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Reciprocal Diffusions and Stochastic Differential Equations of Second Order*

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We develop a theory of second order diffusion processes and associated stochastic differential equations of second order. We show that equations of evolution of the density, mean velocity and momentum flux are a family of first order conservation laws similar to those of continuum mechanics. We verify that the theory is satisfied for a large class of reciprocal Gaussian processes.

KEY WORDS: Reciprocal process, second order diffusion, second order stochastic differential equation, conservation law, statistical mechanics, continuum mechanics, stochastic mechanics.

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1. RECIPROCAL DIFFUSIONS

One of the most beautiful parts of modern mathematics is the rich and wonderful interplay between Markov diffusion processes, linear parabolic partial differential equations and stochastic differential equations of first order. We shall describe the foundations of a

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parallel theory involving reciprocal diffusion processes, nonlinear conservation laws and stochastic differential equations of second order.

Let $(\Omega, \mathscr{F}, \Pr)$ be a probability triple consisting of a sample space Ω, σ-algebra of events $\mathscr{F}$ and probability measure Pr. Let E denote expectation with respect to Pr. Throughout we let $x(t)$ denote a stochastic process over this triple defined for $t \in [0, T]$ and taking values in $\mathbb{R}^{n \times 1}$. We assume that

$$E \int_0^T |x(t)|^2 \, dt < \infty.$$

Given $0 \leqq t_0 \leqq t_1 \leqq T$, let $\mathscr{T}(t_0, t_1)$, $\mathscr{E}(t_0, t_1)$ and $\mathscr{B}(t_0, t_1)$ be the σ subalgebras of $\mathscr{F}$ generated by $x(t)$ interior to, exterior to and on the boundary of the interval defined by t_0 and t_1. In other words

$$\mathscr{T}(t_0, t_1) = \sigma\{x(t) : t \in [t_0, t_1]\}$$

$$\mathscr{E}(t_0, t_1) = \sigma\{x(t) : t \in [0, t_0] \cup [t_1, T]\}$$

$$\mathscr{B}(t_0, t_1) = \sigma\{x(t_0), x(t_1)\}.$$

We denote by $\tilde{\mathscr{T}}(t_0, t_1)$, $\tilde{\mathscr{E}}(t_0, t_1)$ and $\tilde{\mathscr{B}}(t_0, t_1)$ the space of square integrable random variables which are measurable with respect to $\mathscr{T}(t_0, t_1)$, $\mathscr{E}(t_0, t_1)$ and $\mathscr{B}(t_0, t_1)$ respectively.

The concept of a reciprocal process was introduced by Bernstein [1] following ideas of Schrödinger [2, 3]. A process $x(t)$ is *reciprocal* if on every subinterval of $[0, T]$, the interior and exterior are conditional independent given the boundary. More precisely, if $\phi \in \tilde{\mathscr{T}}(t_0, t_1)$ and $\psi \in \tilde{\mathscr{E}}(t_0, t_1)$ then

$$E(\phi\psi \mid \mathscr{B}(t_0, t_1)) = E(\phi \mid \mathscr{B}(t_0, t_1)) E(\psi \mid \mathscr{B}(t_0, t_1)).$$

We refer the reader to [4–17] for more detailed discussions of reciprocal processes. The have also been called quasi Markov or Bernstein processes. They are closely related to conditionally Markov processes. Following Schrödinger's original motivation and Nelson's stochastic mechanics, Zambrini [20–22] has related reciprocal processes to quantum mechanics.

It follows immediately from the definition that Markov processes are reciprocal but not vice versa [5].

To define reciprocal diffusions we must introduce some notation. Given a process $x(t)$ and a small time increment $dt > 0$, define the centered average evaluation $\bar{x}$, centered first difference dx and centered second difference d^2x as

$$\bar{x}(t; dt) = \frac{x(t+dt) + x(t-dt)}{2}$$

$$dx(t; dt) = \frac{x(t+dt) - x(t-dt)}{2}$$

$$d^2x(t; dt) = x(t+dt) - 2x(t) + x(t-dt).$$

Frequently when the context is clear we suppress the argument dt as in $\bar{x}(t)$, or both t and dt as in dx. We also have the forward d^+x and backward d^-x first differences

