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THE STRUCTURE OF SMALL-TIME REACHABLE SETS IN LOW DIMENSIONS*

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Abstract. This paper outlines a general method to determine the geometric structure of small-time reachable sets for a single-input control system with a bounded linear control. The authors' analysis relies on free nilpotent systems as a guide, and hence their techniques only apply to nondegenerate situations. The paper illustrates the effectiveness of the method in low dimensions. Among other results is given a precise description of the small-time reachable set for a system $\dot{x}=f(x)+g(x)u$, $|u|\leqq 1$ in dimension four, under the generic assumption that the constant controls $u\equiv +1$ and $u\equiv -1$ are not singular. As a corollary, a local synthesis is obtained in dimension three for the time-optimal control problem under the analogous generic condition.

Key words. nonlinear systems, nilpotent approximation, reachable sets, bang-bang trajectories, singular arcs

AMS(MOS) subject classifications. 49B10, 93B10

1. Introduction. In this paper we study the qualitative structure of small-time reachable sets in low dimensions for a single-input system with a bounded linear control. More precisely, we consider a system of the form

$$\Sigma: \dot{x}=f(x)+g(x)u, \quad |u|\leqq 1, \quad x\in\mathbb{R}^n \tag{1}$$

where f and g are smooth (C^∞) or analytic vector fields and admissible controls are measurable functions with values in $[-1, 1]$ almost everywhere. A trajectory of the system corresponding to a control $u(\cdot)$ is an absolutely continuous curve $x(\cdot)$ such that $\dot{x}(t)=f(x(t))+g(x(t))u(t)$ almost everywhere. We say a point q is reachable from a point p within time T if and only if there exists a trajectory $x(\cdot)$ defined on an interval $[0, t]$, $t\leqq T$, such that $x(0)=p$ and $x(t)=q$. The set of all such points q is denoted by Reach $(p, \leqq T)$; Reach (p, T) denotes the set of points that are reachable exactly at time T. The reachable set from p, Reach (p), is the set of all points that are reachable from p within some time T.

Reachable sets play an important role in control theory. If a system can be stabilized to a given point by a feedback control law, then that point must be in the reachable set of every other point. In optimal control problems, if the cost is added as another coordinate, then the optimal trajectories must lie in the boundary of the set of reachable points. For this reason the Pontryagin Maximum Principle plays an important role in studying the boundaries of reachable sets.

The problem of describing a reachable set and the extremal trajectories that generate its boundary is closely related to the problem of regular synthesis in the sense of Boltyansky [1] and others [5], [18]. While the problem has been studied extensively for many years, only a few examples of regular syntheses have been described, for instance, [24]. Even in low dimensions, the reachable set of a general control system can be extremely complicated.

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We shall attempt to avoid this difficulty by considering only "nondegenerate" systems. By a nondegenerate system we mean one where (i) f, g, and the low-order Lie brackets of f and g span as many dimensions as is possible given the dimensions of the state space; and where (ii) no nontrivial equality relations hold between those vector fields (for instance, if n is the space dimension, then any relation saying that n vector fields are dependent at a point is considered a nontrivial equality relation, whereas a relation that simply expresses the fact that a vector field can be written in terms of a basis is considered trivial).

This is in the spirit of Lobry [14], who described the small-time reachable set of (1) in dimension three under the assumption that f, g, and $[f, g]$ are linearly independent. The method described below is an attempt to extend Lobry's result to higher dimensions. As will be seen, it is successful in the four-dimensional case, but in higher-dimensional cases obstacles still have to be overcome. These obstacles, however, are not due to our general approach, but they lie in the fact that, at the moment, too little is known about the structure of extremal trajectories. We shall return to this question at the end of the paper. In the paper we shall give a precise description of the small-time reachable set in dimension four assuming that the constant controls $u=+1$ and $u=-1$ are not singular on the boundary of the reachable set. It can easily be seen (cf. § 4) that this is equivalent to an independence assumption on the vector fields f, g, $[f, g]$, and $[f+g, [f, g]]$, respectively, $[f-g, [f, g]]$. As a corollary we are able to improve on recent results of Bressan [4], Schättler [17], and Sussmann [21] on time-optimal control in dimension three.

Throughout this paper we will use nilpotent systems as a guide to the general situation. A system is nilpotent of order k if all brackets of orders greater than k vanish and if k is the smallest integer with this property. In a certain sense these systems play the same role as the polynomials do within the class of smooth functions. Nilpotent systems are the low-order part of the coordinate free Taylor series expansion of a general system.

To be more precise, we must define the Lie jet of system (1). At a point p the Lie jet consists of a list of the values at p of the Lie brackets of f and g written down in some prescribed order. Of course, because of the skew-symmetry and Jacobi relation

$$[f, g]+[g, f]=0, \qquad [f, [g, h]]+[g, [h, f]]+[h, [f, g]]=0,$$

we need only consider a list of distinct brackets. These brackets can be partially ordered by the total number of vector fields involved; for example, f is a bracket of order one and $[f, g]$ is of order two. The Lie jet of order k is a list of values at p of the distinct brackets of f and g of order less than or equal to k. The Lie jets of orders one through four are given below:

Order one: $\{f(p), g(p)\}$,
Order two: $\{f(p), g(p), [f, g](p)\}$,
Order three: $\{f(p), g(p), [f, g](p), [f, [f, g]](p), [g, [f, g]](p)\}$,
Order four: $\{f(p), g(p), [f, g](p), [f, [f, g]](p), [g, [f, g]](p),$
$[f, [f, [f, g]]](p), [f, [g, [f, p]]](p), [g, [g, [f, g]]](p)\}$.

If $N(k)$ is the number of distinct brackets of f and g of order k or less, then the kth-order Lie jet of (1) at p is a point in the vector bundle consisting of the Whitney sum of $N(k)$ copies of the tangent bundle.

A basic result of Krener [12], later proved in other contexts by Rothschild and Stein [15], Hermes [10], Crouch [8], Bressan [3], and Sussmann [20], [21] is that for analytic systems of the form (1), the kth-order Lie jet at p determines the trajectories emanating from p up to order $O(t^{k+1})$ and up to diffeomorphisms of the state space.

Sussmann [22], [23], Bressan [4], and Schättler [16], [17] have shown that the local structure of time-optimal controls in dimension two or three is determined in nondegenerate situations by the second, respectively, third-order Lie jet at a reference point. In degenerate situations higher-order jets need to be considered [16], [17], [23].

On the basis of these results we might conjecture that in nondegenerate situations the kth-order Lie jet at p determines the structure of the set of small-time reachable points where the Hörmander or controllability condition is satisfied, i.e., the rank of the kth-order Lie jet at p equals the dimension of the state space. And maybe the qualitative structure of the reachable set can be obtained by looking at a kth-order nilpotent approximation. Unfortunately, as we mention in the last section, these conjectures are not completely true, but they do motivate much of our work.

The paper is organized as follows. The next section reviews the Pontryagin Maximum Principle as applied to the system (1). This also gives us a chance to introduce some notation and terminology. In § 3, we will describe the main ideas and outline the general structure of our techniques by looking at the trivial two-dimensional case. We will also give a brief proof of Lobry's three-dimensional result. The main part of the paper is § 4, where we determine the geometric structure of the small-time reachable set for the nondegenerate four-dimensional system (assuming that both quadruples $(f, g, [f, g], [f+g, [f, g]])$ and $(f, g, [f, g], [f-g, [f, g]])$ consist of independent vectors at p). We also draw the obvious corollaries about time-optimal control in dimension three. Section 5 concludes with a brief discussion of the free nilpotent five-dimensional system and explains why the general nondegenerate five-dimensional case is different from this one.

2. The maximum principle. The Maximum Principle [13] gives necessary conditions for a point to lie on the boundary of the reachable set. Let $u(\cdot)$ be an admissible control defined on an interval $[0, T]$ and let $x(\cdot)$ be the corresponding trajectory starting at p. If $x(T) \in \partial$ Reach (p), then $x(t) \in \partial$ Reach (p) for all $t \in [0, T]$ and there exists an absolutely continuous curve $\lambda : [0, T] \to \mathbb{R}^n$, which does not vanish anywhere such that

$$\dot{\lambda}(t)^T = -\lambda(t)^T (Df(x(t)) + Dg(x(t)) \cdot u(t)), \tag{2}$$

$$\langle \lambda(t), g(x(t))\rangle u(t) = \operatorname*{Min}_{|v| \leq 1} \langle \lambda(t), g(x(t))\rangle v, \tag{3}$$

$$H = \langle \lambda(t), f(x(t)) + g(x(t))u(t)\rangle \equiv 0 \tag{4}$$

almost everywhere on $[0, T]$. (We write vectors as columns, $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product on $\mathbb{R}^n$, and Df and Dg denote the Jacobian matrices of f and g, respectively.) Any trajectory for which an adjoint variable $\lambda(\cdot)$ exists such that (2)-(4) are satisfied is called an extremal trajectory. The optimality condition (3) determines the control $u(t)$ whenever $\phi(t) := \langle \lambda(t), g(x(t))\rangle \neq 0$; ϕ is called the switching function and $u \equiv -1$ ($u \equiv +1$) on intervals where ϕ is positive (negative). Trajectories corresponding to these constant controls are called bang arcs and are denoted by X $(=f-g)$ and Y $(=f+g)$, respectively. A concatenation of bang arcs is a *bang-bang trajectory*. Observe that $\langle \lambda(t), f(x(t))\rangle = 0$ at switching times t, i.e., where $\langle \lambda(t), g(x(t))\rangle = 0$. At these times (3) gives no information about the optimal control. If, however, ϕ vanishes on an open interval I, then all the derivatives of ϕ also vanish on I and this may determine the control u. We have

$$\dot{\phi}(t) = \langle \lambda(t), [f, g](x(t))\rangle,$$

$$\ddot{\phi}(t) = \langle \lambda(t), [f+gu, [f, g]](x(t))\rangle,$$

and if $\langle\lambda(t), [g, [f, g]](x(t))\rangle$ does not vanish on I, we can solve for u in $\ddot{\phi} = 0$ as follows:

$$u(t) = -\frac{\langle\lambda(t), [f, [f, g]](x(t))\rangle}{\langle\lambda(t), [g, [f, g]](x(t))\rangle}.$$

A control of this type is called singular and the corresponding trajectory is a *singular arc*.

This suggests that concatenations of bang and singular arcs are the natural candidates for trajectories in the boundary of the reachable set (but of course no such regularity statement can be drawn from the Maximum Principle alone). We denote concatenations of bang and singular arcs by the corresponding letter sequence; for instance, we simply write XSY for a concatenation of an X-arc, followed by a singular arc and a Y-trajectory, etc.

3. The main ideas of the technique: the nondegenerate two- and three-dimensional cases. In this section we analyze the (well-known) structure of small-time reachable sets in a nondegenerate situation in dimensions two and three. These cases are easy and give us an opportunity to outline the general ideas of our technique without getting preoccupied with technical details.

Suppose Σ is a system of the form (1) in dimension two and assume that f and g are independent at a reference point p (see Fig. 1). It is clear how the small-time reachable set from p will look. If we let Γ^+ (respectively, Γ^-) be the integral curves of the vector fields $f+g$ (respectively, $f-g$) for positive times, then for sufficiently small T, Reach $(p, \leqq T)$ is the union of Γ^+, Γ^-, and the open sector R between Γ^+ and Γ^- into which $f(p)$ points. It is easy to see that any point in R is reachable from p; for instance, if $q \in R$, just run a trajectory of Σ corresponding to the control $u \equiv +1$ backward in time until it hits Γ^-. The important point is that this is all of the small-time reachable set. This follows immediately from the Maximum Principle since only trajectories corresponding to the constant controls $u \equiv +1$ or $u \equiv -1$ can lie in the boundary of the reachable set. (There cannot be a junction, since then both $\langle\lambda(t), f(x(t))\rangle$ and $\langle\lambda(t), g(x(t))\rangle$ vanish, contradicting the nontriviality of λ.)

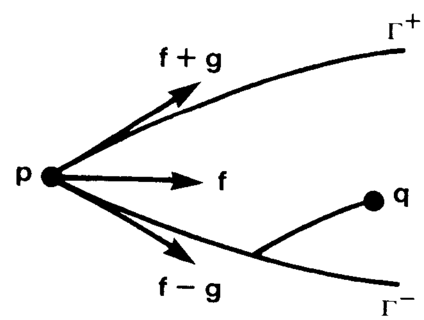

FIG. 1

Generalized to higher dimensions, the quintessence of this argument is to have two hypersurfaces Γ^* and Γ_* which are generated by extremal trajectories, have a common relative boundary and "enclose" a region R. Then, to prove that R is actually the reachable set Reach $(p, \leqq T)$, we must show (i) trajectories cannot leave R through Γ^* or Γ_*, and (ii) all points in the sector are reachable. The latter is immediate if we have a drift vector field f with $f(p) \neq 0$. This is exactly the same argument as in the two-dimensional case. Take any point q inside R and run a trajectory of Σ corresponding to the control $u \equiv 0$ (or for that matter corresponding to any control) backward in time. Since $f(p) \neq 0$, this trajectory will hit Γ^* or Γ_*. So basically (i) must be checked; this is mostly a matter of computing tangent spaces, as will be shown below. This is the general strategy of our technique.

