates. $\phi^{(2)}$ is a polynomial of degree 2, the monomial coefficients of which are to be found.

The time derivative of (3) yields:

$$\dot{x} = \dot{z} + \frac{\partial \phi^{(2)}(z)}{\partial z}\dot{z} \tag{4}$$

We solve for the differential equation in the new coordinates z:

$$\dot{z} = \left(I + \frac{\partial \phi^{(2)}(z)}{\partial z}\right)^{-1} \dot{x} \tag{5}$$

To evaluate (5), the functions f(x) and g(x) are expanded in a Taylor series, and (3) is introduced:

$$\begin{aligned} f(x) &= f^{(1)}(x) + f^{(2)}(x) + O(x)^3 \\ &= f^{(1)}(z + \phi^{(2)}(z)) + f^{(2)}(z) + O(z)^3 \\ &= Az + A\phi^{(2)}(z) + f^{(2)}(z) + O(z)^3 \end{aligned} \tag{6}$$

$$\begin{aligned} g(x) &= g^{(0)}(x) + g^{(1)}(x) + O(x)^2 \\ &= B + g^{(1)}(z) + O(z)^2 \end{aligned} \tag{7}$$

The term $(I + \frac{\partial \phi^{(2)}}{\partial z})^{-1}$ in (5) is expanded in a series around z = 0 as:

$$\left(I + \frac{\partial \phi^{(2)}}{\partial z}\right)^{-1} = \left(I - \frac{\partial \phi^{(2)}}{\partial z} + \left(\frac{\partial \phi^{(2)}}{\partial z}\right)^2 - \dots \right). \tag{8}$$

Then, combining (6), (7), and (8) in (5) and expanding we get:

$$\begin{aligned} \dot{z} = {} & Az + Bu + A\phi^{(2)}(z) + f^{(2)}(z) - \frac{\partial \phi^{(2)}}{\partial z}Az + g^{(1)}(z)u - \frac{\partial \phi^{(2)}}{\partial z}Bu \\ & + O(z,u)^3 \end{aligned} \tag{9}$$

Now we introduce some notation. The Lie bracket of two vector fields is another vector field defined by:

$$[f,g] = \frac{\partial g}{\partial x}f - \frac{\partial f}{\partial x}g$$

So (9) can be written as:

$$\begin{aligned} \dot{z} = {} & Az + Bu + f^{(2)}(z) - [Az,\phi^{(2)}(z)] + g^{(1)}(z)u - [Bu,\phi^{(2)}(z)] \\ & + O(z,u)^3 \end{aligned} \tag{10}$$

With the following choice for $\phi^{(2)}$ all the second order terms of (10) will vanish and the approximation will be accurate to second order:

$$f^{(2)}(z) = [Az,\phi^{(2)}(z)] \tag{11a}$$

$$g^{(1)}(z)u = [Bu,\phi^{(2)}(z)] \tag{11b}$$

which must hold for all constant u. Eqn. (11a) is called the *Homological Equation* [6]. A solution to (11) has to be found by using the freedom in the choice of u; we use a feedback of the following form [2]:

$$u = \alpha^{(2)}(x) + (I + \beta^{(1)}(x))v \tag{12}$$

where $\alpha^{(2)}(x)$ is an m×1 vector of order 2 polynomials, I is an m×m identity matrix, and $\beta^{(1)}(x)$ is an m×m matrix of first order terms. A new input in the linearized coordinates is designated as v. Note that v agrees with u to first order. With the introduction of this feedback, f and g of Eqn. (1) are redefined:

$$\tilde{f}(x) = f(x) + g(x)\alpha^{(2)}(x) \tag{13a}$$

$$\tilde{g}(x) = g(x) + g(x)\beta^{(1)}(x) \tag{13b}$$

The Taylor expansion of (13) yields

$$\tilde{f}(x) = Ax + B\alpha^{(2)}(x) + f^{(2)}(x) + O^3(x) \tag{14a}$$

$$\tilde{g}(x) = B + g^{(1)}(x) + B\beta^{(1)}(x) + O^2(x) \tag{14b}$$

and

$$\tilde{f}^{(2)}(x) = B\alpha^{(2)}(x) + f^{(2)}(x) \tag{15a}$$

$$\tilde{g}^{(1)}(x) = g^{(1)}(x) + B\beta^{(1)}(x) \tag{15b}$$

Reiterating the steps of Eqns. (3) through (11) we find:

$$\tilde{f}^{(2)}(z) = [Az,\phi^{(2)}(z)] \tag{16a}$$

$$\tilde{g}^{(1)}(z) = [Bu,\phi^{(2)}(z)] \tag{16b}$$

The distinction between Eqns. (11) and (16) is seen when (16) is rewritten as:

$$f^{(2)}(z) = -B\alpha^{(2)}(z) + [Az,\phi^{(2)}(z)] \tag{17a}$$

$$g^{(1)}(z)v = -B\beta^{(1)}(z)v + [Bv,\phi^{(2)}(z)] \tag{17b}$$

In the generalized homological equation of (17), the second order terms $f^{(2)}(z)$ and $g^{(1)}(z)v$ can be cancelled out under certain solvability conditions by proper choice of $\phi^{(2)}(z)$, $\alpha^{(2)}(z)$, and $\beta^{(1)}(z)$. For this second order linearization we have a system of $n^2(n+1)/2 + n^2m$ linear algebraic equations in $n^2(n+1)/2 + mn(n+1)/2 + m^2n$ unknowns. When a solution can be found, the resulting system becomes:

$$\dot{z} = Az + Bv + O(z,v)^3 \tag{18}$$

In order to find an approximation of the next higher order, we rewrite (18) by reverting to the variables x and u:

$$\dot{x} = Ax + Bu + O(x,u)^3 \tag{18'}$$

Now we are asuming that in the given nonlinear system the second order terms have been already been canceled as outlined above. Then we seek a new transformation of the form:

$$x = z + \phi^{(3)}(z) \tag{19}$$

Note that transformation (19) will not introduce any terms of degree less than 3. Then the same procedure outlined above is repeated, with the feedback:

$$u = \alpha^{(3)}(x) + (I + \beta^{(2)}(x))v \tag{20}$$

which results in:

$$f^{(3)}(z) = -B\alpha^{(3)}(z) + [Az,\phi^{(3)}(z)] \tag{21a}$$

$$g^{(2)}(z)v = -B\beta^{(2)}(z)v + [Bv,\phi^{(3)}(z)] \tag{21b}$$

These results can be generalized as follows. Given a system which is accurate to only order p-1, i. e.

$$\dot{x} = Ax + Bu + O(x,u)^p \tag{22}$$

a coordinate change is sought as:

$$x = z + \phi^{(p)}(z) \tag{23}$$

along with feedback:

$$u = \alpha^{(p)}(x) + (I + \beta^{(p-1)}(x))v \tag{24}$$

which yields the homological equation to be solved:

$$f^{(p)}(z) = -B\alpha^{(p)}(z) + [Az,\phi^{(p)}(z)] \tag{25a}$$

$$g^{(p-1)}(z)v = -B\beta^{(p-1)}(z)v + [Bv,\phi^{(p)}(z)] \tag{25b}$$

In (25), $\phi^{(p)}$, $f^{(p)}$, $\alpha^{(p)}$, $g^{(p-1)}$ and $\beta^{(p-1)}$ are, respectively, homogeneous vector fields of orders corresponding to their superscripts. The resulting system is accurate up to order p:

$$\dot{z} = Az + Bv + O(z,v)^{p+1} \tag{26}$$

3. Linearizing Transformations for Systems with Small Parameters

In this section, we consider a control system of the form:

$$\dot{x} = f(x,\varepsilon) + g(x,\varepsilon)u \tag{27a}$$

$$x(0) = x^{*}. \tag{27b}$$

where ε is a small parameter that characterizes the way parasitic effects or disturbances enter into the system. We will develop a method of linearizing transformation for this type of system, similar to that of Section 2. First, (27) is expanded as follows:

$$\dot{x} = Ax + Bu + \varepsilon(f^{(1)}(x) + g^{(1)}(x)u) + O(\varepsilon)^2 \tag{28}$$

In (28), the nonlinear function is expanded and grouped in powers of ε. Thus, the superscripts of f and g now correspond to the powers of ε these functions multiply, in contrast with the notation of Section 2. A coordinate change is assumed of the following form:

$$x = z + \varepsilon\phi^{(1)}(z) \tag{29}$$

where the form and the polynomial order of the function $\phi^{(1)}(z)$ is not determined yet. Repeating the calculations similar to the steps of Eqns. (4) through (9) of Section 2 yields:

$$\dot{z} = Az + Bu + \varepsilon(f^{(1)}(z) - [Az,\phi^{(1)}(z)] + g^{(1)}(z)u - [Bu,\phi^{(1)}(z)]) + O(\varepsilon)^2 \tag{30}$$

An input for the control system of Eqn. (28) is chosen as:

$$u = v + \varepsilon(\alpha^{(1)}(x) + \beta^{(1)}(x)v) \tag{31}$$

After a sequence of calculations similar to Eqns. (13) through (17), the homological equations are found:

$$f^{(1)}(z) = -B\alpha^{(1)}(z) + [Az,\phi^{(1)}(z)] \tag{32a}$$

$$g^{(1)}(z)v = -B\beta^{(1)}(z)v + [Bv,\phi^{(1)}(z)] \tag{32b}$$

This result can be generalized for an arbitrary power of ε in the same fashion: A solution to

$$f^{(p)}(z) = -B\alpha^{(p)}(z) + [Az,\phi^{(p)}(z)] \tag{33a}$$

$$g^{(p)}(z)v = -B\beta^{(p)}(z)v + [Bv,\phi^{(p)}(z)] \tag{33b}$$

will yield

$$\dot{z} = Az + Bv + O(\varepsilon)^{p+1} \tag{34}$$

Even though Eqns. (33) and (25) look very similar, there are some fundamental differences. All the variables in Eqn. (33) have different definitions than those of Eqn. (25), as mentioned at the beginning of this section. Moreover, the solvability conditions of (33) are not the same as the conditions of Eqn. (25). Actually, both (32) and (33) may represent an infinite family of equations as opposed to the finite dimensional set of expressions that arise from (25).