$$d^+x(t; dt) = x(t+dt) - x(t)$$

$$d^-x(t; dt) = x(t) - x(t-dt).$$

In contrast to the standard conditional expectation of Markov theory,

$$E_x(\cdot) = E_{x(t)}(\cdot) = E(\cdot \mid x(t) = x)$$

we shall utilize

$$E_{\bar{x}}(\cdot) = E_{\bar{x}(t;dt)}(\cdot) = E(\cdot \mid \bar{x}(t; dt) = x)$$

and occasionally the stronger conditioning

$$E_{\bar{x},dx}(\cdot) = E_{x(t \pm dt)}(\cdot) = E(\cdot \mid x(t \pm dt) = x \pm v\,dt)$$

$$= E(\cdot \mid \bar{x}(t; dt) = x, dx(t; dt) = v\,dt).$$

We now give the second order analogs of the Feller postulates for a first order diffusion. A stochastic process $x(t)$ is a *second order diffusion* if there exist functions $f_i(x,t)$, $g_{ij}(x,t)$, $h_{ik}(x,t)$, $u_i(x,t)$ for $i, j=1,\dots,m$ such that

$$E_{\bar{x}}(dx_i)=u_i(x,t)\,dt+o(dt) \tag{1.1a}$$

$$E_{\bar{x}}(dx_i\,dx_j)=\tfrac{1}{2}h_{ik}(x,t)h_{jk}(x,t)\,dt+\pi_{ij}(x,t)\,dt^2+o(dt)^2 \tag{1.1b}$$

$$E_{\bar{x}}(d^2x_i)=(f_i(x,t)+g_{ij}(x,t)u_j(x,t))\,dt^2+o(dt)^2 \tag{1.1c}$$

$$E_{\bar{x}}(d^2x_i\,d^2x_j)=2h_{ik}(x,t)h_{jk}(x,t)\,dt+o(dt)^2 \tag{1.1d}$$

$$E_{\bar{x}}(d^2x_i\,dx_j)=\tfrac{1}{2}g_{ik}(x,t)h_{kl}(x,t)h_{jl}(x,t)\,dt^2+o(dt)^2 \tag{1.1e}$$

The higher conditional moments of dx and d^2x agree to the lowest nonzero powers of dt with those of Gaussians with the above first and second moments. (1.1f)

In the above we have utilized the summation convention. Conditioned on $\bar{x}(t;dt)=x$ for fixed x,t, the expression $o(dt)^k$ is a deterministic quantity $y(dt;x,t)$ which vanishes faster than dt^k as $dt\to 0$ uniformly in x and t. In other words for every $\varepsilon>0$ there exist $\delta>0$ such that $|y(dt;x,t)|<\varepsilon\,dt^k$ for all $dt<\delta, x\in\mathbb{R}^n$ and $t\in(0,T)$. We denote by $0(dt)^k$ a quantity $y(dt;x,t)$ for which there exist $\varepsilon,\delta>0$ such that $|y(dt;x,t)|<\varepsilon\,dt^k$ if $dt<\delta$.

If $x(t)$ is both a reciprocal process and a second order diffusion then we say it is a *reciprocal diffusion.*

A stochastic process $x(t)$ satisfies the *second order stochastic differential equation*

$$d^2x=f(x,t)\,dt^2+g(x,t)\,dx\,dt+h(x,t)\,d^2w \tag{1.2}$$

where $w(t)$ is a standard m dimensional Wiener process if $x(t)$ is a reciprocal diffusion satisfying (1.1a–f). Actually (1.2) is a mnemonic description of (1.1) in the same way that the first order stochastic differential equation

$$d^+x=f(x,t)\,dt+h(x,t)\,d^+w \tag{1.3}$$

is a mnemonic for the axioms of a first order diffusion.

$$E_x(d^+x_i) = f_i(x,t)\,dt + o(dt) \tag{1.4a}$$

$$E_x(d^+x_i\,d^+x_j) = h_{ik}(x,t)h_{jk}(x,t)\,dt + o(dt) \tag{1.4b}$$

all higher centered conditional moments of dx vanish like $o(dt)$. (1.4c)

