Analysis and Control of Nonlinear Systems
C.I. Byrnes, C.F. Martin and R.E. Saeks (editors)



# Reciprocal Processes, Second Order Stochastic Differential Equations and PDE's of Conservation and Balance

Arthur J. Krener
Department of Mathematics
and
Institute of Theoretical Dynamics
University of California
Davis, CA 95616

## 1. INTRODUCTION

The close connection between Markov processes, diffusions and parabolic partial differential equations is of course well–known. In this paper we shall describe the beginning of a new theory which links reciprocal processes, second order diffusions and the partial differential equations of fluid mechanics, i.e., the continuity, Euler and energy balance equations.

## 2. RECIPROCAL PROCESSES

In the early thirties E. Schrödinger [2,3] introduced a new class of stochastic processes in attempt to formalize the stochastic aspects of Quantum Mechanics. This concept was formalized by S. Bernstein [1] in an address to the International Congress of Mathematicians in Zurich in 1932. Bernstein defined a reciprocal process $x(t)$ as one where conditioned on the values $x(t_0)$ and $x(t_1)$ of the process at two times $t_0 \leq t_1$, the process exterior to $[t_0,t_1]$ is independent of the process interior to $[t_0,t_1]$. This is readily seen to be a generalization of the Markov property, i.e., conditioned on single time $t_0$ the process before $t_0$ is independent of the process after $t_0$. Hence every Markov process is reciprocal but the converse is not true.

The reciprocal property is the specialization to one dimension of P. Levy's definition of a Markov random field [20]. There are two other ways of viewing the reciprocal property. Suppose $x(t)$ is a random process taking values in $\mathbb{R}^n$ and defined for $t \in [0,T]$. We define another process $X(t_0,t_1) = (x(t_0),x(t_1))$ taking values in $\mathbb{R}^{2n}$. We view this process as parametrized by pairs $(t_0,t_1)$ where $t_0 \leq t_1$ or equivalently by subintervals $(t_0,t_1)$. Subintervals are partially ordered by inclusion. It is easy to see

*Research supported in part by NSF under DMS–8601635 and AFOSR under NM85–0267.

that the original process $x(t)$ is reciprocal iff the two time process $X(t_0,t_1)$ is Markov relative to this partial ordering.

Alternatively we can view a reciprocal process as being conditionally Markov in the following sense. Given any $t_0 \in [0,T]$ and $x^0 \in \mathbb{R}^n$, we define a conditioned process $\tilde{x}(t\,|\,t_0,x^0)$ consisting of all sample paths of $x(t)$ satisfying $x(t_0) = x^0$ with the conditional probabilty measure. The process $x(t)$ is said to be conditionally Markov if every $\tilde{x}(t\,|\,t_0,x^0)$ is Markov for $t \in [0,t_0]$ and is also Markov for $t \in [t_0,T]$. (It need not be Markov on $[0,T]$.) It is straightforward to note that a process $x(t)$ is reciprocal iff it is conditionally Markov.

To essentially specify a stochastic process one must describe all finite dimensional distributions of the process, e.g., give the probability distribution of $x(t_0),x(t_1), \ldots ,x(t_n)$ where $0 \leq t_1 \leq \ldots \leq t_n \leq T$. One reason that Markov processes are so well–studied is that they are completely determined by only two functions. The first is $\rho_0(x^0)$, the probability density of $x(0)$. (Throughout we assume that probability densities exist although the discussion can be easily extended using probability distributions.) The second $p(s,x;t,y)$ is the Markov transition density of $x(t) = y$ given that $x(s) = x$. By Bayes' formula the probability density of $x(t_1) = x^1, \ldots , x(t_n) = x^n$ where $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n \leq T$ is given by

$$\rho(t_1,x^1, \ldots , t_n,x^n) = \int \rho_0(x^0)\, p(0,x^0;t_1,x^1) \ldots p(t_{n-1},x^{n-1};t_n,x^n)\, dx^0.$$

A function $p(s,x;t,y)$ is a Markov transition density iff it satisfies the well–known Chapman–Kolmogorov relations, i.e.,

$$\int p(s,x;t,y)\, dy = 1$$

and

$$p(s,x;u,z) = \int_{\mathbb{R}^n} p(s,x;t,y)\, p(t,y;u,z)\, dy$$

where $0 \leq s \leq t \leq u \leq T$.