All technical issues left aside for a moment, the key question is how to come up with the surfaces Γ^* and Γ_*. We propose an inductive procedure. Let us explain it at the next step, which is the case of a three-dimensional system Σ, where we assume that f, g, and $[f, g]$ are independent at a reference point p. (This is the example considered by Lobry [14].)

Choose coordinates $x = (x_1, x_2, x_3)$ such that $\langle dx, (f(p), g(p), [f, g](p)) \rangle = \mathrm{Id}$, the identity matrix. The projection of Σ into the (x_1, x_2)-plane is then the two-dimensional system considered above and we know the structure of its small-time reachable set. Our aim is to find two hypersurfaces Γ^* and Γ_* consisting of extremal trajectories that project onto the reachable set $\tilde{R}$ of the two-dimensional system in dimension three. If Γ^* and Γ_* have a common relative boundary that projects onto $\partial\tilde{R}$ and if Γ^* and Γ_* do not intersect in their relative interior, then it is clear that these surfaces "enclose" a region R. Then we must check whether trajectories can leave R. If this is impossible, R is the small-time reachable set.

The Maximum Principle gives preliminary information about Γ^* and Γ_* because it describes necessary conditions for trajectories to lie in the boundary of the reachable set. In this three-dimensional case it actually determines Γ^* and Γ_* precisely, but in higher dimensions this is no longer true. It is then that we will use nilpotent systems as our guide to find candidates for Γ^* and Γ_*. More on that appears in § 4.

Now that we have outlined the general approach, let us also illustrate the basic technical arguments by reproving Lobry's result. It follows from the Maximum Principle that all trajectories that lie on the boundary of the reachable set are bang-bang. For, if the switching function vanishes at some t, i.e., if $\langle \lambda(t), g(x(t)) \rangle = 0$, then also $\langle \lambda(t), f(x(t)) \rangle = 0$, and hence $\dot{\phi}_\Gamma(t) = \langle \lambda(t), [f, g](x(t)) \rangle$ cannot vanish by the independence of f, g, and $[f, g]$ and the nontriviality of λ. For dimensionality reasons it is therefore reasonable to consider the following two surfaces as candidates for Γ^* and Γ_*:

$$\Gamma^* = \{p \exp(s_1(f-g)) \exp(s_2(f+g)) \colon s_i \geqq 0,\ s_1 + s_2 \text{ small}\},$$

$$\Gamma_* = \{p \exp(t_1(f+g)) \exp(t_2(f-g)) \colon t_i \geqq 0,\ t_1 + t_2 \text{ small}\}.$$

We write flows of vector fields as exponentials and we let the diffeomorphisms act on the right, i.e., $p \exp(tf)$ denotes the point obtained by following the integral curve of f that passes through p at time zero for t units of time.

It is clear that Γ^* and Γ_* are two-dimensional surfaces with boundary. In both cases the boundary consists of the two curves corresponding to the trajectories of $f+g$ and $f-g$ and the point p. Furthermore, by the Campbell-Hausdorff formula [11]

$$\begin{aligned} &p \exp(s_1(f-g)) \exp(s_2(f+g)) \\ &\quad = p \exp((s_1+s_2)f + (s_2 - s_1)g + s_1 s_2 [f, g] + s_1 s_2 \cdot O(T)), \end{aligned}$$

$$p \exp((t_1(f+g)) \exp(t_2(f-g)) = p \exp(t_1 + t_2)f + (t_1 - t_2)g - t_1 t_2 [f, g] + t_1 t_2 \cdot O(T))$$

where $O(T)$ stands for terms that are linear in the total time T. This shows that Γ^* and Γ_* do not intersect in their relative interior. So Γ^* and Γ_* enclose a region R.

To prove that the enclosed sector R is the small-time reachable set we must show that there cannot be any other points in the reachable set. As in the two-dimensional case we have two options: either we show that we have exhausted all trajectories that

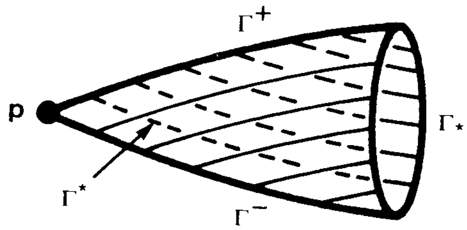

FIG. 2

possibly can lie on the boundary of the reachable set, or we show that trajectories starting at points on Γ^*, Γ_*, Γ^+, or Γ_- cannot leave $R\cup\Gamma^*\cup\Gamma_*\cup\Gamma^+\cup\Gamma_-$. As it turns out, this is the same argument, only viewed differently.

Let us first show that we have exhausted all possible trajectories that can lie in the boundary of the small-time reachable set, i.e., that such a trajectory is bang-bang with at most one switching. Let γ be a bang-bang trajectory with two switches, say of the form XYX, with junctions p_0 and p_1 at times $t_0<t_1$. If $\bar\lambda=\lambda(t_1)$, then we have $\langle\bar\lambda, g(p_1)\rangle=0$ and $\langle\bar\lambda, f(p_1)\rangle=0$. Also $\langle\lambda(t_0), g(p_0)\rangle=0$ or, equivalently, if we move g ahead along the flow of the vector field Y we get $\langle\bar\lambda, \exp(-(t_1-t_0)\operatorname{ad} Y)X(p_0)\rangle=0$. But $\bar\lambda\neq 0$ and so these three vectors are dependent: p_0 and p_1 are *conjugate points* (Sussmann [22]). Therefore

$$X(p_1)\wedge Y(p_1)\wedge\exp(-\Delta t\operatorname{ad} Y)X(p_0)=0$$

i.e., $X(p_1)\wedge Y(p_1)\wedge[X, Y](p_1)+O(\Delta t)=0$, where $\Delta t=t_1-t_0$. But such a relation cannot hold in small time by the independence of X, Y, and $[X, Y]$. Similarly it follows that YXY-concatenations cannot satisfy the Maximum Principle.

This computation can also be viewed in the following way. Define a map $F:(t_1, t_2, t_3)\mapsto p\exp(t_1X)\exp(t_2Y)\exp(t_3X)$ for t_i small. Then this map has full rank if $t_i>0$. For, if we compute the tangent space to the image, but pull back to $p\exp(t_1X)\exp(t_2Y)$, we get exactly the vectors $\exp(-t_2\operatorname{ad} Y)X$, Y, and X. Therefore $F(t_1, t_2, t_3)$ is an interior point of the reachable set. Finally, if we pull back the tangent space one step further to $p\exp(t_1X)$ we have the vectors X, Y, and $\exp(t_2\operatorname{ad} Y)X=X-t_2[X, Y]+O(t_2^2)$. The minus sign at $[X, Y]$ implies that X-trajectories point inside R at points on Γ^*. Similarly, it follows that Y-trajectories steer the system into R from Γ^*. And this proves that trajectories of the system cannot leave R through Γ^*, Γ_*, Γ^+, or Γ_-. (Because of the Maximum Principle we can restrict ourselves to just looking at these regular controls instead of having to consider arbitrary measurable functions. For, if any trajectory would leave R, then there will also have to be additional trajectories lying on the boundary of the reachable set and these must be bang-bang.)

The structure of the small-time reachable set as a stratified set can easily be described using the following notation. For $n\in\mathbb{N}$ let

$$S_{n,-}:=\{p\exp(s_1X)\exp(s_2Y)\exp(s_3X)\cdots\exp(s_nB): s_i>0, B=X \text{ if } n \text{ is odd}, B=Y \text{ if } n \text{ is even}\},$$

$$S_{n,+}:=\{p\exp(t_1Y)\exp(t_2X)\exp(t_3Y)\cdots\exp(t_nB): t_i>0, B=X \text{ of } n \text{ is even } B=Y \text{ if } n \text{ is odd}\}.$$

In a nondegenerate situation each of the $S_{n,\pm}$ is a n-dimensional smooth manifold. (Certainly this will be true in all the cases we consider here.) In the three-dimensional case the boundary of the small-time reachable set consists of the two two-dimensional strata $S_{2,\pm}$ which have in their boundary the two one-dimensional strata $S_{1,\pm}$ and the zero-dimensional stratum $S_0 = \{p\}$. S_0 also lies in the boundary of $S_{1,\pm}$. If we restrict the total time to be $\leqq T$ we must make the obvious adjustments. In particular, we must add the strata $\hat{S}_{n,\pm} := S_{n,\pm} \cap \text{Reach}\,(p, T)$ for $n = 1, 2$.

4. The nondegenerate four-dimensional systems. In this section we determine the geometric structure of the small-time reachable sets from a point p for a system Σ of the form (1) in dimension four, where we assume that the constant controls $u \equiv +1$ and $u \equiv -1$ are not singular. These conditions can easily be expressed in terms of independence assumptions on f, g, and lower-order brackets of f and g. For, a constant control $u \equiv u^0$ is singular on an interval I if and only if there exists an adjoint multiplier λ such that $\langle \lambda, f\rangle$, $\langle \lambda, g\rangle$, $\langle \lambda, [f, g]\rangle$, and $\langle \lambda [f + gu^0, [f, g]]\rangle$ vanish identically on I. By the nontriviality of λ this is impossible if f, g, $[f, g]$, and $[f + gu^0, [f, g]]$ are independent. Therefore in terms of the vector fields X and Y our conditions are equivalent to

(A) X, Y, $[X, Y]$ and $[X, [X, Y]]$ are independent near p;

(B) X, Y, $[X, Y]$ and $[Y, [X, Y]]$ are independent near p.

If we write $[X, [X, Y]]$ as a linear combination of X, Y, $[X, Y]$ and $[Y, [X, Y]]$ as

$$[X, [X, Y]] = \alpha X + \beta Y + \gamma [X, Y] + \delta [Y, [X, Y]],$$

then (A) is equivalent to $\delta \neq 0$.

The cases $\delta > 0$ and $\delta < 0$ are significantly different: if $\delta > 0$ only bang-bang trajectories can lie in the boundary of the reachable set, if $\delta < 0$ singular arcs are possible. Intuitively this is clear. If u is singular on an interval I, then (omitting the arguments t and $x(t)$)

$$\begin{aligned}\ddot{\phi} &= \langle \lambda, [f + gu, [f, g]]\rangle \\ &= \tfrac{1}{4}\langle \lambda, (1-u)[X, [X, Y]] + (1+u)[Y, [X, Y]]\rangle \\ &= \tfrac{1}{4}((1-u)\delta + (1+u)) \cdot \langle \lambda, [Y[X, Y]]\rangle \neq 0\end{aligned}$$

and so $u = (\delta + 1)/(\delta - 1)$. This is an admissible control only if $\delta \leqq 0$. Note that the singular vector field is given in feedback form as

$$S = f + \frac{\delta + 1}{\delta - 1} g = \frac{1}{1-\delta} X + \frac{-\delta}{1-\delta} Y, \quad \delta < 0.$$

4.1. The totally bang-bang case: $\delta > 0$. This is the generalization of Lobry's example to dimension four. We treat only the general case here, but we remark that the structure of the small-time reachable set is the same as for a nilpotent system where f, g, $[f, g]$, and $[f, [f, g]]$ form a basis and all other brackets vanish. In appropriate coordinates the latter system is linear.

The key observation again is that the Maximum Principle precisely determines the possible trajectories that can lie in the boundary of the small-time reachable set.

LEMMA 1. *If γ is a trajectory that lies in the boundary of the small-time reachable set, then γ is bang-bang with at most two switches.*

Proof. We first exclude bang-bang trajectories with more switches. Let γ be a $YXYX$-trajectory with switching points p_1, p_2, and p_3 and let s_1, s_2, s_3, s_4 be the length of the times along the respective X-arcs or Y-arcs. At every junction we have $\langle \lambda, X(p_i)\rangle = 0$ and $\langle \lambda, Y(p_i)\rangle = 0$. This gives rise to four conditions on λ.

If $\tilde{\lambda}$ is the value of the adjoint vector at the switching time at p_2, we have

$$\langle \tilde{\lambda}, X(p_2)\rangle = \langle \tilde{\lambda}, Y(p_2)\rangle = 0,$$

$$\langle \tilde{\lambda}, \exp(-s_2 \operatorname{ad} Y) X(p_1)\rangle = 0,$$

and

$$\langle \tilde{\lambda}, \exp(s_3 \operatorname{ad} X) Y(p_3)\rangle = 0.$$

Again, the nontriviality of $\tilde{\lambda}$ implies that these four vectors are dependent ("conjugate points"). So we get (dividing out s_2 and s_3)

$$0 = X \wedge Y \wedge \left(\frac{\exp(s_3 \operatorname{ad} X) - 1}{s_3}\right) Y \wedge \left(\frac{\exp(-s_2 \operatorname{ad} Y) - 1}{-s_2}\right) X$$

(5) $$= X \wedge Y \wedge [X, Y] + \frac{1}{2}s_3[X,[X, Y]] + O(s_3^2) \wedge -[X, Y] + \frac{1}{2}s_2[Y,[X, Y]] + O(T^2)$$

$$= \frac{1}{2}\sigma(s_2, s_3)(X \wedge Y \wedge [X, Y] \wedge [Y,[X, Y]])_{|p_2}$$

where T is the total time along γ and $O(T^2)$ stands for terms that are quadratic in T; σ is a smooth function of s_2 and s_3. If we express $[X,[X, Y]]$ in terms of X, Y, $[X, Y]$, and $[Y,[X, Y]]$, we see that

(6) $$\sigma(s_2, s_3) = s_2 + s_3\delta + O(T^2)$$

where δ is evaluated at p_2. In a sufficiently small neighborhood of p, δ is bounded away from zero and so the linear terms dominate quadratic remainders in small time. Hence $\sigma(s_2, s_3)$ is positive for s_i small; in particular, it cannot vanish, a contradiction.