In particular (1.1c) asserts that conditional mean acceleration equals $f + gu$ where u is the conditional mean velocity given by (1.1a). Note the difference between the conditional expectations in (1.2) and (1.4). Conditioning on $\bar{x}(t;dt) = x$ is an essential part of the second order stochastic calculus. If we were to condition on $x(t) = x$, we would find that generally $E_x(d^2x)$ is order dt rather than dt^2. In fact it is precisely this conditioning which distinguishes our work from that of Nelson [18] and Zambrini [20–22]. In Nelson's stochastic mechanics the order dt part of $E_x(dx)$ is a vector field $v(x,t)$ called the current velocity while the order dt part of $1/2E_x(d^2x)$ is another vector field $u(x,t)$ called the osmotic velocity. In our work the order dt part of $E_{\bar{x}}(dx)$ is a vector field $u(x,t)$ called the conditional mean velocity. For a Gaussian process with a smooth covariance, Nelson's current velocity equals our conditional mean velocity and we suspect that this is true whenever both exist. On the other hand in our theory $E_{\bar{x}}(d^2x)$ is postulated to be of order dt^2. Hence the coefficient of dt^2 can be viewed as an acceleration. For this reason it differs from Nelson's osmotic velocity. Zambrini's work also uses the Nelson framework.

We call the $n \times 1$ vector field $u(x,t)$ the *mean velocity*. The density of $x(t)$ is denoted by $\rho(x,t)$. The $n \times n$ tensor field $\rho(x,t)\ \pi(x,t)$ is called the *momentum flux tensor*. A related $n \times n$ tensor field, $\rho(x,t)\ \sigma_{ij}(x,t) = \rho(x,t)(u_i(x,t)\ u_j(x,t) - \pi_{ij}(x,t))$, is called the *stress tensor*. The reason for these names will become apparent in Section Four.

Formulas (1.1, 2) suggest that the random parts of dx and d^2x conditioned on $\bar{x}(t;dt) = x$ are given by

$$\widetilde{dx} = dx - E_{\bar{x}}(dx) = dx - u(x,t)\,dt$$

$$\widetilde{d^2x} = d^2x - E_{\bar{x}}(d^2x) = g(x,t)\widetilde{dx} + h(x,t)\,d^2w.$$

We use x^* denote the transpose of x and x^{*2} to denote the symmetric square or outer product, $x^{*2} = xx^*$.

Note that $u(x,t)$ and $\pi(x,t)$ from (1.1) do not appear explicitly in (1.2). As we shall see in Section 4, this is because these quantities and the density $\rho(x,t)$ satisfy a system of nonlinear conservation laws determined by f, g and h. This system of four first order partial differential equations is very similar to the equations of fluid and continuum mechanics. They express conservation of probability, balance of momentum and balance of a tensor form of work in two time scales. They replace the familiar Fokker–Planck equation for a first order diffusion (1.3, 4).

Equations (1.1a–e) assert that the conditional moments of dx and d^2x can be expanded in powers of dt as shown. These formulas give names to the coefficients. The only constraints on the coefficients are found by comparing (1.1b, d and e). Equation (1.1a) asserts that the conditional mean velocity exists and gives it a name $u(x,t)$. Equation (1.1b) describes the variance of dx. The order dt part arises from the fluctuation of the second difference $d^2w\ (t;dt)$ of the Wiener process that appears in (1.2). The factors of 1/2 and 2 in the dt part of (1.1b, d) are explained by

$$E_{\bar{x}}(dw)^{*2}=\tfrac{1}{2}I\,dt$$

$$E_{\bar{x}}(d^2w)^{*2}=2I\,dt.$$

The second order part $\pi\,dt^2$ of (1.1b) has a deterministic contribution $u^{*2}\,dt^2$ from (1.1a) and a stochastic contribution $-\sigma\,dt^2=(\pi-u^{*2})\,dt^2$ from the noise throughout $[0,T]$.

The immediate question that arises is "Are there any second order or reciprocal diffusions?". We answer this in the affirmative in the next section by showing that under mild technical assumptions Gaussian processes with smooth covariances are second order diffusions, and Gaussian reciprocal processes with smooth covariances are reciprocal diffusions. Of course the latter includes Gauss–Markov processes. We derive explicit formulas for the quantities appearing in (1.1) in terms of the covariance matrix of the process.