There is a similar development for reciprocal process due to Schrödinger [2] and Jamison [5]. A reciprocal process $x(t)$ is completely determined by the joint density $\rho_{0,T}(x^0,x^T)$ of the end points $x(0)$ and $x(T)$ and a reciprocal transition density $q(s,x;t,y;u,z)$. The latter is the probability density of $x(t) = y$ given that $x(s) = x$ and $x(u) = z$ where $0 \leq s \leq t \leq u \leq T$. The finite dimensional densities of $x(t)$ are then given by

$$\rho(t_1,x^1,\ldots,t_n,x^n) = \int \rho_{0,T}(x^0,x^T)\, q(t_0,x^0;t_1,x^1;T,x^T)$$

$$q(t_1,x^1;t_2,x^2;T,x^T)\ldots q(t_{n-1},x^{n-1};t_n,x^n;T,x^T)\, dx^0 dx^T$$

To be a reciprocal transition density, $q(s,x;t,y;u,z)$ must satisfy the Schrödinger–Jamison relations

$$\int q(s,x;t,y;u,z)\, dy = 1$$

and

$$q(r,w;s,x;u,z)\, q(s,x;t,y;u,z) = q(r,w;t,y;u,z)\, q(r,w;s,x;t,y)$$

where $0 \le r \le s \le t \le u \le T$ and $w,x,y,z \in \mathbb{R}^n$.

Suppose $x(t)$ is a reciprocal process and $X(t_0,t_1)$ is the associated two time process which is Markov relative to the inclusion partial ordering. One can show that the Chapman–Kolmogorov relations for the Markov transition density of $X(t_0,t_1)$ are equivalent to the Schrödinger–Jamison relations for the reciprocal transition density of $x(t)$.

Schrödinger realized that there is Bayesian way of constructing a reciprocal transition density q from a Markov transition density p,

$$q(s,x;t,y;u,z) = \frac{p(s,x;t,y)\, p(t,y;u,z)}{p(s,x;t,y)}$$

Of course the conditionally Markov property allows one to reverse the process and define a Markov transition density p from a reciprocal transition density q,

$$p(s,x;t,y) = q(s,x;t,y;T,x^T).$$

If we start with a reciprocal transition density q, which we use to define a Markov transition density p which we use to define another reciprocal density $\bar{q}$ then by the second Schrödinger–Jamison relation, $\bar{q} = q$. If we start with a Markov transition density p which we use to define a reciprocal transition density q which we use to define another Markov transition density $\bar{p}$, it does not follow that $\bar{p} = p$.

Schrödinger used a Markov transition density p to construct reciprocal transition density q. With this and an end point density $\rho_{0,T}$ he was able to construct reciprocal processes. Jamison [6] showed that the resulting reciprocal process is actually Markov iff the end point density satisfies

$$\rho_{0,T}(x^0,x^T) = \pi_0(x^0)\ \pi_T(x^T)\ p(0,x^0;T,x^T)$$

for some nonnegative functions $\pi_0(x^0)$ and $\pi_T(x^T)$.

Jamison [6] also studied one dimensional stationary Gaussian reciprocal processes. He showed that covariance r(t) of such a process must satisfy a second order linear differential equation

$$\frac{d^2}{dt^2} r = a\ r$$

where a is a constant. He then used this in an attempt to classify all such processes and this program was successfully completed by Chay [8] and Carmichael–Masse–Theodorescu [11].

The author became interested in reciprocal process through his study of acausal linear systems [14] driven by white noise and satisfying independent random boundary conditions of the form

$$dx = A(t)\ x\ dt + B(t)\ dw$$

$$v = V^0\ x(0) + V^1\ x(t).$$

Here x(t) is an n dimensional Gaussian process, w(t) is a standard n dimensional Wiener process and v is an n dimensional random vector independent of w(t). We assume that the above boundary value problem is well–posed so that the Green's matrix $\Gamma(t,s)$ exists. We can express the solution of the stochastic differential equation as

$$x(t) = \Phi(t,0)\ v + \int_0^T \Gamma(t,s)\ B(s)\ dw(s)$$

where the integral is a Wiener integral and $\Phi(t,s)$ is the fundamental matrix solution of $\dot{x} = Ax$. We have normalized so that $V^0 + V^1\ \Phi(T,0) = I$.