Analogously, if $\tilde{\gamma}$ is a $XYXY$-concatenation with switching points q_1, q_2, and q_3 and if t_1, t_2, t_3, t_4 are the times along the respective trajectories, then we get

(7) $$0 = X \wedge Y \wedge \left(\frac{\exp(-t_2 \operatorname{ad} X) - 1}{-t_2}\right) Y \wedge \left(\frac{\exp(t_3 \operatorname{ad} Y) - 1}{t_3}\right) X$$
$$= \frac{1}{2}\tau(t_2, t_3)(X \wedge Y \wedge [X, Y] \wedge [Y,[X, Y]])_{|q_2}$$

where

(8) $$\tau(t_2, t_3) = -t_3 - t_2\delta + O(T^2)$$

is a smooth function of t_2 and t_3 near the origin. Again, since δ is bounded away from zero near p this function is negative for small times, a contradiction.

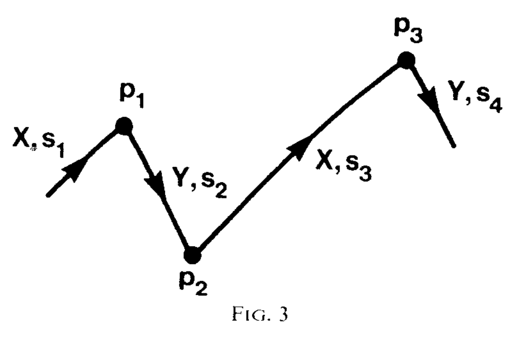

FIG. 3

It now follows that, in fact, any trajectory that lies in the boundary of the small-time reachable set is bang-bang. This is an easy but slightly technical argument. We will do it here rigorously since we will need the computations later on anyway. The point is that we do not have a priori knowledge about regularity properties of the controls, e.g., that they are piecewise constant. This is the case if and only if the zero set $Z(\phi)$ of the switching function ϕ is finite. If it were infinite, then the set N_ϕ of limit points of $Z(\phi)$ would be nonempty. In fact, it is a closed, nowhere dense, perfect set. (If $t_1 < t_2$ are points in $N(\phi)$ then, since ϕ cannot vanish identically, ϕ is different from zero somewhere in (t_1, t_2) and by continuity it is different from zero on a whole interval. It is perfect, i.e., every point $t \in N(\phi)$ is a limit point of points $t_n \in N(\phi)$, $t_n \neq t$, since $N(\phi)$ cannot have isolated points. We can see that this is so, since we know already that bang-bang trajectories with more than three switchings do not lie in the boundary of the small-time reachable set!) Suppose $t_1 < t_2$ are times in $N(\phi)$. There exists a $\tilde{t} \in (t_1, t_2)$ such that $\phi(\tilde{t}) \neq 0$. Let $\tilde{t}_1 := \sup([t_1, \tilde{t}] \cap N(\phi))$ and let $\tilde{t}_2 := \inf([\tilde{t}_1, t_2] \cap N(\phi))$. Then $\tilde{t}_1 < \tilde{t}_2$, $\tilde{t}_i \in N(\phi)$, and $Z(\phi) \cap [\tilde{t}_1, \tilde{t}_2]$ is finite. This implies that γ contains subarcs of the form $*B\cdot$ and $\cdot B*$, where B denotes a bang arc (X or Y), $\cdot$ stands for any switching, and $*$ stands for a junction in $N(\phi)$. Observe that $\dot{\phi}(t) = 0$ if $t \in N(\phi)$. We will now show that none of these concatenations can lie in the boundary of the reachable set and this will prove the lemma.

Without loss of generality we consider a concatenation of the form $*X\cdot$ with switching points p_0 and p_1 and let t be the time along X. Then, if $\tilde{\lambda}$ is the value of the adjoint vector at the switching time corresponding to p_0, we have

$$\langle \tilde{\lambda}, X(p_0) \rangle = \langle \tilde{\lambda}, Y(p_0) \rangle = \langle \tilde{\lambda}, [X, Y](p_0) \rangle = 0.$$

Also $\langle \tilde{\lambda}, \exp(-t \operatorname{ad} X) Y(p_1) \rangle = 0$ and so by nontriviality of $\tilde{\lambda}$ we again get

$$\begin{aligned} 0 &= X \wedge Y \wedge [X, Y] \wedge Y - t[X, Y] + \tfrac{1}{2}t^2[X, [X, Y]] + O(t^3) \\ &= \tfrac{1}{2}t^2(1 + O(t))(X \wedge Y \wedge [X, Y] \wedge [Y, [X, Y]])_{|p_0}. \end{aligned} \tag{9}$$

This cannot hold in small time. Analogously it follows that no $*B\cdot$ or $\cdot B*$ concatenation can lie in the boundary of the small-time reachable set if $\delta \neq 0$. This proves the lemma (and note that the argument is valid in general under assumptions (A) and (B)). □

It is now clear that the surfaces Γ^* and Γ_* must be as follows:

$$\Gamma^* = \{p \exp(s_1 X) \exp(s_2 Y) \exp(s_3 X) \colon s_i \geqq 0, \text{small}\},$$

$$\Gamma_* = \{p \exp(t_1 Y) \exp(t_2 X) \exp(t_3 Y) \colon t_i \geqq 0, \text{small}\}.$$

Γ^* and Γ_* are three-dimensional surfaces with common boundary C that has precisely the structure of the boundary of the small-time reachable set in dimension three. It is the union of two two-dimensional surfaces made out of XY- and YX-trajectories respectively, glued together along the X- and Y-trajectories.

We will now show that Γ^* and Γ_* do not intersect away from C, in particular that they enclose an open region that will be the interior of the small-time reachable set.

Definition. We say a point q is an entry point (respectively, an exit point) of a (closed) set S for a vector field Z if for some $\varepsilon > 0$, $S \cap \{q \exp(tZ) \colon -\varepsilon \leqq t \leqq 0\} = \{q\}$ (respectively, if $S \cap \{q \exp(tZ) \colon 0 \leqq t \leqq \varepsilon\} = \{q\}$).

Lemma 2. *For sufficiently small T the points in Γ^* are entry points for the small-time reachable set from p for $[Y, [X, Y]]$. The points in Γ_* are exit points.*

Proof. If q is an exit (entry) point for Reach$(p, \leqq T)$ that does not lie in Reach(p, T), i.e., exit or entry is not due to the time restriction, then the corresponding trajectory is extremal and the adjoint multiplier satisfies the transversality condition $\langle\lambda, [Y,[X, Y]](q)\rangle \leqq 0$ ($\langle\lambda, [Y,[X, Y]](q)\rangle \geqq 0$). We claim that necessarily

$$q \in \Gamma_* \ (q \in \Gamma^*).$$

Recall that the second derivative of the switching function is given by

$$\begin{aligned}\ddot{\phi}(t) &= \langle\lambda(t), [f+gu, [f, g]](x(t))\rangle \\ &= \tfrac{1}{4}(1-u(t))\langle\lambda, [X,[X, Y]](x(t))\rangle \\ &\quad + \tfrac{1}{4}(1+u(t))\langle\lambda, [Y,[X, Y]](x(t))\rangle.\end{aligned} \tag{10}$$

Expressing $[X,[X, Y]]$ in terms of X, Y, $[X, Y]$, and $[Y,[X, Y]]$, we get a linear combination of terms $\langle\lambda, X\rangle$, $\langle\lambda, Y\rangle$, $\langle\lambda, [X, Y]\rangle$, and $\langle\lambda, [Y,[X, Y]]\rangle$, where the coefficient at $\langle\lambda, [Y,[X, Y]]\rangle$ is

$$\tfrac{1}{2}(1-u)\delta + \tfrac{1}{2}(1+u) \geqq \operatorname{Min}(1, \delta) > 0.$$

Suppose γ is a bang-bang trajectory with two junctions. Then the two junctions determine a multiplier λ up to a positive constant multiple. Normalize such that $\|\lambda(0)\|_2 = 1$. Because γ has two junctions $\langle\lambda, X\rangle$, $\langle\lambda, Y\rangle$, and $\langle\lambda, [X, Y]\rangle$ vanish somewhere on $[0, T]$, $T = t_1 + t_2 + t_3$. For sufficiently small T these functions will be bounded in absolute value on $[0, T]$ by any $\varepsilon > 0$. Because of (B) $|\langle\lambda(t), [Y,[X, Y]](x(t))\rangle|$ can be bounded away from zero on $[0, T]$. By choosing ε, i.e., T small enough, $\langle\lambda, [Y,[X, Y]]\rangle$ dominates all other terms in (10), that is, we have in small time: $\ddot{\phi}$ has constant sign equal to sign ($\langle\lambda, [Y,[X, Y]]\rangle$). But $\langle\lambda, [Y,[X, Y]]\rangle > 0$ allows only for XYX-trajectories and $\langle\lambda, [Y,[X, Y]]\rangle < 0$ permits only YXY-concatenations. This proves our claim.

We still need to show that points in Γ^* and Γ_* in fact have these optimization properties. Suppose γ is a XYX trajectory. Then the tangent space at the endpoint is spanned by X, $\exp(-t_3 \operatorname{ad} X)Y$ and $\exp(-t_3 \operatorname{ad} X)\exp(-t_2 \operatorname{ad} Y)X$. Note that $[Y,[X, Y]]$ always points to one side of the tangent space since

$$\begin{aligned}&X \wedge \exp(-t_3 \operatorname{ad} X)Y \wedge \exp(-t_3 \operatorname{ad} X)\exp(-t_2 \operatorname{ad} Y)X \wedge [Y,[X, Y]] \\ &= -t_2\left(X \wedge \exp(-t_3 \operatorname{ad} X)Y \wedge \exp(-t_3 \operatorname{ad} X)\left(\frac{\exp(-t_2 \operatorname{ad} Y) - 1}{-t_2}\right)X \right. \\ &\qquad\qquad \left. \wedge [Y,[X, Y]]\right) \\ &= t_2(X \wedge Y - t_3[X, Y] + O(t_3^2) \wedge [X, Y] + O(T) \wedge [Y,[X, Y]]) \\ &= t_2(1 + O(T))(X \wedge Y \wedge [X, Y] \wedge [Y,[X, Y]]).\end{aligned} \tag{11}$$

If we write the defining equations for Γ^* and Γ_* in terms of canonical coordinates of the second kind, that is, as products of the flows of the vector fields X, Y, $[X, Y]$, $[Y,[X, Y]]$ in the form

$$p \exp(x_1 X)\exp(x_2 Y)\exp(x_3[X, Y])\exp(x_4[Y,[X, Y]]), \tag{12}$$

then this implies that we can think of Γ^* as the graph of a function $x_4 = \psi(x_1, x_2, x_3)$. It also follows from (12) that the integral curve of $[Y,[X, Y]]$ through p and the compact set Reach (p, T) are disjoint for small positive T. Therefore, given T, there exists a $\tilde{T} \leqq T$ with the following property. Any integral curve of $[Y,[X, Y]]$ that passes through a point on $\Gamma^*(\tilde{T})$, the set of all trajectories in Γ^* of total time $\leqq \tilde{T}$, does not meet Reach (p, T). This implies that the points on $\Gamma^*(\tilde{T})$ are entry points for the small-time reachable set. For, if $q \in \Gamma^*(\tilde{T})$ is not an entry point, then by compactness there exists an entry point of Reach $(p, \leqq T)$ of the form $q \exp r[Y, [X, Y]]$. Since this flow does not meet Reach (p, T) this point must lie on Γ^* and this contradicts the graph property. Analogously the result follows for Γ_*. □

An easy computation shows that, if Γ^* and Γ_* would intersect away from C, then it would have to happen transversally. This would contradict Lemma 2.

The geometric structure of the small-time reachable set is now clear. It is the exact analogue of Figs. 1 and 2 in four dimensions. Its boundary consists of the surfaces Γ_* and Γ^* that match up along C, the set of points reachable by a bang-bang trajectory with at most one switch. The open region enclosed by Γ^* and Γ_* is the interior of the reachable set. A stratification of its boundary is given by S_0 and $S_{n,\pm}$ for $n = 1, 2, 3$ (see § 3).

Remark. This qualitative structure of the small-time reachable set for a totally bang-bang system generalizes to arbitrary dimensions under the conditions of Krener's and Sussmann's nonlinear bang-bang theorem [19]. Suppose that the vector fields f and $\mathrm{ad}^i f(g)$, $i = 0, \cdots, n-1$ are independent at p and that for $i = 0, \cdots, n-1$ there exist smooth functions α_{ij} and β_i with $|\beta_i(p)| < 1$ such that

$$[g, \mathrm{ad}^i f(g)] = \sum_{j=0}^{i} \alpha_{ij}\, \mathrm{ad}^j f(g) + \beta_i\, \mathrm{ad}^{i+1} f(g).$$

Then it follows that for sufficiently small-time T all trajectories that lie in the boundary of the reachable set from p are bang-bang with at most n switchings. A stratification of the boundary is given by the strata $S_0 = \{p\}$ and $S_{k,\pm}$, $k = 1, \cdots, n$. In particular, points in $S_{n,+}$ are exit points of the reachable set for $(-1)^{n-1}\, \mathrm{ad}^{n-1} f(g)$, points in $S_{n,-}$ are entry points. Given the results on the structure of trajectories in the boundary, this is a straightforward generalization of the argument above. All the difficult work has been carried out by Sussmann in [19], specifically in the proof of Lemma 3 there.