In Section Three we explore how second order diffusions transform under change of variables and in Section Four we derive the conservation and balance laws which are described above. In Section 5 we verify that Gaussian reciprocal processes of Section 2 satisfy these laws.

2. RECIPROCAL AND GAUSSIAN PROCESSES

Let $x(t)$ be a Gaussian process defined on $[0, T]$ and taking values in $\mathbb{R}^{n\times 1}$. For convenience we assume $x(t)$ is zero mean and we denote by $R(t,s)$ its covariance matrix,

$$E(x(t))=0$$

$$E(x(t)x^*(s))=R(t,s).$$

We shall assume that $R(t,s)$ is a smooth (C^∞) function of t,s in the triangle $0\leqq s\leqq t\leqq T$ and the limits of R and its partial derivatives exist and are continuous on the closed triangle. Because $R(t,s)=R^*(s,t)$ we need only consider R in this triangle.

We shall also assume that

$$R(t,t)=I \tag{2.1a}$$

$$\begin{bmatrix} I & R^*(t+\tau,t-\tau) \\ R(t+\tau,t-\tau) & I \end{bmatrix} \text{is nonsingular for small } \tau>0 \tag{2.1b}$$

$$\frac{\partial R}{\partial t}(t,t)+\frac{\partial R^*}{\partial s}(t,t)<0. \tag{2.1c}$$

All evaluations of R and its partials at $t=s$ are limits of values in the interior of the triangle $0\leqq s\leqq t\leqq T$. Notice that (2.1a) is merely a normalization assuming $R(t,t)$ is invertible. Moreover (2.1c) essentially implies (2.1a, b) holds for almost all t.

In [15] we showed that any stationary reciprocal Gaussian process satisfying (2.1) has a C^∞ covariance in the above sense and moreover the covariance $R(t-s)=R(t,s)$ satisfies a pair of second order matrix differential equations

$$\ddot{R}=G\dot{R}+FR \tag{2.2a}$$

$$\ddot{R}^*=-G\dot{R}^*+FR^*. \tag{2.2b}$$

We now extend this to the nonstationary case. We refer the reader to [14, 15] for full details.

Let $0 \leqq s < t < T$ then by the Gaussian and reciprocal properties the covariance $R(t,s)$ satisfies for $\tau > 0$ sufficiently small

$$R(t,s) = [R(t,t-\tau)R(t,t+\tau)]\begin{bmatrix} I & R^*(t+\tau,t-\tau) \\ R(t+\tau,t-\tau) & I \end{bmatrix}^{-1} \times \begin{bmatrix} R(t-\tau,s) \\ R(t+t,s) \end{bmatrix}. \tag{2.3}$$

We define $K_i(t,\tau)$ in the obvious fashion so that this becomes

$$R(t,s) = [K_1(t,\tau)K_2(t,\tau)]\begin{bmatrix} R(t-\tau,s) \\ R(t+\tau,s) \end{bmatrix}$$

We differentiate this twice with respect to τ to obtain

$$0 = [K_1(t,\tau)K_2(t,\tau)]\begin{bmatrix} \frac{\partial^2}{\partial t^2}R(t-\tau,s) \\ \frac{\partial^2}{\partial t^2}R(t+\tau,s) \end{bmatrix} + 2\left[\frac{\partial K_1}{\partial \tau}(t,\tau)\frac{\partial K_2}{\partial \tau}(t,\tau)\right]\begin{bmatrix} \frac{\partial}{\partial t}R(t-\tau,s) \\ \frac{\partial}{\partial t}R(t+\tau,s) \end{bmatrix} + \left[\frac{\partial^2 K_1}{\partial \tau^2}(t,\tau)\frac{\partial^2 K_2}{\partial \tau^2}(t,\tau)\right]\begin{bmatrix} R(t-\tau,s) \\ R(t+\tau,s) \end{bmatrix}.$$