We have proved [14] that the solution of such a stochastic boundary value problem is a reciprocal process and we speculated that every Gaussian reciprocal process is the solution of such a stochastic boundary value problem. This conjecture was motivated by the fact that every Gaussian Markov process is the solution of a stochastic initial value problem, i.e., $V^0 = I$ and $V^1 = 0$. This conjecture is not true and this led us to discover a theory of reciprocal diffusions and stochastic differential equations of second order.

## 3. Diffusions

We recall the Feller postulates for a Markov diffusion x(t). First some notation, let

$x^*$ denote the transpose of a n dimensional column vector x and $x^{*2}$ the outer product of x with itself $x^{*2} = xx^*$, this is n x n matrix. The forward difference operator $d^+$ is defined by

$$d^+x(t;dt) = x(t+dt) - x(t)$$

where $dt > 0$ is a small positive quantity. Typically we suppress arguments as in $d^+x$. Conditional expectation given that $x(t) = x$ is denoted by

$$E_{x(t)}(\cdot) = E(\cdot \mid x(t) = x)$$

The symbol $O(dt)^k$ denotes a function of x,t and dt for which there exist $\epsilon,\delta>0$ such that if $dt < \delta$ then $|O(dt)| < \epsilon\, dt^k$ for all $x \in \mathbb{R}^n$ and $t \in (0,T)$. The symbol $o(dt)^k$ denotes function of x,t and dt which for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $dt < \delta$ then $|o(dt)^k| < \epsilon\, dt^k$.

A Markov process x(t) is a Markov diffusion if there exists n x 1 and n x m valued functions f(x,t) and g(x,t) such that

(MD1) $\quad \text{Prob}\{|x(t+dt) - x| > \epsilon \mid x(t) = x\} = O(dt)$

(MD2) $\quad E_{x(t)}(d^+x) = f(x,t)\,dt + o(dt)$

(MD3) $\quad E_{x(t)}(d^+x)^{*2} = (g(x,t))^{*2}\,dt + o(dt)$

(MD4) Third and higher centered conditional moments of dx vanish like o(dt).

The interpertation of these postulates is that conditioned on $x(t) = x$, the forward increment $d^+x$ of the process has a mean value approximately equal to f dt and variance approximately equal to $g^{*2}$ dt. In other words x(t) is mean differential but the individual sample paths are not for they have an extremely large standard deviation $O(dt)^{1/2}$.

From these postulates one can deduce that the density $\rho(x,t)$ of x(t) satisfies the Fokker–Plank equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho f_i) - \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}(\rho\, g_{ik} g_{jk}) = 0.$$

Moreover using the Ito stochastic integral we can realize x(t) as the solution of the stochastic differential equation

$$d^+x = f(x,t)\,dt + g(x,t)\,d^+w$$

$$x(0) = x^0.$$

We now sketch out the foundations of a parallel theory of reciprocal diffusions that we have recently developed. More details can be found in [21]. We need some more notation. We define the centered average, centered first difference and centered second difference as

$$\bar{x}(t;dt) = \frac{x(t+dt) + x(t-dt)}{2}$$

$$dx(t;dt) = \frac{x(t+dt) - x(t-dt)}{2}$$

$$d^2x(t;dt) = x(t+dt) - 2\,x(t) + x(t-dt)$$

Frequently we suppress argument as in $\bar{x}(t)$ or dx. We also introduce another conditional expectation operation

$$E_{\bar{x}(t)}(\cdot) = E(\,\cdot \mid \bar{x}(t;dt) = x)$$

A reciprocal process x(t) is a reciprocal diffusion if there exists n x 1 valued functions f(x,t) and u(x,t), n x n valued functions g(x,t) and $\pi(x,t)$ and n x m valued function h(x,t) such that

(RD1) $\mathrm{Prob}\{\, |x(t) - x| > \epsilon \mid \bar{x}(t;dt) = x \,\} = O(dt)$

(RD2) $E_{\bar{x}(t)}(dx) = u(x,t)\,dt + o(dt)$

(RD3) $E_{\bar{x}(t)}(d^2x) = (f(x,t) + g(x,t)\,u(x,t))\,dt^2 + o(dt)^2$

(RD4) $E_{\bar{x}(t)}(dx)^{*2} = \frac{1}{2}(h(x,t))^{*2}\,dt + \pi(x,t)\,dt^2 + o(dt)^2$

