REALIZATIONS OF RECIPROCAL PROCESSES

Arthur J. Krener*
Department of Mathematics and
the Institute of Theoretical Dynamics
University of California
Davis, CA 95616
USA

Abstract We review the concept of a reciprocal process and show that a stationary Gaussian reciprocal process, which satisfies a certain technical assumption, can be realized by a linear stochastic differential equation with independent initial condition.

*Research supported in part by NSF under DMS-8601635.

1. Reciprocal Processes. Suppose x(t) is an n vector valued stochastic process where t ranges over a subset of the reals or the integers. The process x(t) is called reciprocal (or quasi-Markov) if given any $\tau_0 \le \tau_1$ the values of the process within $[\tau_0,\tau_1]$ are independent of the values of the process outside of $[\tau_0,\tau_1]$ conditioned on $x(\tau_0)$ and $x(\tau_1)$.

In particular a Gaussian process x(t) is reciprocal if for any $t_1,\ldots, t_k \le \tau_0 \le s_1,\ldots,s_m \le \tau_1 \le t_{k+1},\ldots,t_l$ we have for $i = 1,\ldots,l$

$$E(x(t_i)|x(\tau_0),x(\tau_1)) \tag{1.1a}$$
$$= E(x(t_i)|x(\tau_0),x(\tau_1),x(s_1),\ldots,x(s_m))$$

and for $j = 1,\ldots,m$

$$E(x(s_j)|x(\tau_0),x(\tau_1))$$

(1.1b)

$$= E(x(s_j)|x(\tau_0),x(\tau_1),x(t_1),\ldots x(t_l)).$$

This definition was formulated by Serge Bernstein [1] as a generalization of the concept of a Markov process. Recall that a process x(t) is <u>Markov</u> if for any τ_0 the values of the process to the left of τ_0 are independent of the values to the right conditioned on $x(\tau_0)$. A Gaussian process x(t) is Markov if for any $t_1,\ldots,t_k \le \tau_0 \le s_1,\ldots,s_m$ we have for $i = 1,\ldots,k$

(1.2a) $$E(x(t_i)|x(\tau_0)) = E(x(t_i)|x(\tau_0),x(s_1),\ldots x(s_m))$$

and for $j = 1,\ldots m$

(1.2b) $$E(x(s_j)|x(\tau_0)) = E(x(s_0)|x(\tau_0),x(t_1),\ldots,x(t_k)).$$

It is easy to see that Markov processes are reciprocal but the converse is not true. Throughout this paper we will restrict our attention to zero-mean Gaussian processes, and often we shall further restrict our attention to stationary zero mean Gaussian processes. Because of the zero-mean Gaussian assumption, all the probablistic information about the process x(t) is contained in its covariance

(1.3) $$R_x(t,s) = E(x(t)x^*(s))$$

where * denotes transpose. This is a nxn matrix valued function.

A process x(t) is <u>nonsingular</u> <u>of</u> <u>order</u> <u>one</u> if $R_x(\tau_0,\tau_0)$ is nonsingular for every τ_0. Such a process is Markov iff its covariance satisfies

$$(1.4) \quad R_x(t,s) = R_x(t,\tau_0)R_x^{-1}(\tau_0,\tau_0)R_x(\tau_0,s)$$

for any $t \le \tau_0 \le s$.

Let τ denote the ordered k-tuple $(\tau_0,\ldots,\tau_{k-1})$ where $\tau_0 \le \tau_1 \le\ldots\le \tau_k$. Define an $k\cdot n$ dimensional random vector $X(\tau)$ by

$$(1.5) \quad X(\tau) = \begin{pmatrix} x(\tau_0) \\ \cdot \\ \cdot \\ \cdot \\ x(\tau_{k-1}) \end{pmatrix}$$

A process $x(t)$ is nonsingular of order k if for any $\tau_0 < \tau_1 <\ldots< \tau_{k-1}$ the covariance of the random vector $X(\tau)$ is positive definite.