By (2.1) and arguments similar to those of [15] we verify that the limits of $K(t,\tau)$, $\partial/\partial\tau K(t,\tau)$ and $(\partial/\partial\tau)^2 K(t,\tau)$ exist as $\tau \to 0$. In particular $K_i(t,0) = \frac{1}{2}I$ and so we obtain for all $0 \leqq s < t < T$

$$\frac{\partial^2}{\partial t^2}R(t,s) = G(t)\frac{\partial R}{\partial t}(t,s) + F(t)R(t,s) \tag{2.4}$$

where

$$G(t) = -2\left(\frac{\partial K_1}{\partial \tau}(t,0) - \frac{\partial K_2}{\partial \tau}(t,0)\right) \tag{2.5a}$$

$$F(t) = -\left(\frac{\partial^2 K_1}{\partial \tau^2}(t,0) + \frac{\partial^2 K_2}{\partial \tau^2}(t,0)\right). \tag{2.5b}$$

But notice in our derivation of (2.4) we did not use the fact that $t>s$, we only used the fact that $t \in (t-\tau, t+\tau)$ and $s \notin (t-\tau, t+\tau)$. Hence (2.4) must also hold for $0<t<s \leqq T$. Since $R(t,s) = R^*(s,t)$ we conclude that for $0<t<s \leqq T$,

$$\frac{\partial^2}{\partial t^2} R^*(s,t) = G(t)\frac{\partial R^*}{\partial t}(s,t) + F(t)R^*(s,t).$$

By interchanging the symbols t and s, we see that for $0<s<t \leqq T$ we have

$$\frac{\partial^2}{\partial s^2} R^*(t,s) = G(s)\frac{\partial R^*}{\partial s}(t,s) + F(s)R^*(t,s) \tag{2.6}$$

By continuity (2.4) and (2.6) must hold for $0 \leqq s \leqq t \leqq T$. We also obtain alternative formulas for $G(t)$ and $F(t)$, namely

$$G(t) = \left(\frac{\partial^2 R}{\partial t^2}(t,t) - \frac{\partial^2 R^*}{\partial s^2}(t,t)\right)\left(\frac{\partial R}{\partial t}(t,t) - \frac{\partial R^*}{\partial s}(t,t)\right)^{-1} \tag{2.7a}$$

$$F(t) = \frac{\partial^2 R}{\partial t^2}(t,t) - G(t)\frac{\partial R}{\partial t}(t,t) = \frac{\partial^2 R^*}{\partial s^2}(t,t) - G(t)\frac{\partial R^*}{\partial s}(t,t) \tag{2.7b}$$

We define $H(t)H^*(t)$ by

$$H(t)H^*(t) = -\left(\frac{\partial R}{\partial t}(t,t) - \frac{\partial R^*}{\partial s}(t,t)\right). \tag{2.7c}$$

We have proved

THEOREM 2.1 *Let $x(t)$ be a zero mean Gaussian reciprocal process with smooth covariance $R(t,s)$ satisfying* (2.1). *Then $R(t,s)$ satisfies the matrix differential equations* (2.4) *and* (2.6) *on $0 \leqq s \leqq t \leqq T$ where $F(t)$ and $G(t)$ are given by* (2.7*a*, *b*).

Every Gauss–Markov process $x(t)$ with a smooth covariance is an Ornstein–Uhlenbeck process, i.e., a solution of a first order linear stochastic differential equation of the form

$$d^+x = F(t)x\,dt + H(t)\,d^+w.$$

The next theorem shows that every Gaussian reciprocal process $x(t)$ with smooth covariance satisfying (2.1) is a solution of a second order linear stochastic differential equation

$$d^2x = F(t)x\,dt^2 + G(t)\,dx\,dt + H(t)\,d^2w.$$

THEOREM 2.2 *Let $x(t)$ be a Gaussian process with smooth covariance $R(t,s)$ satisfying* (2.1). *Then $x(t)$ is a second order diffusion. If $x(t)$ is also reciprocal then it is a reciprocal diffusion. In either case $f(x,t)=F(t)x$, $g(x,t)=G(t)$ and $h(x,t)=H(t)$ of* (2.7) *and $u(x,t)$, $\pi(x,t)$ are given by*

$$u(x,t) = U(t)x \tag{2.8a}$$

$$U(t) = \frac{1}{2}\left(\frac{\partial R}{\partial t}(t,t) + \frac{\partial R^*}{\partial s}(t,t)\right) \tag{2.8b}$$

$$\pi(x,t) = u(x,t)u^*(x,t) - \sigma(x,t) \tag{2.8c}$$

$$\sigma(x,t) = \sigma(t) = -\frac{1}{2}\left(\frac{\partial^2 R}{\partial t\,\partial s}(t,t) + \frac{\partial^2 R^*}{\partial t\,\partial s}(t,t)\right) + U(t)U^*(t). \tag{2.8d}$$

Proof The proof of this theorem is a straightforward exercise in computing conditional expectations of Gaussian random variables. We shall sketch the details.