(RD5) $E_{\bar{x}(t)}(d^2x)^{*2} = 2\,(h(x,t))^{*2}\,dt + o(dt)^2$

(RD6) $E_{\bar{x}(t)}(d^2x\,dx^*) = \frac{1}{2}g(x,t)(h(x,t))^{*2}\,dt^2 + o(dt)^2$

(RD7) Third and higher joint centered conditional moments of dx and $d^2x$ vanish like $o(dt)^2$

Basically these postulates assert that the first and second joint conditional moments of dx and $d^2x$ exist and have the indicated expansions in power series in dt. They define the coefficients f,g,h,u and $\pi$ of the power series and they imply certain relation between these coefficients. These definitions and relations are as follows:

(i) The dt part of RD2 defines u.

(ii) The $dt^2$ part of RD3 defines f + g u.

(iii) The dt and $dt^2$ parts of RD4 defines $h^{*2}$ and $\pi$.

(iv) The $dt^2$ part of RD6 defines g $h^{*2}$.

(v) The dt part of RD3 vanishes.

(vi) The $dt^2$ part of RD5 vanishes and the dt part is four times the dt part of RD4.

Any process satisfying (RD1–6) is called a second order diffusion.

We refer to u as the mean velocity, $\rho$ u as the mean momentum, f + g u as the mean acceleration, h as the noise coefficient and $\rho\ \pi$ as the mean momentum flux of the process x(t). A related quantity $\rho\ \sigma = \rho(u\ u^* - \pi)$ is called the stress tensor. The reason for the terminology will become apparent in a moment.

A reciprocal diffusion satisfying RD1–7 is said to be a solution of the second order stochastic differential equation.

$$d^2x = f(x,t)\ dt^2 + g(x,t)\ dx\ dt + h(x,t)\ d^2w$$

where w(t) is a standard m dimensional Wiener process. This is a (partial) mnemonic for the above portulates. In particular applying $E_{\bar{x}(t)}(\cdot)$ we obtain RD3 from RD2 under the assumption that $d^2w$ is independent of $\bar{x}(t)$. Applying $E_{\bar{x}(t)}(\cdot)$ to $(d^2x)^{*2}$ yields RD5. Finally RD6 follows from applying $E_{\bar{x}(t)}(\cdot)$ to $d^2x\ dx^*$ using RD4.

To get a feeling for these axioms it is convenient to introduce another conditional expectation

$$E_{x(t\pm dt)}(\cdot) = E(\ |\ x(t\pm dt) = x\pm v\ dt\ )$$

Suppose x(t) is a reciprocal diffusion which also satisfies the stronger conditions.

(RD3*) $E_{x(t\pm dt)}(d^2x) = (f(x,t) + g(x,t)\ v)\ dt^2 + o(dt)^2$

(RD4*) $E_{x(t\pm dt)}(d^2x)^{*2} = 2(h(x,t))^{*2}\ dt + o(dt)^2$

(RD6) $E_{\bar{x}(t)}(d^2x\ dx^*) = \frac{1}{2}\ g(x,t)(h(x,t))^{*2}\ dt^2$
$+ f(x,t)\ u(x,t)^* + g(x,t)\ \pi(x,t)\ dt^3 + o(dt)^3$

Then x(t) is called a strongly reciprocal diffusion.

Conditioned on $x(t\pm dt) = x\pm v\ dt$ the mean sample path of the process over the time interval [t–dt, t+dt] traces out a parabola in (t,x) space passing through (t±dt, x ± v dt) and with second derivative equal to f(x,t) + g(x,t) v. Hence the mean path deviates from the straight line between (t±dt, x±v dt) by $O(dt)^2$. Compared to this, the standard deviation of sample paths from the mean path is very large, $O(dt)^{1/2}$.

Conditioning on $\bar{x}(t;dt) = x$ rather than x(t) = x is crucial to the above development. Even for very nice processes, such as an Ornstein Uhlenbeck process, the quantity

$E_{x(t)}(d^2x)$ is $O(dt)$ rather than $O(dt)^2$. In the stochastic mechanics of Nelson [18] the dt part of this quantity is twice the osmotic velocity. Nelson's current velocity, the dt part of $E_{x(t)}(dx)$ is generally equal to our mean velocity $u(x,t)$ from (RD2).