Suppose $x(t)$ is nonsingular of order 2, then $x(t)$ is reciprocal iff its covariance satisfies

$$(1.6) \quad R_x(t,s) =$$

$$[R_x(t,\tau_0)\ R_x(t,\tau_1)] \begin{bmatrix} R_x(\tau_0,\tau_0) & R_x(\tau_0,\tau_1) \\ R_x(\tau_1,\tau_0) & R_x(\tau_1,\tau_1) \end{bmatrix}^{-1} \begin{bmatrix} R_x(\tau_0,s) \\ R_x(\tau_1,s) \end{bmatrix}$$

for all $t \le \tau_0 \le s \le \tau_1$ and for all $\tau_0 \le s \le \tau_1 \le t$.

If $\tau = (\tau_0,\tau_1)$ and $\sigma = (\sigma_0,\sigma_1)$ then we can define a partial ordering by $\tau \ge \sigma$ if $\tau_0 \le \sigma_0 \le \sigma_1 \le \tau_1$. Let $X(\tau)$ be defined by (1.5). The process $x(t)$ is reciprocal if the process $X(\tau)$ is Markov relative to this partial ordering.

Mehr and McFadden [2] noted that reciprocal processes are <u>conditionally Markov</u>. If we condition an $x(\tau_1)$ then the conditional process is Markov to the left of τ_1 and if we condition on $x(\tau_0)$ the conditional process is Markov to the right of τ_0.

2. Examples. We review the classification of all one dimensional, stationary, Gaussian, reciprocal processes where $t \in \mathbb{R}$. This is due to Jamison [3], Chay [4] and Carmichael-Masse-Theodorescu [5]. Essentially there are only six families of such processes,

1a. Ornstein Uhlenbeck Processes
1b. Cosh Processes
1c. Sinh Processes
2. Slepian Processes
3a. Cosine Processes
3b. Shifted Cosine Processes

The Ornstein Uhlenbeck processes are the only ones that are Markov. They have covariance $R_x(t,s) = R_x(t-s)$ given by

$$R_x(t) = e^{-|At|} R_x(0) \tag{2.1a}$$

Such processes have an infinite lifetime, i.e. they can be defined for all $t \in \mathbb{R}$. Of course one can restrict t to lie in some proper subset of $\mathbb{R}$. If $A = 0$ then the process is constant with respect to t and hence singular of order two. If $R_x(0) = 0$ then the process is identically zero and singular of order one. Otherwise the process is nonsingular of every order $k \geq 0$.

The remaining one dimensional stationary Gaussian reciprocal processes are not Markov. A Cosh process has covariance

$$R_x(t) = \frac{\cosh A(T/2-t)}{\cosh AT/2} R_x(0) \tag{2.1b}$$

where A, T > 0. A Cosh process has a finite lifetime because any covariance must satisfy the Cauchy-Schwartz inequality, $|R_x(t)| \le R_x(0)$. But $R_x(t)$ given by (2.1b) violates this for t > T. Since $R_x(T) = R_x(0)$, it is a <u>cyclic process</u>, $x(0) = x(T)$ a.s.

A <u>Sinh process</u> has covariance

$$R_x(t) = \frac{\sinh A(T/2-t)}{\sinh AT/2} R_x(0) \tag{2.1c}$$

where A, T > 0. It also has a finite lifetime of length at most T. Since $R_x(T) = -R_x(0)$, it is an <u>anticyclic process</u> $x(0) = -x(T)$ a.s.

A <u>Slepian process</u> has covariance of the form

$$R_x(t) = (1-2t/T)\, R_x(0) \tag{2.2}$$

where T > 0. Again it has a finite lifetime of length at most T. It also is anticyclic, $x(0) = -x(T)$ a.s.

A <u>Cosine process</u> has covariance

$$R_x(t) = (\cos At)\, R_x(0). \tag{2.3a}$$

It has an infinite lifetime.

Since $R_x(t)$ is periodic with period $2T = 2\pi/A$, the process is also periodic $x(t) = x(t+2T)$ a.s. Furthermore, it is antiperiodic $x(t) = -x(t+T)$ a.s.

A Shifted Cosine process has covariance of the form

$$(2.3b) \qquad R_x(t) = \frac{\cos A(t+t_0)}{\cos At_0} R_x(0)$$

where $0 < t_0 < \pi/2A$. It has a finite maximum lifetime $T = \pi/A-2t_0$ and it is anticyclic, $x(0) = x(-T)$ a.s.