4.2. The bang-bang singular case: $\delta < 0$. This case is a nontrivial extension of Lobry's result. Here not all the extremal trajectories actually lie in the boundary of the small-time reachable set. It is therefore not clear how we should choose Γ^* and Γ_*. We now use the structure of the small-time reachable set for the corresponding free nilpotent system as a guide. The only reasonable nilpotent approximation to choose is one where all brackets of orders greater than or equal to 4 vanish. Note that f, g, $[f, g]$, and $[g, [f, g]]$ are always independent in this case. Since we want to work with a system as simple as possible, we also assume $[f, [f, g]] \equiv 0$. This is an equality relation in the third-order Lie jet, but in a slightly more general setup (weighted Lie algebra) this would be a free nilpotent system. Therefore we refer to this system as the "free" nilpotent case. We will first analyze a model of this "free" nilpotent case, and then we will show that the general case has the same qualitative behavior.

4.2.1. The reachable set in the "free" nilpotent case. To simplify some computations we restrict ourselves to the following model $\tilde{\Sigma}$:

$$\dot{x}_0 = 1, \quad \dot{x}_1 = u, \quad \dot{x}_2 = x_1, \quad \dot{x}_3 = \tfrac{1}{2}x_1^2. \tag{13}$$

Note that $[g, f](x) = (\partial/\partial x_2) + x_1(\partial/\partial x_3)$, $[g, [g, f]] \equiv \partial/\partial x_3$ and all other brackets vanish identically. It is clear that the qualitative structure of the reachable set from the origin at any time is the same as for the small-time reachable set: one is a rescaling of the other. (If u is a control defined on $[0, T]$ and x is the corresponding trajectory, then the time 1 reachable set can be obtained from the time T reachable set by letting $\bar{u}(t) := u(t/T)$ and $\bar{x}_i(t) := T^i x_i(t/T)$ for $i = 1, 2, 3$.) To determine the reachable set it therefore suffices to look at time slices $T = \text{constant}$, and without loss of generality we can assume $T = 1$.

If $\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3)^T$ is an adjoint vector for an extremal trajectory $x(\cdot)$, then λ_1 is the switching function and

$$\dot{\lambda}_1 = -\lambda_2 - \lambda_3 x_1, \quad \dot{\lambda}_2 = 0, \quad \dot{\lambda}_3 = 0,$$

and in particular $\ddot{\lambda}_1 = \lambda_3 u$, i.e., $u = 0$ is the only singular control. Note that, if $\lambda_3 = 0$, then λ_1 is a linear function and the extremal trajectory is uniquely determined. By a theorem of Bressan [2] this implies that the reachable set is convex in direction of $(0, 0, 0, 1)^T$ or equivalently in the direction of $[g, [g, f]] = \frac{1}{2}[X, [X, Y]]$; that is, if (p_0, p_1, p_2, a) and (p_0, p_1, p_2, b) lie in the reachable set, then the whole segment $\{(p_0, p_1, p_2, c)\colon a \leqq c \leqq b\}$ lies in the reachable set. It is therefore clear what the surfaces Γ^* and Γ_* have to be: Γ^* consists of trajectories which are exit points for $[X, [X, Y]]$ and Γ_* of those which are entry points. Equivalently, we can speak of trajectories that maximize/minimize the coordinate x_3.

For extremal trajectories that give rise to entry/exit points for $[X, [X, Y]]$, an additional transversality condition was to hold. One of the directions $\pm[X, [X, Y]]$ can be separated from an approximating cone to the reachable set at this point. In our case these conditions simply say that $\lambda_3 \geqq 0$ for trajectories that minimize x_3 and $\lambda_3 \leqq 0$ for those that maximize x_3. In particular $\lambda_3 = 0$ for those that do both and these trajectories are bang-bang with at most one switching. So again the common boundary of Γ^* and Γ_* will be a set C that has the structure of the boundary of the small-time reachable set in dimension three.

We now determine Γ_*. We can assume $\lambda_3 > 0$ and without loss of generality normalize λ_3 to 1. Thus, $\ddot{\lambda}_1 = -u$ and so λ_1 is strictly convex and positive along X, strictly concave and negative along Y. Singular controls satisfy the generalized Legendre-Clebsch condition [13]: $\langle \lambda, [g, [f, g]] \rangle = -\lambda_3 < 0$. It follows that the only extremal trajectories are concatenations of a bang arc, followed by a singular arc and another bang arc. We now restrict to the time slice $T = 1$. Define

$$\Gamma_{-0-} := \{0 \exp(s_1 X) \exp(s_2 f) \exp(s_3 X)\colon s_i \geqq 0,\ s_1 + s_2 + s_3 = 1\},$$

$$\Gamma_{-0+} := \{0 \exp(s_1 X) \exp(s_2 f) \exp(s_3 Y)\colon s_i \geqq 0,\ s_1 + s_2 + s_3 = 1\},$$

$$\Gamma_{+0-} := \{0 \exp(t_1 Y) \exp(t_2 f) \exp(t_3 X)\colon t_i \geqq 0,\ t_1 + t_2 + t_3 = 1\},$$

$$\Gamma_{+0+} := \{0 \exp(t_1 Y) \exp(t_2 f) \exp(t_3 Y)\colon t_i \geqq 0,\ t_1 + t_2 + t_3 = 1\}.$$

We will show that these are two-dimensional surfaces with boundary which match up and together form Γ_* with

$$\partial\Gamma_* = \{0 \exp(s_1 X) \exp(s_2 Y)\colon s_i \geqq 0,\ s_1 + s_2 = 1\}$$
$$\cup \{0 \exp(t_1 Y) \exp(t_2 X)\colon t_i \geqq 0,\ t_1 + t_2 = 1\}.$$

LEMMA 3. *Each of the sets $\Gamma_{\pm 0 \pm}$ is a two-dimensional surface with boundary. For any two of them the images of the open simplices are disjoint. Furthermore,*

$$\Gamma_{-0-} \cap \Gamma_{-0+} = \Gamma_{-0} = \{0 \exp(s_1 X) \exp(s_2 f)\colon s_i \geqq 0,\ s_1 + s_2 = 1\},$$

$$\Gamma_{-0-} \cap \Gamma_{+0-} = \Gamma_{0-} = \{0 \exp(s_1 f) \exp(s_2 X)\colon s_i \geqq 0,\ s_1 + s_2 = 1\},$$

$$\Gamma_{-0-} \cap \Gamma_{+0+} = \Gamma_0 = \{0 \exp(sf)\colon 0 \leqq s \leqq 1\} = \Gamma_{-0+} \cap \Gamma_{+0-},$$

$$\Gamma_{-0+} \cap \Gamma_{+0+} = \Gamma_{0+} = \{0 \exp(s_1 f) \exp(s_2 Y)\colon s_i \geqq 0,\ s_1 + s_2 = 1\},$$

$$\Gamma_{+0-} \cap \Gamma_{+0+} = \Gamma_{+0} = \{0 \exp(s_1 Y) \exp(s_2 f)\colon s_i \geqq 0,\ s_1 + s_2 = 1\}.$$

Graphically, these relations can be illustrated as shown in Fig. 4.

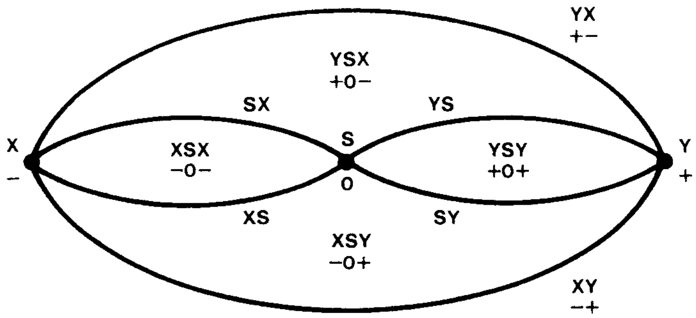

FIG. 4

The proof of the lemma consists of straightforward computations that we shall only illustrate in one case. It is easy to see that all the maps are regular with rank 2 in the interior, and it is clear how the maps behave on the boundary. So the $\Gamma_{\pm 0 \pm}$ are two-dimensional surfaces with boundary. To prove that the images of the open simplex under different maps are disjoint, we choose a way that does not use the specific form of the equations, but works with a basis provided by the vector fields f, g, $[f, g]$, and $[g, [f, g]]$. This also gives an idea how the analogous argument in the general case runs. We rewrite the defining equations in terms of canonical coordinates of the second kind as products of the flows of the vector fields f, g, $[f, g]$, and $[g, [f, g]]$. Since in this case

$$\exp(f+g) = \exp([g,[f,g]]/3)\exp([f,g]/2)\exp(g)\exp(f), \tag{14}$$

we get, for instance, for Γ_{+0+}:

$$\begin{aligned}
&0 \exp(t_1(f+g))\exp(t_2 f)\exp(t_3(f+g)) \\
&\quad = 0 \exp((\tfrac{1}{3}t_1^3[g,[f,g]])\exp(\tfrac{1}{2}t_1^2[f,g])\exp(t_1 g)\exp((t_1+t_2)f)) \\
&\qquad \times \exp(\tfrac{1}{3}t_3^3[g,[f,g]])\exp(\tfrac{1}{2}t_3^2[f,g])\exp(t_3 g)\exp(t_3 f) \\
&\quad = 0 \exp((\tfrac{1}{6}(t_1+t_3)^3 + t_2 t_3(t_1+\tfrac{1}{2}t_3))[g,[f,g]])\exp((\tfrac{1}{2}(t_1+t_3)^2 + t_2 t_3)[f,g]) \\
&\qquad \times \exp((t_1+t_3)g)\exp(f).
\end{aligned}$$

Analogously we have for Γ_{-0+}:

$$\begin{aligned}
&0 \exp(s_1(f-g))\exp(s_2 f)\exp(s_3(f+g)) \\
&\quad = 0 \exp((\tfrac{1}{3}s_1^3 - s_1^2 s_3 + \tfrac{1}{3}s_3^3 + \tfrac{1}{2}s_2 s_3^2 - s_1 s_2 s_3)[g,[f,g]]) \\
&\qquad \times \exp((-\tfrac{1}{2}s_1^2 + \tfrac{1}{2}s_3^2 + (s_1+s_2)s_3)[f,g])\exp((s_3-s_1)g)\exp(f).
\end{aligned}$$

A simple computation shows that the equations we obtain by equating the coordinates have no positive solution. Similarly this is shown for all pairs of surfaces. The statements about the intersections are then clear. □

This shows that Γ_* is a two-dimensional stratified set with its one-dimensional relative boundary $\partial\Gamma_*$ made out of bang-bang trajectories with at most one switching. Figure 4 gives a precise description of the stratification. We now show that the points on Γ_* are, in fact, the points that have the smallest x_3 coordinate among all points of Reach $(0, 1)$ with a fixed (x_0, x_1, x_2).

Let us first compute the tangent spaces to the surfaces $\Gamma_{\pm 0 \pm}$. Note that in each case the pullback of the tangent space to the endpoint of the singular arc simply consists of the space spanned by the vectors g and $[f, g]$ evaluated there (remember that we are working in the time slice $T=1$). This implies that $[X,[X, Y]]=2[g,[g,f]]$ always points to one side of the tangent space. In fact,

$$\exp(-t \operatorname{ad}(f \pm g))g \wedge \exp(-t \operatorname{ad}(f \pm g))[f, g] \wedge [g,[f, g]] \equiv 1(g \wedge [f, g] \wedge [g,[f, g]]).$$

In the limit this also holds for the one-dimensional strata. Therefore $[g,[g,f]]$ always points to one side of the stratified surface Γ_*. It is easy to see that, in fact, we can think of Γ_* as the graph of a piecewise defined function $x_3=\psi(x_1, x_2)$. (The projections of the images onto (x_1, x_2) intersect only along the projections of the intersections of the surfaces $\Gamma_{\pm 0 \pm}$.) Since we have exhausted all possible extremal trajectories that can minimize the coordinate x_3 with Γ_*, it is now clear that given $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \Gamma_*$ any other point $(x_1, x_2, x_3) \in \operatorname{Reach}(0,1)$ with $x_1=\bar{x}_1$ and $x_2=\bar{x}_2$ must satisfy $x_3>\bar{x}_3$. This concludes the analysis of Γ_*.

Next we will determine Γ^*. Here we can assume $\lambda_3=-1$ and so $\ddot{\lambda}_1=u$, i.e., the switching function ϕ is convex when ϕ is negative and concave when ϕ is positive. This clearly suggests bang-bang extremals. However, now the situation is significantly different from all previous cases: it will turn out that the times along bang arcs are no longer free, which in turn will mean that we cannot a priori exclude bang-bang trajectories with a large number of switchings. In general, it is a very difficult problem to eliminate extremal trajectories with a large number of switchings (cf. [4] or [16]). It turns out that in our approach we do not even have to address this issue.