The first question that comes to mind is "Are there any reciprocal diffusions?". In [21] we showed that answer is decidely yes. In particular we showed that any reciprocal Gaussian process with smooth covariance $R(t,s)$ satisfying certain technical conditions is a strongly reciprocal diffusion. This includes such Markov processes. For a Gaussian reciprocal process the second order stochastic differential is linear of the form

$$d^2x = F(t)\, x\, dt^2 + G(t)\, dx\, dt + H(t)\, d^2w$$

where

$$f(x,t) = F(t)\, x = \left(\frac{\partial^2 R}{\partial t^2}(t,t) - G(t)\frac{\partial R}{\partial t}(t,t)\right) x$$

$$g(x,t) = G(t) = \left(\frac{\partial^2 R}{\partial t^2}(t,t) - \frac{\partial^2 R^*}{\partial s^2}(t,t)\right)\left(\frac{\partial R}{\partial t}(t,t) - \frac{\partial R^*}{\partial s}(t,t)\right)^{-1}$$

$$(h(x,t))^{*2} = (H(t))^{*2} = -\left(\frac{\partial R}{\partial t}(t,t) - \frac{\partial R^*}{\partial s}(t,t)\right)$$

The principle technical conditions are that $R(t,t) = I$ and $H(t)^{*2}$ is invertible. The other quantities $u(x,t)$ and $\pi(x,t)$ are given by

$$u(x,t) = U(t)\, x = \frac{1}{2}\left(\frac{\partial R}{\partial t}(t,t) + \frac{\partial R^*}{\partial s}(t,t)\right) x$$

$$\pi(x,t) = u(x,t)\, u^*(x,t) - \sigma(x,t)$$

$$\sigma(x,t) = -\frac{1}{2}\left(\frac{\partial^2 R}{\partial t\, \partial s}(t,t) + \frac{\partial^2 R^*}{\partial t\, \partial s}(t,t)\right) + U(t)\, U^*(t).$$

All of the above evaluations are at $s = t^-$

Suppose $x(t)$ is Gaussian process and a solution of the first order stochastic boundary value problem

$$dx = A(t)\, x\, dt + B(t)\, dw$$

$$v = V^0 x(0) + V^1 x(t)$$

in the sense defined above using the Green's matrix. Assume $R(t,t) = I$ and $B(t)$ is invertible. Then $x(t)$ is a reciprocal diffusion satisfying the second order linear stochastic differential equation above with

$$H(t) = B(t)$$
$$G(t) = -(A^2(t) - A^{*2}(t) + \dot{A}(t) - \dot{A}^*(t))\,(B(t)\,B^*(t))^{-1}$$
$$F(t) = A^2(t) + \dot{A}(t) - G(t)\,A(t).$$

Because of the complexity of these relations, it is possible for a process to satisfy a relatively simple first order equation and a relatively complicated second order equation or vice versa. The latter is the case for the Brownian Bridge or pinned Wiener process $x(t)$ which satisfies the first order equation

$$d^+x = \frac{-1}{(1-t)}\,x\,dt + d^+w$$

$$x(0) = 0$$

and the second order equation

$$d^2x = d^2w$$

$$x(0) = x(1) = 0.$$

The density $\rho$ of a Markov diffusion satisfies the Fokker–Plank equation. For a strongly reciprocal diffusion the density $\rho$, mean momentum $\rho\, u$ and mean momentum flux $\rho\, \pi$ satisfy at least in a weak sense a system of hyperbolic conservation laws similar to the continuity, Euler and kinetic energy balance equations of fluid mechanics. They are

$$\frac{\partial}{\partial t}\rho = -\frac{\partial}{\partial x_k}(\rho\, u_k)$$

$$\frac{\partial}{\partial t}(\rho\, u_i) = \rho\,(f + g\,u)_i - \frac{\partial}{\partial x_k}(\rho\, \pi_{ik})$$

and

$$\frac{\partial}{\partial t}(\rho\, \pi_{ij}) = \rho\,(f\,u^* + uf^* + g\pi + \pi g^*)_{ij}$$

$$-\frac{\partial}{\partial x_k}(\rho\,(u_i\,u_j\,u_k - \sigma_{ij}\,u_k - \sigma_{ik}\,u_j - \sigma_{jk}\,u_i))$$

with summation on repeated indices understood. A second order or reciprocal diffusion need only satisfy the first two of these equations.