The Cosh, Sinh, Slepian, Cosine and Shifted Cosine processes are all nonsingular of order two on any interval of length less than T. Since in each case, $x(t) = \pm\, x(t+T)$, they are singular of order two on intervals of length T. All of the above processes except for the Cosine processes are nonsingular of arbitrary order on any interval of length less than T. A Cosine process is singular of order 3. This means that the behavior of such a process is completely determined by its values $x(\tau_0)$ and $x(\tau_1)$ at two times where $\tau_1-\tau_0$ is not an integer multiple of T.

3. Realization Theory. It is well known [6] that if R(t) is a continuous covariance of stationary Gauss Markov process then R(t) is C^∞ and it satisfies a first order linear differential equation

$$(3.1) \qquad \dot{R}(t) = AR(t)$$

for $t \geq 0$. Furthermore if B is an nxn matrix such that $BB^* = -(\dot{R}(0)+\dot{R}^*(0))$ then the process $x(t)$ defined for $t \geq 0$ by the stochastic differential equation

$$(3.2a) \qquad dx = A\,x\,dt + Bdw$$

$$(3.2b) \qquad x(0) \simeq N(0,R(0)),$$

where w is standard m dimensional Wiener process independent of $x(0)$, has covariance $R(t)$. Note $\dot{R}(0) = \dot{R}(0^+)$.

In this section we shall show that certain continuous stationary Gaussian reciprocal covariances can be realized by second order linear stochastic differential equations driven by white Gaussian noise with independent initial conditions. This partially confirms a conjecture of ours made in [7].

The first step is to show that a continuous stationary Gaussian covariance $R(t,s) = R(t-s)$ must be C^∞. We did this in [7] but we shall repeat the proof here. We assume $R(t)$ is defined for $|t| \leq T$ and is nonsingular of order two for $|t| < T$.

For a stationary reciprocal covariance, (1.6) becomes

$$(3.3) \qquad R(t-s) =$$

$$[R(t-\tau_0)\ R(t-\tau_1)] \begin{bmatrix} R(0) & R(\tau_0-\tau_1) \\ R(\tau_1-\tau_0) & R(0) \end{bmatrix}^{-1} \begin{bmatrix} R(\tau_0-s) \\ R(\tau_1-s) \end{bmatrix}$$

and this holds for $\tau_0 \le s \le \tau_1$ and either $t \le \tau_0$ or $t \ge \tau_1$. Assume that $\tau_1 - \tau_0 < T$. If we integrate with respect to t over $[t_0 - \delta, \tau_0]$ where $0 < \delta < T - \tau_1 + \tau_0$, we obtain

$$\int_{\tau_0-\delta}^{\tau_0} R(t-s)dt = \int_{\tau_0-\delta-s}^{\tau_0-s} R(t)dt = [\delta I + \sigma(\delta) \quad \sigma(\delta)] \begin{bmatrix} R(\tau_0-s) \\ R(\tau_1-s) \end{bmatrix}$$

where $\sigma(\delta)/\delta \to 0$ as $\delta \to 0$. If we integrate (3.1) with respect to t over $[\tau_1,\tau_1+\delta]$ we obtain

$$\int_{\tau_1}^{\tau_1+\delta} R(t-s)dt = \int_{\tau_1-s}^{\tau_1+\delta-s} R(t)dt = [\sigma(\delta) \quad \delta I + \sigma(\delta)] \begin{bmatrix} R(\tau_0-s) \\ R(\tau_1-s) \end{bmatrix}$$

Putting these together we have

$$(3.4) \quad \begin{bmatrix} \int_{\tau_0-\delta-s}^{\tau_0-s} R(t)dt \\ \int_{\tau_1-s}^{\tau_1+\delta-s} R(t)dt \end{bmatrix} \begin{bmatrix} \delta I + \sigma(\delta) & \sigma(\delta) \\ \sigma(\delta) & \delta I + \sigma(\delta) \end{bmatrix} \begin{bmatrix} R(\tau_0-s) \\ R(\tau_1-s) \end{bmatrix}$$