Let us start by showing that the times along bang arcs can no longer vary freely. Suppose we have a concatenation of a Y-trajectory followed by an X-arc with switchings at the beginning and the end $(\cdot XY \cdot)$. Call the switching points p_0, p_1, and p_2 and let s and t be the times along X and Y, respectively. Then p_0, p_1, and p_2 are conjugate points and therefore

$$\begin{aligned} 0 &= \exp(-s \operatorname{ad} X) Y \wedge X \wedge Y \wedge \exp(t \operatorname{ad} Y) X \\ &= \left(\frac{\exp(-s \operatorname{ad} X)-1}{-s}\right) Y \wedge X \wedge Y \wedge \left(\frac{\exp(t \operatorname{ad} Y)-1}{t}\right) X \\ &= X \wedge Y \wedge [X, Y] + s[g,[f, g]] \wedge [Y, X] - t[g,[f, g]] \\ &= (s-t)(X \wedge Y \wedge [X, Y] \wedge [g,[f, g]]). \end{aligned} \tag{15}$$

Hence $s=t$ and the same is true for a $\cdot YX \cdot$-concatenation. Therefore, so as not to violate the Maximum Principle, and since we do not expect any degeneracies in the structure of the reachable set, we restrict ourselves to the following two surfaces:

$$\tilde{\Gamma}^- = \{0 \exp(s_1 X) \exp(s_2 Y) \exp(s_3 X) \colon s_i \geqq 0,\ s_1+s_2+s_3=1,\ s_1 \leqq s_2,\ s_3 \leqq s_2\},$$

$$\tilde{\Gamma}^+ = \{0 \exp(t_1 Y) \exp(t_2 X) \exp(t_3 Y) \colon t_i \geqq 0,\ t_1+t_2+t_3=1,\ t_1 \leqq t_2,\ t_3 \leqq t_2\}.$$

Our aim is to build Γ^* out of trajectories from $\tilde{\Gamma}^+$ and $\tilde{\Gamma}^-$. However, as they are at the moment, we still have too many extremal trajectories. The surfaces $\tilde{\Gamma}^-$ and $\tilde{\Gamma}^+$ have a nontrivial intersection $\tilde{\gamma}$. To see this let us rewrite the defining maps in terms of canonical coordinates as follows:

$$\begin{aligned} 0 \exp(s_1 X) \exp(s_2 Y) \exp(s_3 X) = {} & 0 \exp(s_1 s_2(s_2-s_1)[g,[f, g]]) \exp(s_1 s_2[X, Y]) \\ & \times \exp(s_2 Y) \exp((s_1+s_3) X), \end{aligned}$$

$$0 \exp(t_1 Y) \exp(t_2 X) \exp(t_3 Y) = 0 \exp(t_2 t_3 (2t_1 - t_2 + t_3)[g, [f, g]]) \exp(t_2 t_3 [X, Y]) \times \exp((t_1 + t_3) Y) \exp(t_2 X).$$

If we equate the coordinates, it follows easily that $s_1 = t_3$, $s_2 = t_2$, and $s_3 = t_1$. It follows that $\tilde{\Gamma}^+$ and $\tilde{\Gamma}^-$ also intersect along the one-dimensional curve

$$\tilde{\gamma} = \{0 \exp(sX) \exp(Y/2) \exp((\tfrac{1}{2} - s)X) : 0 \leqq s \leqq \tfrac{1}{2}\}.$$

We need to analyze the intersection more closely. Let

$$q = 0 \exp(s_1 X) \exp(s_2 Y) \exp(s_3 X) \in \tilde{\gamma}.$$

Then the tangent space to $\tilde{\Gamma}^-$ at q is spanned by (recall that $s_3 = 1 - s_1 - s_2$)

$$\exp(-s_3 \operatorname{ad} X) \exp(-s_2 \operatorname{ad} Y) X - X = s_2([X, Y] + (2s_3 - s_2)[g, [f, g]]),$$

$$\exp(-s_3 \operatorname{ad} X) Y - X = 2g - s_3[X, Y] - s_3^2[g, [f, g]].$$

The point q also lies on $\tilde{\Gamma}^+$ and a tangent vector to $\tilde{\Gamma}^+$ at q is

$$\ell = \exp(-t_3 \operatorname{ad} Y) X - Y = -2g + t_3[X, Y] - t_3^2[g, [f, g]].$$

In the intersection $t_3 = s_1 =: s$, $s_2 = \frac{1}{2}$ and $s_3 = \frac{1}{2} - s$. Thus

$$T_q \tilde{\Gamma}^- \wedge \ell = A(2g \wedge [X, Y] \wedge [g, [f, g]])$$

where

$$A = \begin{vmatrix} 1 & s - \frac{1}{2} & -(s - \frac{1}{2})^2 \\ 0 & 1 & \frac{1}{2} - 2s \\ -1 & s & -s^2 \end{vmatrix} = 2s(s - \tfrac{1}{2}) \leqq 0.$$

Hence $\tilde{\Gamma}^-$ and $\tilde{\Gamma}^+$ intersect transversally except at the endpoints of $\tilde{\gamma}$ ($s = 0$, $s = \frac{1}{2}$). Observe that the endpoints are characterized by the condition that the conjugate point relation $s = t$ $(= \frac{1}{2})$ holds. We need to know which surface has a larger x_3-coordinate. It follows from

$$T_q \tilde{\Gamma}^- \wedge [g, [g, f]] \equiv -2g \wedge [X, Y] \wedge [g, [f, g]]$$

that ℓ and $[g, [g, f]]$ point to the same side of $\tilde{\Gamma}^-$ at q. Observe that $x_1 = 0$ for points on $\tilde{\gamma}$. Since the coefficient of ℓ at g is negative, the points of $\tilde{\Gamma}^+$ for which $x_1 < 0$ have a larger x_3-coordinate than those points on $\tilde{\Gamma}^-$. Conversely for $x_1 > 0$ the x_3-coordinate of points on $\tilde{\Gamma}^-$ is larger. Therefore we define

$$\Gamma^- := \{0 \exp(s_1 X) \exp(s_2 Y) \exp(s_3 X) : s_i \geqq 0, s_1 + s_2 + s_3 = 1, s_2 \geqq \tfrac{1}{2}\},$$

$$\Gamma^+ := \{0 \exp(t_1 Y) \exp(t_2 X) \exp(t_3 Y) : t_i \geqq 0, t_1 + t_2 + t_3 = 1, t_2 \geqq \tfrac{1}{2}\}.$$

Observe that Γ^- has the Y-trajectory in its boundary and that the X-trajectory lies in the boundary of Γ^+. Define $\Gamma^* := \Gamma^- \cup \Gamma^+$. It follows from above that $[X, [X, Y]] = 2[g, [g, f]]$ always points to one side of $\tilde{\Gamma}^-$, and similarly this holds for $\tilde{\Gamma}^+$. Since $x_1 \geqq 0$ for points in Γ^-, $x_1 \leqq 0$ for points in Γ^+ and $x_1 = 0$ exactly on the intersection, it follows that Γ^* is a piecewise defined function $x_3 = \psi(x_1, x_2)$.

It is obvious that $\partial\Gamma^*$ consists of all trajectories that are bang-bang with at most one switching, i.e., $\partial\Gamma^* = \partial\Gamma_*$. Graphically, the structure is illustrated in Fig. 5.

By directional convexity it is clear that the whole set R between Γ_* and Γ^* lies in Reach (0, 1). We need to show that it lies nowhere else. The points of $\tilde{\Gamma}^+$ and $\tilde{\Gamma}^-$ that we deleted lie in the interior of R. (We deleted those points on $\tilde{\Gamma}^+$, respectively, $\tilde{\Gamma}^-$ that lie below $\tilde{\Gamma}^-$, respectively, $\tilde{\Gamma}^+$ in the direction of $[X, [X, Y]]$.) But this implies that the endpoints of bang-bang trajectories with more than two switchings lie in the interior of the reachable set. Suppose we have an extremal $XYXY$-trajectory with

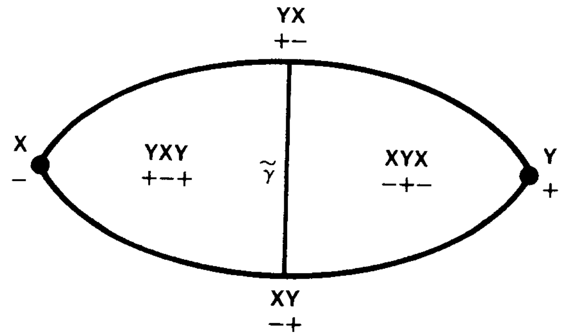

FIG. 5

times s_1, s_2, s_3, and s_4 along the trajectories. Then $s_2 = s_3$ by the conjugate point relation, and thus $s_2 < s_1 + s_3$. By the invariance of the structure of the reachable set it follows that $0 \exp(s_1 X) \exp(s_2 Y) \exp(s_3 X) \in \operatorname{int} \mathrm{Reach}(0, s_1 + s_2 + s_3)$. (This is a point of the type we deleted!) Hence the trajectories that define Γ^+ and Γ^- are the only extremal trajectories that can lie on the boundary of the reachable set. This proves $R = \mathrm{Reach}(0, 1)$.

Summary. For every time t the time $-t-$ reachable set is a stratified set that is topologically a sphere. Its boundary consists of two hemispheres $\Gamma^*(t)$ and $\Gamma_*(t)$ whose common relative boundary $\partial\Gamma^*(t)$ consists of all points reachable in time t by a bang-bang trajectory with at most one switch. $\Gamma^*(t)$ consists of all bang-bang trajectories with at most two switchings for which the time along the intermediate arc is greater than or equal to the sum of the times of the adjacent arcs. $\Gamma_*(t)$ consists of all trajectories that are concatenations of a bang arc, followed by a singular arc and another bang arc, where the times along these trajectories are free subject to $0 \leqq \text{time} \leqq t$. The stratification of its boundary is given in Figs. 4 and 5.

4.2.2. The general case. We now show that the qualitative structure of the small-time reachable set does not change in the general case. Clearly, some of the arguments will have to be adjusted; for instance, the correct generalization of the arguments using directional convexity now use the integral curves of $[X, [X, Y]]$. However, finding a general version for the explicit computations in the analysis of the bang-bang extremal trajectories is crucial.

We first define Γ_*. Recall that the singular control is given in feedback form as $u = (\delta + 1)/(\delta - 1)$ and since $\delta < 0$ we have no problems with u hitting the control constraint $|u| = 1$ in small time. Let $\rho = 1/(1-\delta)$, $\rho \in (0, 1)$, and let $S := f + (\delta + 1)/(\delta - 1) g = \rho X + (1 - \rho) Y$, be the singular vector field. Define

$$\Gamma_{-s-} := \{p \exp(s_1 X) \exp(s_2 S) \exp(s_3 X) : s_i \geqq 0, \text{small}\},$$

$$\Gamma_{-s+} := \{p \exp(s_1 X) \exp(s_2 S) \exp(s_3 Y) : s_i \geqq 0, \text{small}\},$$

$$\Gamma_{+s-} := \{p \exp(t_1 Y) \exp(t_2 S) \exp(t_3 X) : t_i \geqq 0, \text{small}\},$$

$$\Gamma_{+s+} := \{p \exp(t_1 Y) \exp(t_2 S) \exp(t_3 Y) : t_i \geqq 0, \text{small}\},$$

$$\Gamma_* := \Gamma_{-s-} \cup \Gamma_{-s+} \cup \Gamma_{+s-} \cup \Gamma_{+s+}.$$

If we replace f by S in Lemma 3, then the statement stays true verbatim for $\Gamma_{\pm s \pm}$ instead of $\Gamma_{+0\pm}$. (The computations are a straightforward though somewhat messy extension of the computation in the "free" nilpotent case and we omit them.) So again Γ_* is a stratified two-dimensional surface; its one-dimensional relative boundary $\partial\Gamma_*$ is made out of the bang-bang trajectories with at most one switching.

LEMMA 4. *For sufficiently small T the points on Γ_* are entry points of* Reach $(p, \leqq T)$ *for* $[X, [X, Y]]$.

Proof. The strategy is the same as in the proof of Lemma 2. We first show that the extremals on Γ_* satisfy the necessary transversality condition for entry points (which are not due to the time constraint). Then we show that Γ_* actually is a graph with the coefficient of the flow of $[X, [X, Y]]$ as dependent variable. As in Lemma 2 this suffices to prove our result.