Suppose we consider a volume with boundary in x–space. If we integrate $\rho$ over this volume we obtain the probability measure of the volume. The first equation states that the time rate of change of the probability of the volume is equal to the flux of particles through the boundary due to the mean velocity.

If we integrate $\rho$ u over the volume, we obtain the total momentum in the volume. The second equation states that the time rate of change of momentum in the volume is equal to the forces acting on the particles in the volume plus the net flux of momentum through the boundary.

If we integrate $\rho\ \pi$ over the volume we obtain the total momentum flux in the volume. Physically this is somewhat hard to comprehend but for smooth processes the contraction $\frac{1}{2}\rho\ \pi_{ii}$ is the kinetic energy. Hence we view $\frac{1}{2}\rho\ \pi_{ij}$ as a tensor form of kinetic energy. More precisely, if $\lambda_i$ is a constant n vector then the scalar valued process $z(t) = \lambda_i x_i(t)$ has kinetic energy equal to $\frac{1}{2}\rho\ \pi_{ij}\ \lambda_i\lambda_j$. With this interpertation the third equation states that time rate of change of tensor kinetic energy in the volume is equal to the mean work done on the particles in the volume by the force $d^2x/dt^2$ acting through the distance dx plus the flux of tensor kinetic energy through the surface of the volume. This flux is due to mean tensor kinetic energy $\frac{1}{2}\rho\ u_i\ u_j$ (called internal energy) transported by mean velocity $u_k$, random tensor kinetic energy $-\frac{1}{2}\rho\ \sigma_{ij}$ transported by mean velocity $u_k$ and mixed random/mean kinetic energy transported by random velocity. The latter represented by the last two terms of the flux are usually described as viscosity or stress terms in fluid dynamics. They represent the transport of energy due to random jumps of particles between regions of differing mean velocity.

The third equation expresses kinetic energy balance at the standard time scale, i.e., the $dt^2$ part of $E_{\bar{x}(t)}\ (dx)^{*2}$. There is also a form of energy at a fast time scale, i.e., the dt part $E_{\bar{x}(t)}\ (dx)^{*2}$. We call this hyperkinetic energy and its balance is described by another conservation law

$$\frac{\partial}{\partial t}(\rho\ h\ h^*)_{ij} = \frac{\rho}{2}(g\ h\ h^* + h\ h^*\ g^*)_{ij}$$

$$-\frac{\partial}{\partial x_k}(\rho\ (h\ \ h^*)_{ij}\ u_k)$$

which is also satisfied by second order diffusions.

Notice that the first three equations can be viewed independently of this last. We chose the name "hyperkinetic" to suggest a hyperkinetic child sitting at his school desk whose endless fidgeting is to no net effect (except possibly on his teacher).

In [21] we formally derived the four conservation laws from the postulates of a strongly reciprocal diffusion. Although they can be thought of in physical terms, they are not consequences of physical principles or assumptions. We verified that these conservation laws are satisfied by the reciprocal Gaussian processes discussed above.

REFERENCES

[1] Bernstein, S., Sur les laisons entre les grandeurs aleatoires, Proc. of Int. Cong. of Math., Zurich (1932) 288–309.

[2] Schrödinger, E., Uber die Umkehrung der Naturgesetze, Sitz. Ber der Preuss. Akad. Wissen., Berlin Phys. Math. 144 (1931).

[3] Schrödinger, E., Theorie relativiste de l'electron et l'interpretation de la mechanique quantique, Ann. Inst. H. Poincare 2, 269–310 (1932).

[4] Slepian, D., First passage time for a particular Gaussian process, Ann. Math. Statist. 32, 610–612 (1961).

[5] Jamison, B., Reciprocal Processes, Z. Wahrsch. Gebiete 30 (1974) 65–86.

[6] Jamison, B., The Markov Processes of Schrödinger, Z. Wahrsch. Gebiete 32 (1975) 323–331.

[7] Jamison, D., Reciprocal processes: the stationary Gaussian case, Ann. Math. Stat. 41 (1970) 1624–1630.

[8] Chay, S. C., On quasi–Markov random fields, J. Multivar. Anal. 2 (1972) 14–76.