Since $R(t)$ is C^0, the left side of (3.4) in C^1 in s for $s \epsilon [\tau_0,\tau_1]$. By this we mean the left (right) derivative exists and is continuous at $\tau_0(\tau_1)$. For sufficiently small $\delta > 0$ the first matrix on the right is invertible hence we conclude that $R(\tau_0-s)$ and $R(\tau_1-s)$ are C^1 in $s \epsilon [\tau_0,\tau_1]$. Since τ_0 and τ_1 are arbitrary except that $0 < \tau_1 - \tau_0 < T$ we conclude $R(t)$ is C^1 on $[0,T]$. By repeating the argument we conclude $R(t)$ is C^∞ on $[0,T)$. Since $R(-t) = R^*(t)$ it follows that $R(t)$ is also C^∞ on $(-T,0]$. By continuity $R(0^+) = R(0^-)$. The left and right derivatives need not agree at 0, instead $-\dot{R}(0^-) = \dot{R}^*(0^+)$, $\ddot{R}(0^-) = \ddot{R}^*(0^+)$, etc. Henceforth we shall take $\dot{R}(0)$ as $\dot{R}(0^+)$, $\ddot{R}(0)$ as $\ddot{R}(0^+)$, etc.

The next step is to show that $R(t)$ satisfies two second order matrix differential equations. Let $\tau_0 = s - \sigma$ and $\tau_1 = s + \sigma$ for $\sigma > 0$ then (3.3) becomes

$$(3.5a) \qquad R(t-s) = [R(t-s+\sigma) \;\; R(t-s-\sigma)] \begin{bmatrix} H_1(\sigma) \\ H_2(\sigma) \end{bmatrix}$$

where $H_1(\sigma)$ and $H_2(\sigma)$ are determined for $\sigma > 0$ by

$$(3.5b) \qquad \begin{bmatrix} R(0) & R^*(2\sigma) \\ R(2\sigma) & R(0) \end{bmatrix} \begin{bmatrix} H_1(\sigma) \\ H_2(\sigma) \end{bmatrix} = \begin{bmatrix} R^*(\sigma) \\ R(\sigma) \end{bmatrix}$$

since $R(t)$ is assumed to be the covariance of a process which is nonsingular of order two. For convenience, we make a change of coordinates, $x_{new}(t) = R^{-1/2}(0)x_{old}(t)$ and thereby normalize $R(0) = I$. We would like to study the limit of $H_1(\sigma)$ and $H_2(\sigma)$ and their derivatives as $\sigma \to 0$. From (3.5b) we obtain for $\sigma > 0$

$$(3.6a) \qquad H_1(\sigma) = G^{-1}(\sigma) \; F(\sigma)$$

$$(3.6b) \qquad H_2(\sigma) = R(\sigma) - R(2\sigma) \; H_1(\sigma)$$

where $F(\sigma)$ and $G(\sigma)$ are C^∞ for $\sigma \geq 0$ and given by

(3.6b) $$F(\sigma) = R^*(\sigma) - R^*(2\sigma)\, R(\sigma)$$

(3.6c) $$G(\sigma) = I - R^*(2\sigma)\, R(2\sigma).$$

Since $F(0) = G(0) = 0$, (3.6a) is indeterminate at $\sigma = 0$. We define

$$\bar{F}(\sigma) = \begin{cases} F(\sigma)/\sigma & \sigma > 0 \\ \dot{F}(0) & \sigma = 0 \end{cases}$$

$$\bar{G}(\sigma) = \begin{cases} G(\sigma)/\sigma & \sigma > 0 \\ \dot{G}(0) & \sigma = 0 \end{cases}.$$

By repeated application of L'Hopital's rule it is easy to verify that $\bar{F}(\sigma)$ and $\bar{G}(\sigma)$ are C^∞ for $\sigma \geq 0$.

Henceforth we shall invoke the assumption that

(3.7) $$\bar{G}(0) = \dot{G}(0) = -2(\dot{R}(0) + \dot{R}^*(0))$$

is invertible. Rewriting (3.6a) we have for $\sigma > 0$

$$H_1(\sigma) = (G(\sigma)/\sigma)^{-1}(F(\sigma)/\sigma) = \bar{G}(\sigma)/\bar{F}(\sigma)$$

and hence $H_1(\sigma)$ has a C^∞ extension to $\sigma \geq 0$. Equation (3.6b) defines a C^∞ extension of $H_2(\sigma)$ to $\sigma \geq 0$. By straightforward differentiation of (3.6) we obtain

(3.8a) $H_1(0) = H_2(0) = \frac{1}{2} I$

(3.8b) $\dot{H}_1(0) = -\dot{H}_2(0) = \frac{-1}{4} (\dot{R}(0)+\dot{R}^*(0))^{-1}(\ddot{R}(0)-\ddot{R}^*(0))$

(3.8c) $\ddot{H}_1(0) + \ddot{H}_2(0) = -\ddot{R}(0) - 4\dot{R}(0)\dot{H}_1(0)$.