If γ is any trajectory containing a singular arc then, for sufficiently small time, $\langle\lambda, [X, [X, Y]]\rangle$ will dominate $\langle\lambda, X\rangle$, $\langle\lambda, Y\rangle$, and $\langle\lambda, [X, Y]\rangle$, in particular, it has constant sign. Along the singular arc $\langle\lambda, [X, [X, Y]]\rangle = 2\delta/(1-\delta)\ \langle\lambda, [g, [X, Y]]\rangle$ and the generalized Legendre–Clebsch condition implies that $\langle\lambda, [X, [X, Y]]\rangle$ is positive. This shows that points in Γ_* satisfy the necessary transversality condition. An argument analogous to the one made in the proof of Lemma 1 shows that, in fact, any extremal trajectory for which $\langle\lambda, [X, [X, Y]]\rangle$ is positive has to be of the form *BSB*, that is, we have exhausted all possible candidates. To prove that indeed each point on Γ_* has the entry property, we show again that we can think of Γ_* as the graph of a piecewise defined function $x_3 = \psi(x_0, x_1, x_2)$, where (x_0, x_1, x_2, x_3) are canonical coordinates of the second kind, and x_3 is the coefficient at the flow of $[X, [X, Y]]$. Let us consider, for instance, Γ_{+s-}. It is easier to compute the pullback of the tangent space to the endpoint of the singular arc. It is spanned by X, S, and $\exp(-t_2 \operatorname{ad} S)X$. Note that $S = \rho X + (1+\rho)Y$ and it follows by induction that $\operatorname{ad}^n S(X) = \alpha_n X + \beta_n Y + \gamma_n [X, Y]$ with smooth functions $\alpha_n, \beta_n, \gamma_n$:

$$[S, \operatorname{ad}^{n-1} S(X)] = [\rho X + (1-\rho)Y, \alpha_{n-1}X + \beta_{n-1}Y + \gamma_{n-1}[X, Y]]$$
$$= \gamma_{n-1}\underbrace{(\rho[X, [X, Y]] + (1-\rho)[Y, [X, Y]])}_{=\rho(\alpha X + \beta Y + \gamma[X, Y])} + f, g \text{ or } [f, g] \text{ terms}$$

Also $[S, X] = [\rho X + (1-\rho)Y, X] = 2L_x(\rho)g + (\rho - 1)[X, Y]$. Therefore

$$X \wedge S \wedge \exp(-t_2 \operatorname{ad} S)X = (1-\rho)^2 t_2(1 + O(t_2)) \cdot (f \wedge g \wedge [X, Y]).$$

Now if we take the wedge-product with $[X, [X, Y]]$ pulled back along X, t_3 this yields

$$X \wedge S \wedge \exp(-t_2 \operatorname{ad} S)X \wedge \exp(t_3 \operatorname{ad} X)([X, [X, Y]])$$
$$= (1-\rho)^2 t_2(1 + O(T)) \cdot (f \wedge g \wedge [f, g] \wedge [X, [X, Y]])$$

and there are no problems with dominance since t_2 factors. Hence $[X, [X, Y]]$ always points to one side of Γ_{+s-} in the interior. Analogously it follows for the other surfaces. By continuity this also follows for the one-dimensional strata. Straightforward but slightly more tedious computations show also that the projections of the relative interiors of the sets $\Gamma_{\pm s \pm}$ onto (x_0, x_1, x_2)-space are pairwise disjoint. Therefore Γ_* is a graph in canonical coordinates. This proves the lemma. □

The analysis of the bang-bang extremals is more difficult. We start by computing the conjugate point relations. Suppose γ is a $\cdot XYX\cdot$-concatenation starting at p with junctions at p, p_1, p_2, p_3 and times s_1, s_2, s_3 along the respective trajectories. Then we have (the vector fields are evaluated at p_1):

$$0 = X \wedge Y \wedge \left(\frac{\exp(-s_1 \operatorname{ad} X) - 1}{-s_1}\right) Y \wedge \left(\frac{\exp(s_2 \operatorname{ad} Y) - 1}{s_2}\right) X$$

$$(16) \qquad = X \wedge Y \wedge [X, Y] - \frac{1}{2}s_1[X, [X, Y]] + O(s_1^2) \wedge -[X, Y] - \frac{1}{2}s_2[Y, [X, Y]] + O(s_2^2)$$

$$= \frac{1}{2}\sigma(s_1, s_2)(X \wedge Y \wedge [X, Y] \wedge [Y, [X, Y]])|_{p_1}$$

where $\sigma(s_1, s_2) = -s_1\delta - s_2 + O(2)$.

The equation $\tilde{\sigma}(s_1, s_2)=0$ has a unique solution $\overline{s_1}(s_2)$ and in general XYX-trajectories only satisfy the necessary conditions of the Maximum Principle if $s_1 \leqq \overline{s_1}(s_2)$. Note that $\underset{\sim}{\sigma}(0, s_2)<0$ and so this is equivalent to $\underset{\sim}{\sigma}(s_1, s_2)\leqq 0$. (Using an argument analogous to (9) it can be shown that extremal trajectories do indeed have switchings at $s_1=\overline{s_1}$, but we will not need this.) Furthermore,

$$0=X(p_2)\wedge Y(p_2)\wedge\left(\frac{\exp(-s_2 \operatorname{ad} Y)-1}{-s_2}\right)X(p_1)$$

$$\wedge\left(\frac{\exp(s_3 \operatorname{ad} X)-1}{s_3}\right)Y(p_3)$$

$$=X\wedge Y\wedge -[X, Y]+\frac{1}{2}s_2[Y,[X, Y]]+\cdots\wedge[X, Y]+\frac{1}{2}s_3[X,[X, Y]]+\cdots$$

$$=\frac{1}{2}\tilde{\sigma}(s_2, s_3)(X\wedge Y\wedge[X, Y]\wedge[Y,[X, Y]])|_{p_2}$$

where $\tilde{\sigma}(s_2, s_3)=-s_2-s_3\delta+O(T^2)$.

Again the equation $\tilde{\sigma}(s_2, s_3)=0$ can be solved by $\overline{s_3}(s_2)$, and YXY-concatenations only satisfy the Maximum Principle if $s_3\leqq\overline{s_3}(s_2)$. Since $\tilde{\sigma}(s_2, 0)<0$ this is equivalent to $\tilde{\sigma}(s_2, s_3)\leqq 0$.

Therefore we define

$$\tilde{\Gamma}^-=\{p\exp(s_1X)\exp(s_2Y)\exp(s_3X)\colon s_i\geqq 0,\ \text{small},\ s_2 \text{ is free},$$
$$\underset{\sim}{\sigma}(s_1, s_2)\leqq 0,\ \tilde{\sigma}(s_2, s_3)\leqq 0\}.$$

Analogously we must compute the conjugate point relations along a $\cdot YXY\cdot$-concatenation which yields

$$\tilde{\Gamma}^+:=\{p\exp(t_1Y)\exp(t_2X)\exp(t_3Y)\colon t_1\geqq 0,\ \text{small } t_2 \text{ is free},$$
$$\underset{\sim}{\tau}(t_1, t_2)\geqq 0\Leftrightarrow t_1\leqq\bar{t}_1(t_2)\tilde{\tau}(t_2, t_3)\geqq 0\Leftrightarrow t_3\leqq\bar{t}_3(t_2)\}$$

where

$$\underset{\sim}{\tau}(t_1, t_2)=-t_1-t_2\delta+O(T^2),\qquad \tilde{\tau}(t_2, t_3)=-t_2\delta-t_3+O(T^2)$$

and $\bar{t}_1$ and $\bar{t}_3$ are the solutions of $\underset{\sim}{\tau}=0$ and $\tilde{\tau}=0$, respectively. $\tilde{\Gamma}^+$ and $\tilde{\Gamma}^-$ are three-dimensional surfaces with relative boundary made up entirely of bang-bang trajectories with at most one switch.

LEMMA 5. *The surfaces $\tilde{\Gamma}^-$ and $\tilde{\Gamma}^+$ intersect along a two-dimensional surface $\hat{\Gamma}$. The intersection of $\hat{\Gamma}$ with the relative boundaries $\partial\tilde{\Gamma}^-$ and $\partial\tilde{\Gamma}^+$ are the following one-dimensional curves:*

$$\tilde{\gamma}=\{p\exp(s_1X)\exp(s_2Y)\colon s_2\geqq 0,\ \textit{small},\ s_1=\bar{s}_1(s_2)\},$$
$$\underset{\sim}{\gamma}=\{p\exp(t_1Y)\exp(t_2X)\colon t_2\geqq 0,\ \textit{small},\ t_1=t_1^-(t_2)\}$$

(i.e., the trajectories corresponding to the conjugate points). Away from $\underset{\sim}{\gamma}$ and $\tilde{\gamma}$ the surface entirely lies in the relative interior of $\tilde{\Gamma}^-$, respectively, $\tilde{\Gamma}^+$ and there the intersection is transversal.

Proof. We want to solve the equation

$$p\exp(s_1X)\exp(s_2Y)\exp(s_3X)=p\exp(t_1Y)\exp(t_2X)\exp(t_3Y). \tag{17}$$

Suppose a point q in the relative interior of $\tilde{\Gamma}^+$ or $\tilde{\Gamma}^-$ lies on $\hat{\Gamma}$. We claim that (16) can be solved in terms of t_1 and t_2 near q. This follows from the Implicit Function Theorem if the Jacobian with respect to (s_1, s_2, s_3, t_3) is nonsingular at q. If we compute these derivatives and pull the vectors back along X we get

$$\exp(-s_2 \operatorname{ad} Y)X \wedge Y \wedge X \wedge \exp(s_3 \operatorname{ad} X)Y$$
$$= s_2 s_3\left(X \wedge Y \wedge \left(\frac{\exp(-s_2 \operatorname{ad} Y)-1}{-s_2}\right) X \wedge \left(\frac{\exp(s_3 \operatorname{ad} X)-1}{s_3}\right) Y\right)$$
$$= \frac{1}{2} s_2 s_3 \cdot \tilde{\sigma}(s_2, s_3)(X \wedge Y[X, Y] \wedge [Y,[X, Y]])\big|_{p_2 = p \exp(s_1 X)\exp(s_2 Y)}.$$

But in int $(\tilde{\Gamma}^-)$ s_2 and s_3 are positive and also $\tilde{\sigma}(s_2, s_3) < 0$ since the conjugate point relation does not hold. So we can solve in terms of t_1 and t_2. This computation shows also that $\tilde{\Gamma}^+$ and $\tilde{\Gamma}^-$ intersect transversally in int $(\tilde{\Gamma}^+)$ or int $(\tilde{\Gamma}^-)$.

Next we show that points q of this type exist. For that we rewrite both sides of (17) in terms of canonical coordinates of the second kind. A short computation (cf., for instance, [16]) shows that

$$\begin{aligned} p\exp(s_1 X)\exp(s_2 Y)\exp(s_3 X) = p &\exp(\tfrac{1}{2}s_1 s_2(s_1\delta + s_2 + O(S^2))[Y,[X, Y]]) \\ &\cdot \exp(s_1 s_2(1+O(S))[X, Y]) \\ &\cdot \exp((s_2 + O(S^3))Y)\exp((s_1+s_3+O(S^3))X), \end{aligned}$$
$$\begin{aligned} p\exp(t_1 Y)\exp(t_2 X)\exp(t_3 Y) = p &\exp(\tfrac{1}{2}t_2 t_3(2t_1 + t_3 + t_2\delta + O(T^2))[Y,[X, Y]]) \\ &\cdot \exp(t_2 t_3(1+O(T))[X, Y]) \\ &\cdot \exp((t_1+t_3+O(T^3))Y)\exp((t_2+O(T^3))X) \end{aligned}$$

where $O(S^k)$ or $O(T^k)$ stand for terms of order greater than or equal to k in the total time, $S = s_1+s_2+s_3$, $T = t_1+t_2+t_3$, and δ is evaluated at p. Equating coefficients we get

$$(18)\qquad \begin{aligned} &\text{(i)} && s_1+s_3+O(S^3) = t_2+O(T^3), \\ &\text{(ii)} && s_2+O(S^3) = t_1+t_3+O(T^3), \\ &\text{(iii)} && s_1 s_2(1+O(S)) = t_2 t_3(1+O(T)), \\ &\text{(iv)} && s_1 s_2(s_1\delta+s_2+O(S^2)) = t_2 t_3(2t_1+t_3+t_2\delta+O(T^2)). \end{aligned}$$

If we assume that all switching times are comparable, i.e., of order T, then (18(i), (ii)), and

$$\text{(iv}'\text{)}\qquad s_1\delta + s_2 + O(S^2) = 2t_1 + t_3 + t_2\delta + O(T^2)$$

can easily be solved for s in terms of t modulo higher-order terms:

$$(19)\qquad \begin{aligned} s_1 &= t_2 + \frac{1}{\delta}t_1 + O(T^2), \\ s_2 &= t_1 + t_3 + O(T^3), \\ s_3 &= -\frac{1}{\delta}t_1 + O(T^2). \end{aligned}$$

With these times the conjugate point relations cannot hold since

$$(20)\qquad \tilde{\sigma}(s_2, s_3) = -s_2 - s_3\delta + O(T^2) = -t_3 + O(T^2)$$

is negative. So the corresponding point q lies in fact in the relative interior and therefore it is possible to solve for t_3 in terms of t_1 and t_2:

$$t_3 = -t_1 - \delta t_2 + O(T^2). \tag{21}$$

This gives a solution to (18). Note that

$$t_2 = \rho T + O(T^2) = \frac{T}{1-\delta} + O(T^2). \tag{22}$$

As long as (t_1, t_2, t_3) are bounded away from the boundary of the simplex $t_1 + t_2 + t_3 \leqq T$, the times are comparable, these computations are justified, and we get a two-dimensional intersection that we can parametrize by t_1 and t_2. The problem is whether it extends all the way to the boundary. But the equations (19) and (21) are well defined for $t_1 \to 0$ (in a time-slice $t_1 + t_2 + t_3 = T$ it follows that $t_3 \to -\delta t_2 + O(t_2^2)$, i.e., to a limit of order T. By (20) this implies that the two-dimensional surface defined by these functions of (t_1, t_2) stays away from the conjugate point condition $\tilde{\sigma}(s_2, s_3) = 0$. Hence the implicit function theorem is still applicable.) Therefore $\hat{\Gamma}$ extends all the way out to $t_1 = 0$, i.e., to the XY boundary surface.