[9] Abrahams, J. and Thomas, J. B., "Some comments on conditionally Markov and reciprocal Gaussian processes", IEEE Trans. Information Theory, vol. IT–27, 523–525, 1981.

[10] Adler, R. J., The geometry of random fields. New York: Wiley, 1981.

[11] Carmichael, J. P., Masse, J. C., and Theodorescu, R., "Processus gaussiens stationnaires reciproques sur un intervalle", C. R. Acad. Sci. Paris, Ser. I, vol. 295, 291–293, 1982.

[12] Carmichael, J. P., Masse, J. C., and Theodorescu, R., "Multivariate reciprocal stationary Gaussian processes", Preprint, Laval Univ., Dept. Math., Quebec, 1984.

[13] Carmichael, J. P., Masse, J. C., and Theodorescu, R., Representations for multivariate reciprocal Gaussian processes, Preprint, Laval Univ., Dept. Math., Quebec, 1986.

[14] Krener, A. J., Reciprocal Processes and the Stochastic Realization Problem for Acausal Systems, in Modeling, Identification and Robust Control, C. I. Byrnes and A. Lindquist, eds., North–Holland, Amsterdam, 1986, 197–211.

[15] Krener, A. J., Realizations of Reciprocal Processes, Proceedings IIASA Conf. on Modeling and Adaptive Control, Sopron, 1986.

[16] Minoshin, R. N., (1979). Second–order Markov and reciprocal stationary Gaussian processes. Theor. Prob. Appl. 24, 845–852.

[17] Abrahams, J. (1984). On Miroshin's second–order reciprocal processes. SIAM J. Appl. Math. 44, 190–192.

[18] Nelson, E., Quantum Fluctuations, Princeton Univ. Press, Princeton, NJ, 1985.

[19] Guerra, F. and Morato, L. M., Quantization of dynamical systems and stochastic control theory, Phys. Rev. D. 27 (1983) 1774–1786.

[20] Lévy, P., A special problem of Brownian motion and a general theory of Gaussian random functions, Proc. Third Berkeley Symp. Math Stat. and Prob. 2 (1956) 133–175.

[21] Krener, A. J., Reciprocal diffusions and stochastic differential equations of second order. Preprint, 1987. (Appendixed)

[22] Kushner, H. J., Necessary conditions for continuous parameter stochastic optimization problems, SIAM J. Control and Opt., 10 (1972) 550–565.

[23] Hausmann, U. G., On the stochastic maximum principle, SIAM J. Control and Opt., 19 (1978) 252–269.

[24] Bismut, J. M., Théorie probabiliste du contrôe des diffusions, Mem. AMS 4 (1976) No. 167.

[25] Imre Fényes, Eine wahrscheinlichkeitstheoretische Begründung und Interpretation der Quantenmechanik, Zeitschrift für Physik 132 (1952) 81–106.

[26] Kunio Yasue, Stochastic calculus of variations, J. of Functional Analysis 41 (1981) 327–340.

[27] Krener, A. J., The asymptotic approximation of nonlinear filters by linear filters. In Theory and Application of Nonlinear Control System, C. I. Byrnes and A. Lindquist, eds. North–Holland, Amsterdam (1986) 359–378.

[28] Frezza, R., S. Karahan, A. J. Krener and M. Hubbard. Application of an efficient nonlinear filter, Preprint, submitted for publication.

[29] Phelps, A. and A. J. Krener, Computation of observer normal form using MACSYMA, Preprint, submitted for publication.

[30] Krener, A. J., S. Karahan, M. Hubbard and R. Frezza, Higher order linear approximations to nonlinear control systems. Proceedings, IEEE Conf. on Decision and Control, Los Angeles (1987).

[31] Krener, A. J., Acausal realization theory, Part. 1; Linear deterministic systems, SIAM J. Control and Optimization, 25 (1987) 499–525.

[32] Krener, A. J., Reciprocal processes and the stochastic realization problem for acausal systems. In Modeling, Identification and Robust Control, C. I. Byrnes and A. Lindquist, eds. North–Holland, Amsterdam (1986) 197–210.

[33] Krener, A. J., Realizations of reciprocal processes, Preprint, submitted for publication.

[34] Krener, A. J., Reciprocal diffusions and stochastic differential of second order, Preprint, submitted for publication.