We return to (3.5a) at $s = 0$ and differentiate twice with respect to σ at $\sigma = 0$ to obtain

$$
\begin{aligned}
0 = {} & \ddot{R}(t)(H_1(0)+H_2(0)) \\
& + 2\dot{R}(t)(\dot{H}_1(0)-\dot{H}_2(0)) \\
& + R(t)(\ddot{H}_1(0)+\ddot{H}_2(0))
\end{aligned}
$$

By utilizing (3.8) we obtain

(3.9a) $\ddot{R}(t) = -2\dot{R}(t)M^* + 2R(t)\, N^*$

where

(3.10a) $-2M^* = (\dot{R}(0) + \dot{R}^*(0))^{-1} (\ddot{R}(0) - \ddot{R}^*(0))$

(3.10b) $2N^* = \ddot{R}(0) + 2\dot{R}(0)M^*$

Equation (3.5a) is valid both for $t \geq s + \sigma$ and for $t \leq s + \sigma$. Since $s = \sigma = 0$ this implies that (3.9a) is valid for $t \epsilon [0,T)$ and for $t \epsilon (-T,0]$. The covariance $R(t) = R^*(-t)$ so $\dot{R}(t) = -\dot{R}^*(-t)$ and $\ddot{R}(t) = \ddot{R}^*(-t)$.

We transpose (3.9a) and substitute to obtain

$$(3.9b) \qquad \ddot{R}(t) = 2M\dot{R}(t) + 2NR(t).$$

By adding and subtracting (3.9a,b) we obtain the following.

Theorem 1 Suppose $R(t)$ is the continuous covariance of a stationary Gaussian reciprocal process defined on $[0,T]$ and (WLOG) $R(0) = I$. Suppose that $\dot{R}(0) + \dot{R}^*(0)$ is invertible. Then $R(t)$ is C^∞ on $[0,T)$ and satisfies the differential equation

$$(3.11a) \qquad \ddot{R}(t) = M\dot{R}(t) - \dot{R}(t)M^* + NR(t) + R(t)N^*$$

and the side constraint

$$(3.11b) \qquad 0 = M\dot{R}(t) + \dot{R}(t)M^* + NR(t) - R(t)N^*$$

where M,N are defined by (3.10).

We now construct a process $y(t)$ which realizes the stationary Gaussian reciprocal covariance $R(t)$, under the assumption that $\dot{R}(0) + \dot{R}^*(0)$ is invertible. By the Cauchy-Schwartz inequality $R(0) - R^*(\sigma)R(\sigma)$ is monotone increasing for small $\sigma > 0$ hence $\dot{R}(0) + \dot{R}^*(0)$ is nonpositive definite. Since it is assumed to be invertible, it is negative definite and there exists an invertible nxn matrix B, such that

$$(3.12) \qquad B_1B_1^* = -(\dot{R}(0) + \dot{R}^*(0))$$

Let N and M be as above (3.10). Define a nxn symmetric matrix $\pi(t)$ as the solution of the matrix Riccati differential equation

$$(3.13a) \qquad \frac{d\pi}{dt} = 2N\dot{R}^*(0) + 2\dot{R}(0)N^*$$

$$+ 2M\pi(t) + 2\pi(t)M^*$$

$$+ (\ddot{R}(0) + \pi(t))(B_1B_1^*)^{-1}(\ddot{R}^*(0) + \pi(t))$$

$$(3.13b) \qquad \pi(0) = \dot{R}(0)\dot{R}^*(0)$$

Let $B_2(t)$ be an nxn matrix defined by

$$(3.14) \qquad B_2(t) = -(\ddot{R}(0) + \pi(t))B_1^{*-1}$$

Finally we define a 2n dimensional process $x(t) = (x_1(t), x_2(t))$ by the stochastic differential equation