Any nonlinear system expressed in the form of in Eqn. (1) can always be transformed into the form of (27) as follows: First, consider the expansion of (1) as

$$\dot{x} = Ax + f^{(2)}(x) + Bu + g^{(1)}(x)u + O(x,u)^3 \tag{35}$$

Scale the coordinates and the input with:

$$\xi = \varepsilon^{-1}x$$

$$\mu = \varepsilon^{-1}u$$

introducing the above into (35) yields

$$\dot{\xi} = A\xi + B\mu + \varepsilon(\bar{f}^{(2)}(\xi) + \bar{g}^{(1)}(\xi)\mu) + O(\varepsilon)^2 \tag{36}$$

This equation is of the form of Eqn. (28), except for the difference in the way expansions of f and g are defined. We use the overbar notation to emphasize this point. The input

$$u = \alpha^{(2)}(x) + (I + \beta^{(1)}(x))v \tag{12}$$

is also transformed with an additional scaling $\eta = \varepsilon^{-1}v$:

$$\mu = \eta + \varepsilon(\bar{\alpha}^{(2)}(\xi) + \bar{\beta}^{(1)}(\xi)\eta) \tag{37}$$

With this scaling of coordinates, a linearization problem given as in Section 2 can be alternatively solved with the procedure outlined in this section.

4. Form of the Nonlinear Compensation

After a higher order linearization is obtained, the next step is to choose a feedback law to achieve closed–loop pole assignment. Consider the approximation of Section 2 where

$$\dot{x} = Ax + Bu + O(x,u)^p \tag{22}$$

has been transformed by the coordinate change

$$x = z + \phi^{(p)}(z) \tag{23}$$

and feedback

$$u = \alpha^{(p)}(x) + (I + \beta^{(p-1)}(x))v \tag{24}$$

into

$$\dot{z} = Az + Bv + O(z,v)^{p+1} \tag{26}$$

A closed–loop pole assignment can be made with state feedback of the form

$$v = Fz + r \tag{38}$$

where r is an open loop control. The approximation of (26) then becomes

$$\dot{z} = (A + BF)z + Br. \tag{39}$$

Notice that (23) agrees with the identity transformation up to order p–1, so it is easily inverted at least up to order p:

$$z = x - \phi^{(p)}(x) \tag{40}$$

The input u of Eqn. (24) in the original coordinate becomes, with the aid of (22), (38), and (40):

$$\begin{aligned} u &= \alpha^{(p)}(x) + (I + \beta^{(p-1)}(x))(F(x - \phi^{(p)}(x)) + r) \\ &= Fx + r + \{\alpha^{(p)}(x) + \beta^{(p-1)}(x)(Fx - F\phi^{(p)}(x) + r) \\ &\quad - F\phi^{(p)}(x)\}. \end{aligned} \tag{41}$$

Thus the control function has the form of a pole assignment for the linear part of (22) plus some correction terms of higher order (grouped in the bracket of Eqn. (41)). This result clearly shows the purpose and nature of the nonlinear feedback.

6. Conclusion

In this paper we have presented an alternative approach to the analysis and design of nonlinear control systems. The procedure consists of finding a coordinate change by an appropriate feedback to achieve higher order linear approximations to nonlinear systems. Because of space limitations, we have not presented the details of the solvability conditions. The method of solving for the linearizing transformations is based on the normal forms approach of Poincaré, which is a widely used technique in the analysis of bifurcations in nonlinear vector fields. This suggests the applicability of these powerful bifurcation methods in nonlinear control systems analysis. Aeyels [10] and Abed and Fu [11] have studied the local stabilization problem for nonlinear systems with this approach. In other words, the method is an appropriate tool for the analysis of nonlinear systems in which plant parameter variations cause fundamental changes in the structure of the system. Another important issue is the following: When a solution exists, the functions $\alpha^{(p)}(x)$, $\beta^{(p-1)}(x)$, $\phi^{(p)}(x)$ are not necessarily unique. The question of what is the best choice, or even what is a reasonable choice among the possible solutions needs more investigation.

The equations that need to be solved for finding the transformations are a set of linear algebraic equations. However, the number of equations grow rapidly with increasing orders of linearization and with higher dimensional systems.For example, for a second order linearization and with n states and m inputs we have a system of $n^2(n + 1)/2 + n^2m$ linear algebraic equations in $n^2(n + 1)/2 + mn(n + 1)/2 + m^2n$ unknowns. With the use of symbolic algebraic manipulation packages and with the availibility of increasingly powerful computers, this is not considered as a serious setback. A symbolic algebra program that automatically solves these transformations on the computer is in preparation.

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