A precise characterization of $\tilde{\Gamma}^+ \cap \tilde{\Gamma}^- \cap \{p \exp(s_1 X) \exp(s_2 Y): s_i \geqq 0, \text{small}\}$ is possible. Clearly these are points such that $t_1 = 0$, $t_2 = s_1$, $t_3 = s_2$, and $0 = s_3$. Since $(s_1, s_2, 0) \in \operatorname{dom} \tilde{\Gamma}^-$ we have $\underset{\sim}{\sigma}(s_1, s_2) \leqq 0$, and since $(0, s_1, s_2) \in \operatorname{dom} \tilde{\Gamma}^+$ we have $\tilde{\tau}(s_1, s_2) \geqq 0$. But in this case $\underset{\sim}{\sigma}(s_1, s_2) = \tilde{\tau}(s_1, s_2)$ (cf. (16) and the analogous formula for $\tilde{\tau}$). Therefore $\underset{\sim}{\sigma}(s_1, s_2) = 0$, i.e., $s_1 = \overline{s_1}(s_2)$, the conjugate point relation.

This proves that $\tilde{\Gamma}^- \cap \tilde{\Gamma}^+$ extends all the way out to the XY-boundary surface and that the intersection with the XY-surface is the one-dimensional curve $\tilde{\gamma}$ consisting of the conjugate points.

Analogously we can show that (17) can also be solved in terms of s_1 and s_2 in $\operatorname{int}(\tilde{\Gamma}^-)$. Using these formulas we can show that $\tilde{\Gamma}^- \cap \tilde{\Gamma}^+$ extends all the way up to the YX-boundary surface and that the intersection of $\tilde{\Gamma}^- \cap \tilde{\Gamma}^+$ with the YX-surface consists of the curve $\underset{\sim}{\gamma}$. □

Note that in a time-slice $t_1 + t_2 + t_3 = T$ the qualitative geometric structure of $\tilde{\Gamma}^- \cup \tilde{\Gamma}^+$ is exactly as in the free nilpotent case. Only the condition $t_2 = T/2$ is replaced by $t_2 \doteq (1/(1-\delta))T$ (modulo higher terms) which shifts $\hat{\Gamma}$ away from the center. This is illustrated in Fig. 6.

The surface $\hat{\Gamma}$ bisects $\tilde{\Gamma}^+$ and $\tilde{\Gamma}^-$ and only one of the two components has the Y-, respectively, X-trajectory in its boundary. We define Γ^- and Γ^+ to be these components

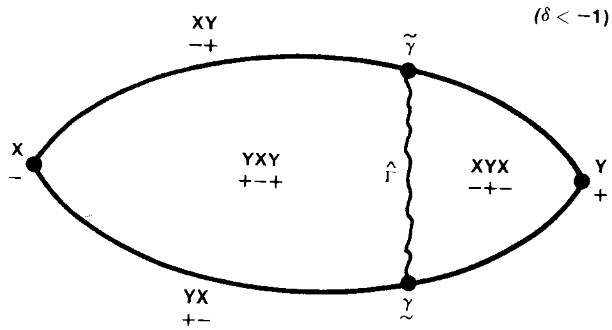

FIG. 6

and let $\Gamma^* = \Gamma^+ \cup \Gamma^-$. It is then clear that Γ^* is a three-dimensional stratified surface whose relative boundary consists of all bang-bang trajectories with at most one switching, i.e., $\partial\Gamma^* = \partial\Gamma_*$.

LEMMA 6. *The points in* Γ^* *are exit points of the small-time reachable set for* $[X, [X, Y]]$.

Proof. It is easy to see (cf. (10)) that, for sufficiently small time, all extremals on $\tilde{\Gamma}^-$ or $\tilde{\Gamma}^+$ satisfy the necessary transversality condition $\langle \lambda, [X, [X, Y]] \rangle \leqq 0$.

We show first that the points that we deleted from $\tilde{\Gamma}^-$ and $\tilde{\Gamma}^+$ are not exit points (see Fig. 7). Let

$$q = p \exp(s_1 X) \exp(s_2 Y) \exp(s_3 X) = p \exp(t_1 Y) \exp(t_2 X) \exp(t_3 Y)$$

be a point in the relative interior of $\hat{\Gamma}$. $\tilde{\Gamma}^-$ and $\tilde{\Gamma}^+$ intersect transversally. It follows as in the proof of Lemma 2 (cf. (11)) that the XYX- and YXY-surfaces are graphs $x_4 = \psi(x_1, x_2, x_3)$ in canonical coordinates of the second kind with x_4 the coefficient at the flow of $[X, [X, Y]]$. This inherits on $\tilde{\Gamma}^-$ and $\tilde{\Gamma}^+$. To prove that the parts of $\tilde{\Gamma}^-$ (respectively, $\tilde{\Gamma}^+$) that we delete are not exit points, it suffices to show that these parts lie below $\tilde{\Gamma}^+$ (respectively, $\tilde{\Gamma}^-$) in direction of $[X, [X, Y]]$.

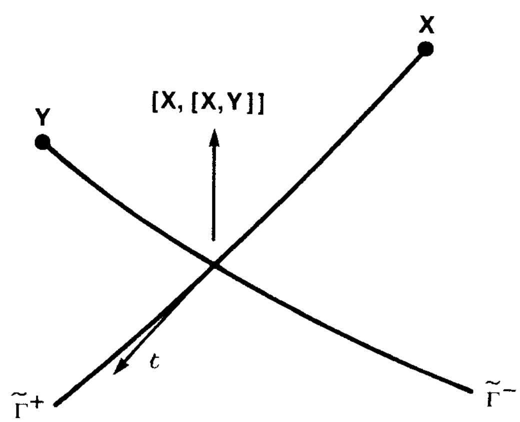

FIG. 7

The tangent space to $\tilde{\Gamma}^-$ at q is spanned by X, $\exp(-s_3 \operatorname{ad} X)Y$ and $\exp(-s_3 \operatorname{ad} X)(\exp((-s_2 \operatorname{ad} Y) - 1)/-s_2)X$. To show that the part of $\tilde{\Gamma}^+$ that we deleted lies below $\tilde{\Gamma}^-$ near q it suffices to show that $[X, [X, Y]]$ and a tangent vector t to $\tilde{\Gamma}^+$ that is oriented toward the sector of $\tilde{\Gamma}^+$ that we deleted point to opposite sides of $T_q\tilde{\Gamma}^-$. We get such a vector t if we lengthen the time along the last Y leg. (We delete the piece that contains in its boundary the trajectories corresponding to the conjugate point relation $t_3 = \bar{t}_3(t_2)$.)

Instead of computing at q we pull back all vectors along X, s_3 and get

$$\begin{aligned}
&\exp(+s_3 \operatorname{ad} X)(T_q\tilde{\Gamma}^-) \wedge \exp(+s_3 \operatorname{ad} X)[X, [X, Y]] \\
&= \left(X \wedge Y \wedge \left(\frac{\exp(-s_2 \operatorname{ad} Y) - 1}{-s_2} \right) X \wedge \exp(s_3 \operatorname{ad} X)[X, [X, Y]] \right) \\
&= -(\delta + O(T))(X \wedge Y \wedge [X, Y] \wedge [Y, [X, Y]])|_{p_2 = p \exp(s_1 X) \exp(s_2 Y)},
\end{aligned}$$

$$\exp(s_3 \operatorname{ad} X)(T_q\tilde{\Gamma}^-) \wedge \exp(s_3 \operatorname{ad} X)Y$$
$$= s_3\left(X \wedge Y \wedge \left(\frac{\exp(-s_2 \operatorname{ad} Y)-1}{-s_2}\right) X \wedge \left(\frac{\exp(s_3 \operatorname{ad} X)-1}{s_3}\right) Y\right)$$
$$= \frac{1}{2} s_3\tilde{\sigma}(s_2, s_3)(X \wedge Y \wedge [X, Y] \wedge [Y, [X, Y]])|_{p_2}.$$

But q is a point in $\hat{\Gamma}$, and $\hat{\Gamma}$ lies entirely in the relative interior of $\tilde{\Gamma}^-$ except for the obvious boundary curves $\tilde{\gamma}$ and $\underset{\sim}{\gamma}$. In particular (cf. also the proof of Lemma 5) the conjugate point relation $s_3 = \bar{s}_3(\tilde{s}_2)$ does not hold, or equivalently, $\tilde{\sigma}(s_2, s_3) < 0$. So these wedge-products have opposite signs, which proves our claim. This also implies that the portion of $\tilde{\Gamma}^-$ that we delete lies below $\tilde{\Gamma}^+$, and since there is no other intersection this holds for all the points we deleted.

The stratified sets Γ^* and Γ_* enclose a region R that lies in the small-time reachable set. In particular, the portions of $\tilde{\Gamma}^-$ and $\tilde{\Gamma}^+$ that we deleted therefore lie in the interior of the reachable set. Since these pieces contain the trajectories corresponding to the conjugate points $t_3 = \bar{t}_3(t_2)$ and $s_3 = \overline{s_3}(s_2)$, it follows that no bang-bang trajectory with more than two switchings lies in the boundary of the small-time reachable set. Hence the points in Γ^* are the only possible exit points of the small-time reachable set for $[X, [X, Y]]$. It follows from the construction of Γ^- and Γ^+ that Γ^* is also a graph. Again, the projections onto (x_1, x_2, x_3)-space are disjoint. Therefore it follows as in Lemma 2 that the points on Γ^* have the exit property for sufficiently small time. □

Finally, Γ^* and Γ_* do not intersect in their relative interiors. It is now clear how the *small-time reachable* set looks: It is the set of points enclosed by the two three-dimensional stratified surfaces Γ^* and Γ_*. Γ^* consists of bang-bang trajectories with at most two switchings such that modulo higher-order terms

$$t_1 + \delta t_2 + t_3 \leqq 0 \tag{23}$$

if t_1, t_2, and t_3 are the consecutive times along a YXY arc and

$$s_1\delta + s_2 + s_3\delta \geqq 0 \tag{24}$$

if s_1, s_2, s_3 are consecutive times along XYX. Γ_* consists of all concatenations of a bang arc, followed by a singular arc and another bang arc where the time along the trajectories is free. Γ^* and Γ_* have a common relative boundary C consisting of all trajectories that are bang-bang with at most one switching. For sufficiently small-time T a time-slice of the reachable set has exactly the same qualitative geometric structure as for the free nilpotent system (13). Furthermore, if $\delta(\cdot)$ is an integral curve of $[X, [X, Y]]$ such that $\delta(t_1)$ and $\delta(t_2)$, $t_1 < t_2$, lie in the small-time reachable set, then so does the whole curve $\delta(t)$, $t_1 \leqq t \leqq t_2$. The points on Γ_* are entry points for $[X, [X, Y]]$; the points on Γ^* are exit points.

Remark. We emphasize that the result is not what might be expected intuitively. From dimensionality we could conjecture the occurrence of bang-bang trajectories with two switchings, respectively, *BSB* trajectories in the boundary of the small-time reachable set. Also, this is essentially what was partially known from earlier results. However, we see no simple reasoning that could explain why, in fact, *some* of these *bang-bang trajectories with two switchings are not a part of the boundary.* This is only revealed by our analysis.

4.3. Time-optimal control in dimension three. Our results have immediate implications on time-optimal control in dimension three. Suppose the triples $(g, [f, g], [f+g, [f, g]])$ and $(g, [f, g], [f-g, [f, g]])$ consist of independent vectors at a point p in $\mathbb{R}^3$.

Equivalently, suppose that the constant controls $u \equiv +1$ and $u \equiv -1$ are not singular. If we augment the three-dimensional system Σ to a four-dimensional system $\hat{\Sigma}$ by introducing time as a coordinate, $\dot{x}_0 = 1$, $x_0(0) = 0$, i.e.,

$$\hat{f} = \begin{pmatrix} 1 \\ f \end{pmatrix}, \qquad \hat{g} = \begin{pmatrix} 0 \\ g \end{pmatrix},$$

then if a Σ-trajectory $x(\cdot):[0, T] \to \mathbb{R}^3$ steering p to q is time-optimal, the augmented trajectory $\hat{x}$ lies in the boundary of the reachable set from p. The augmented system $\hat{\Sigma}$ satisfies our assumptions ($\underline{A}$) and ($\underline{B}$), and therefore time-optimal trajectories are bang-bang with at most two switchings or concatenations of a bang-arc, followed by a singular arc and one more bang arc. Under additional assumptions this result was obtained earlier by Bressan [4], who studied only trajectories emanating from an equilibrium point of f and by Sussmann [22] and Schättler [17] who both assumed in addition also that f, g and $[f, g]$ were independent. Our analysis shows that the vector field f is irrelevant and we do not have to make any assumptions about it. Our results are also more precise in the sense that we can exclude the optimality of those bang-bang trajectories with two switchings that violate (23) (respectively, (24)) in the bang-bang singular case. We summarize in the following corollary.

COROLLARY. *Suppose the vector fields g, $[f, g]$ and $[f+g, [f, g]]$ are independent near a reference point $p \in \mathbb{R}^3$. Write*

$$[f-g, [f, g]] = ag + b[f, g] + c[f+g, [f, g]]$$

and assume that c does not vanish. Then we have in small time:

(i) *If $c > 0$, then time-optimal trajectories are bang-bang with at most 2 switches.*

(ii) *If $c < 0$, then time-optimal trajectories are bang-bang with at most two switchings or are concatenations of a bang arc, a singular arc, and another bang arc. Time-optimal XYX (respectively, YXY) concatenations satisfy modulo higher-order terms*

$$c(s_1 + s_3) + s_2 \geqq 0 \qquad (\textit{resp.}, \ t_1 + t_3 + ct_2 \leqq 0)$$

where s_1, s_2, s_3 (respectively, t_1, t_2, t_3) are the consecutive times along the bang arcs.