$$(3.15a) \qquad \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ 2N & 2M \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} dt + \begin{pmatrix} B_1 \\ B_2(t) \end{pmatrix} dw$$

$$(3.15b) \qquad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} R(0) \\ \dot{R}(0) \end{pmatrix} v \qquad\qquad v \simeq N(0,I)$$

satisfies (3.16b,c). From (3.16a) we obtain for $T > t \geq s \geq 0$,

$$(3.18b) \qquad \frac{\partial P_{11}}{\partial t}(t,s) = P_{21}(t,s)$$

$$(3.18c) \qquad \frac{\partial P_{21}}{\partial t}(t,s) = 2MP_{21}(t,s) + 2NP_{11}(t,s)$$

hence $P_{11}(t,s) = R(t-s)$ and $P_{21}(t,s) = \dot{R}(t-s)$. We have proved the following.

Theorem 2 Suppose $R(t)$ is the nxn continuous covariance of a stationary Gaussian reciprocal process defined on $[0,T]$ and (WLOG) $R(0) = I$. Suppose $\dot{R}(0) + \dot{R}^*(0)$ is invertible. Then $R(t)$ can be realized by a first order 2n dimensional linear stochastic differential equation (3.15a) driven by n dimensional white Gaussian noise with an independent initial condition (3.15b).

4. Conclusion In Sections One and Two we defined and gave examples of reciprocal processes. In Section Three we showed how certain stationary Gaussian reciprocal processes can be realized via stochastic differential equations. The condition that we required was that $\dot{R}(0) + \dot{R}^*(0)$ be invertible, but we believe that this technical condition can be dropped. We hope to prove this in the near future.

where w is an n dimensional standard Wiener process independent of v.

Let

$$P(t,s) = \begin{bmatrix} P_{11}(t,s) & P_{12}(t,s) \\ P_{21}(t,s) & P_{22}(t,s) \end{bmatrix}$$

then P(t,s) satisfies for $T > t \geq s \geq 0$

$$\text{(3.16b)} \qquad \frac{\partial P}{\partial t}(t,s) = AP(t,s)$$

$$\text{(3.16b)} \qquad \frac{d}{dt} P(t,t) = AP(t,t) + P(t,t)A^* + B(t)B^*(t)$$

and

$$\text{(3.16c)} \qquad P(0,0) = \begin{pmatrix} R(0) & \dot{R}^*(0) \\ \dot{R}(0) & \dot{R}(0)\dot{R}^*(0) \end{pmatrix}$$

where

$$A = \begin{bmatrix} 0 & I \\ 2N & 2M \end{bmatrix} \qquad B(t) = \begin{bmatrix} B_1 \\ B_2(t) \end{bmatrix}$$

It is straightforward to verify that

$$\text{(3.17a)} \qquad P(t,t) = \begin{bmatrix} R(0) & \dot{R}^*(0) \\ \dot{R}(0) & \pi(t) \end{bmatrix}$$

REFERENCES

1. Bernstein, S., Sur les laisons entre les grandeurs aleatoires, Proc. of Int. Cong. of Math., Zurich (1932) 288-309.

2. Mehr, C. B. and McFadden, J. A., Certain properties of Gaussian processes and their first passage times, J. Roy. Stat. Soc. B27 (1965) 505-522.

3. Jamison, D., Reciprocal processes: the stationary Gaussian case, Ann. Math. Stat. 41 (1970) 1624-1630.

4. Chay, S. C., On quasi-Markov random fields, J. Multivar. Anal. 2 (1972) 14-76.

5. Carmichael, J-P., Masse, T-C. and Theodorescu, R., Processus gaussiens stationaires reciproques sur un intervalle, C. R. Acad. Sc. Paris, Serie I 295 (1982) 291-293.

6. Doob, J. L., Stochastic Processes (Wiley, N.Y. 1953).

7. Krener, A. J., Reciprocal Processes and the Stochastic Realization Problem for Acausal Systems, in Modeling, Identification and Robust Control, C. I. Byrnes and A. Lindquist, eds., North-Holland, Amsterdam, 1986, 197-211.