5. A brief outlook to higher dimensions. We have outlined a general method to determine the structure of the small-time reachable sets and proved its effectiveness in nondegenerate cases in small dimensions. One of the difficulties that will become more and more prominent in higher dimensions is that the necessary conditions of the Maximum Principle will not restrict the class of extremal trajectories sufficiently enough to give the candidates for Γ^* and Γ_*.

Under assumptions ($\underline{A}$) and ($\underline{B}$) in dimension four, we could overcome this problem by taking a corresponding "free" nilpotent system of the same dimension as a guide. We do not expect this to happen in general. In fact, for the five-dimensional system Σ, where we assume that f, g, $[f, g]$, $[f, [f, g]]$, and $[g, [f, g]]$ are independent, the small-time reachable set has extremal trajectories in its boundary that do not appear in the analogous five-dimensional free nilpotent system. The reason for this lies in a qualitatively different behavior of the singular controls, specifically, in the fact that singular controls can now hit the control constraint $|u| = 1$ and may have to be terminated. Nevertheless, the free nilpotent system contains most of the information about the small-time reachable set, though it does not characterize it completely. To be more specific, we will briefly describe (without proofs) the structure of the reachable set for the free nilpotent system in dimension five and how the general case differs from it.

We take as our model:

$$\dot{x}_0 = 1, \quad \dot{x}_1 = u, \quad \dot{x}_2 = x_1, \quad \dot{x}_3 = x_2, \quad \dot{x}_4 = \tfrac{1}{2}x_1^2.$$

It is no problem whatsoever to carry out the analysis within our technique as in the construction in § 4.2.1. Now the reachable set is convex in direction of $(0, 0, 0, 0, 1)^T = [g, [g, f]]$ and Γ^*, respectively, Γ_* will consist of those trajectories that are exit, respectively, entry points.

It follows from the generalized Legendre-Clebsch condition that Γ_* contains concatenations with singular arcs, whereas Γ^* will consist of bang-bang trajectories only. Singular controls are constant, but now they can take on any value in $[-1, 1]$.

Let $\Gamma_* = \Gamma_{-u-} \cup \Gamma_{-u+} \cup \Gamma_{+u-} \cup \Gamma_{+u+}$, where

$$\Gamma_{-u-} := \{0 \exp(s_1X) \exp(s_2(f+ug)) \exp(s_3X): s_i \geqq 0,\ s_1+s_2+s_3 = 1,\ u \in [-1, 1]\},$$

etc. (By the invariance property of the reachable set we can restrict to the time-slice $T = 1$.) The points on Γ_* are precisely the ones that minimize the coordinate x_4.

For a fixed value u_0 of the singular control, $-1 < u_0 < +1$, the qualitative structure of $\Gamma_{*,u_0} = \Gamma_*$ restricted to values $u = u_0$ is precisely as in 4.2.2, Fig. 4 (see Fig. 8).

For $u_0 = +1$, $\Gamma_{-u-} \upharpoonright u = 1$ reduces to Γ_{-+-} and all other strata become trivial whereas for $u_0 = -1$, $\Gamma_{+u+} \upharpoonright u = -1 = \Gamma_{+-+}$ and the remaining strata are trivial. For each of these two-dimensional surfaces (u_0 fixed) the relative boundary consists of all bang-bang trajectories with at most one switching. The surfaces Γ_{*,u_0} themselves interpolate between Γ_{--+} for $u_0 = -1$ and Γ_{-+-} for $u_0 = 1$. Topologically Γ_* is a stratified sphere

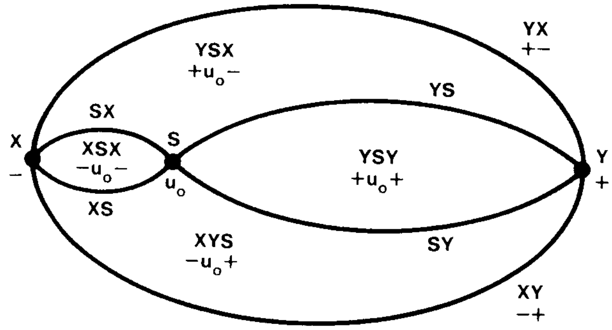

FIG. 8

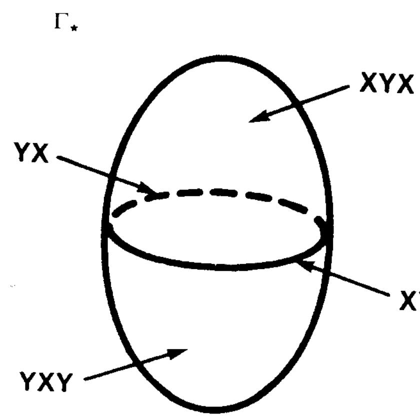

FIG. 9

with $\partial\Gamma_* = \Gamma_{-+-} \cup \Gamma_{+-+}$, i.e., all bang-bang trajectories with at most two switchings (see Fig. 9).

The surface Γ^* consists of bang-bang trajectories analogous to the bang-bang singular case in dimension four. Now

$$\tilde{\Gamma}^- = \{0 \exp(s_1 X) \exp(s_2 Y) \exp(s_3 X) \exp(s_4 Y) \colon s_i \geqq 0,\ s_1 + s_2 + s_3 + s_4 = 1,$$
$$s_1 \leqq s_3,\ s_4 \leqq s_2\text{-conjugate point relations}\},$$

$$\tilde{\Gamma}^+ = \{0 \exp(t_1 Y) \exp(t_2 X) \exp(t_3 Y) \exp(t_4 X) \colon t_i \geqq 0,\ t_1 + t_2 + t_3 + t_4 = 1,$$
$$t_1 \leqq t_3,\ t_4 \leqq t_2\text{-conjugate point relations}\}.$$

$\tilde{\Gamma}^-$ and $\tilde{\Gamma}^+$ intersect in a two-dimensional surface $\hat{\Gamma}$, which consists of those trajectories for which

$$(s_1 + s_3)^2 - (s_1 + s_3) + 2 s_2 s_3 = 0,$$

respectively,

$$(t_1 + t_3)^2 - (t_1 + t_3) + 2 t_2 t_3 = 0.$$

The intersection is transversal except at those points that lie on the relative boundary of $\tilde{\Gamma}^-$ or $\tilde{\Gamma}^+$. These points are again characterized by the conjugate point relation

$$\hat{\Gamma} \cap \Gamma_{-+-} = \{0 \exp(t_1 Y) \exp(t_2 X) \exp(t_3 Y) \exp(t_4 X) \colon t_1 = 0,\ t_4 = t_2\},$$

$$\hat{\Gamma} \cap \Gamma_{+-+} = \{0 \exp(t_1 Y) \exp(t_2 X) \exp(t_3 Y) \exp(t_4 X) \colon t_1 = t_3,\ t_4 = 0\}.$$

We define Γ^- (respectively, Γ^+) as the component of $\tilde{\Gamma}^-$ ($\tilde{\Gamma}^+$) containing the YX-curve $= \{0 \exp(s_2 Y) \exp(s_3 X) \colon s_i \geqq 0,\ s_2 + s_3 = 1\}$ (respectively, the XY-curve) in its boundary. Then $\Gamma^* := \Gamma^- \cup \Gamma^+$ consists precisely of those points that maximize x_4 on the reachable set. Note that topologically Γ^* also is a stratified sphere with $\partial\Gamma^* = \Gamma_{-+-} \cup \Gamma_{+-+}$, the set of all bang-bang trajectories with at most two switchings (see Fig. 10).

The key fact here is that it is still obvious that $\partial\Gamma^*$ and $\partial\Gamma_*$ match up. They are identical. It is therefore clear that Reach $(0, 1)$ is the set of all points that lie between Γ^* and Γ_*.

It is precisely this simple reasoning that breaks down in the general case. The cause for this lies in the structure of the singular controls. The analysis of the bang-bang trajectories carries over to the general case with only one minor change in the structure. Whereas in the free nilpotent system the two curves $\hat{\Gamma} \cap \Gamma_{+-+}$ and $\hat{\Gamma} \cap \Gamma_{-+-}$ both have points corresponding to the X- and Y-trajectories as endpoints, this need no longer be true: $\hat{\Gamma} \cap \Gamma_{+-+}$ is a curve starting at $0 \exp(1 \cdot Y)$ but which in general no longer ends in $0 \exp(1 \cdot X)$ but rather on a point in the XY-curve (respectively, YX-curve). This distortion is due to the presence of fourth-order brackets. One possible case is depicted in Fig. 11.

Still the relative boundary of Γ^* consists of all bang-bang trajectories with at most two switchings. The structure breaks down in the analysis of the singular surface Γ_* for u near ± 1. The reason is that in the presence of fourth-order brackets the singular controls are no longer constant, and thus the analogue of Γ_{*,u_0} for $u_0 = -1$ does not reduce to Γ_{+-+}, i.e., to bang-bang trajectories with two switchings. For instance, it may not be at all possible to start a singular control with $u_0 = -1$. This is the case if $\dot{u} < 0$ at $u_0 = -1$, which happens under generic assumptions on fourth-order brackets.

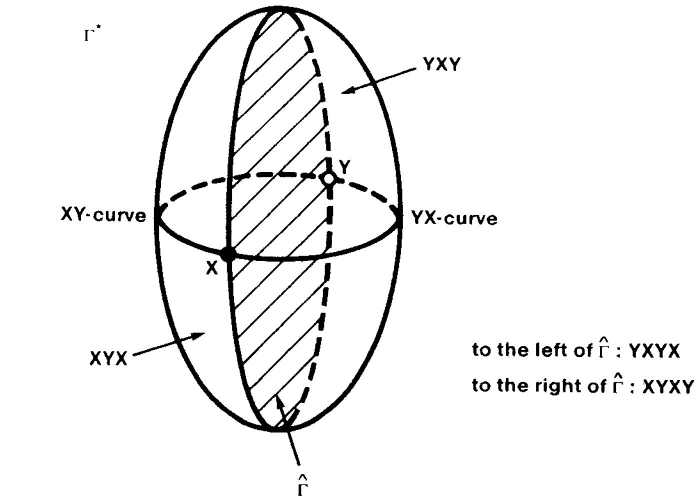

FIG. 10

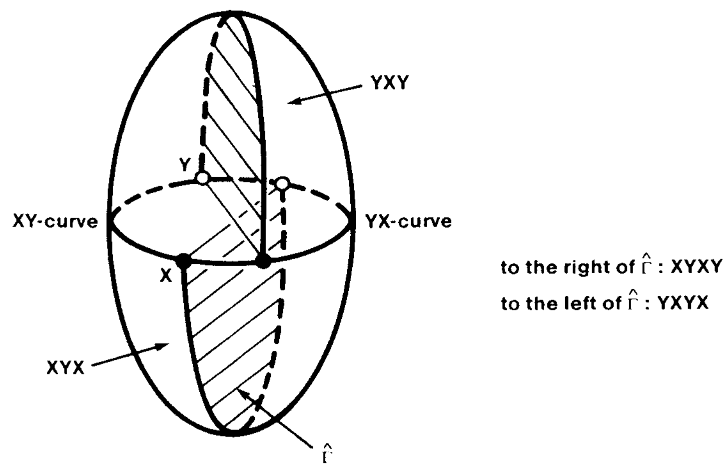

FIG. 11

For the same reason, singular controls with u_0 close to ± 1 may have to be terminated when they become one in absolute value. If the singular control becomes saturated (i.e., hits the constraint and cannot be continued) then this determines the subsequent structure of the trajectory and it is easy to see that concatenations such as *BSBB* or *BBSB*, which are not present in the free nilpotent system, come into play. Therefore Γ_* has trajectories in its relative boundary that contain singular arcs. The main challenge in applying our technique to higher dimensions seems to be finding a way to decide whether structurally different trajectories, such as a bang-bang trajectory, and a concatenation that contains a singular arc steer a system to the same point. Once $\partial\Gamma^*$ and $\partial\Gamma_*$ can be identified, it is clear that the set they enclose is the small-time reachable set.

Note, however, that this structural instability only happens near $\Gamma_{*,-1}$ and $\Gamma_{*,+1}$. The structure of most of the trajectories in the boundary is still the same as in the free nilpotent systems. And it is intuitively clear that the structure of the exceptional trajectories will come up in a higher-dimensional nilpotent system. Therefore, in our view, the study of the structure of the reachable sets for nilpotent systems will be the key to the general problem.

6. Summary. We have described an approach to determining the qualitative structure of the small-time reachable set in a nondegenerate situation. It is a nontrivial extension of a construction done by Lobry in dimension three. In dimension four we succeed completely in determining the small-time reachable set. For higher dimensions obstacles still have to be overcome. However, they do not lie in the general structure of our approach, but in the fact that too little is known about the structure of extremal trajectories in higher dimensions. For instance, in the five-dimensional case, what is the precise structure of extremal trajectories that contain a saturated singular arc? For dimensions six and beyond, the crucial new ingredient appears to be the incorporation of chattering arcs, another structure of extremal trajectories about which little is still known.

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