By assumption (2.1)

$$E(x(t))^{*2} = R(t,t) = I \tag{2.9a}$$

so

$$0 = \frac{\partial R}{\partial t}(t,t) + \frac{\partial R}{\partial s}(t,t) \tag{2.9b}$$

and

$$0 = \frac{\partial^2 R}{\partial t}(t,t) + 2\frac{\partial^2 R}{\partial t\,\partial s}(t,t) + \frac{\partial^2 R}{\partial s^2}(t,t). \tag{2.9c}$$

In particular (2.9b) insures that HH^* (2.7a) is symmetric and $U(t)$ (2.8b) is skew symmetric.

Next

$$E(dx(t))^{*2} = \tfrac{1}{4}(R(t+dt, t+dt) - R(t+dt, t-dt)$$
$$- R^*(t+dt, t-dt) + R(t-dt, t-dt))$$

By Taylor series expansion and (2.9) we obtain

$$E(dx(t))^{*2} = -\frac{1}{2}\left(\frac{\partial R}{\partial t}(t,t) - \frac{\partial R^*}{\partial s}(t,t)\right)dt$$
$$+\frac{1}{2}\left(\frac{\partial^2 R}{\partial t\,\partial s}(t,t) + \frac{\partial^2 R^*}{\partial t\,\partial s}(t,t)\right)dt^2 + o(dt)^2. \tag{2.10a}$$

In a similar fashion we obtain

$$E(\bar{x}(t))^{*2} = I + \frac{1}{2}\left(\frac{\partial R}{\partial t}(t,t) + \frac{\partial R^*}{\partial s}(t,t)\right)dt + o(dt) \tag{2.10b}$$

$$E(d^2x(t))^{*2} = -2\left(\frac{\partial R}{\partial t}(t,t) - \frac{\partial R^*}{\partial s}(t,t)\right)dt + o(dt)^2 \tag{2.10c}$$

$$E(dx(t)\bar{x}(t)^*) = \frac{1}{2}\left(\frac{\partial R}{\partial t}(t,t) + \frac{\partial R^*}{\partial s}(t,t)\right)dt + o(dt) \tag{2.10d}$$

$$E(d^2x(t)\bar{x}(t)^*) = \frac{1}{2}\left(\frac{\partial^2 R}{\partial t^2}(t,t) + \frac{\partial^2 R^*}{\partial s^2}(t' t)\right)dt^2 + o(dt)^2 \tag{2.10e}$$

$$E(d^2x(t)\,dx(t)^*) = -\frac{1}{2}\left(\frac{\partial^2 R}{\partial t^2}(t,t) - \frac{\partial^2 R^*}{\partial s^2}(t,t)\right)dt^2$$

$$+\frac{1}{2}\left(\frac{\partial^3 R}{\partial s\,\partial t^2}(t,t)+\frac{\partial^3 R^*}{\partial t\,\partial s^2}\right)dt^3+o(dt)^3. \quad (2.10f)$$

Therefore we obtain

$$\begin{aligned} E_{\bar{x}}(dx) &= E(dx\bar{x}(t)^*)(E(\bar{x}(t))^{*2})^{-1}x \\ &= \frac{1}{2}\left(\frac{\partial R}{\partial t}(t,t)+\frac{\partial R^*}{\partial s}(t,t)\right)x\,dt+o(dt) \\ &= U(t)x\,dt+o(dt) \qquad (2.11a) \end{aligned}$$

and in a similar fashion

$$\begin{aligned} E_{\bar{x}}(d^2x) &= \frac{1}{2}\left(\frac{\partial^2 R}{\partial t^2}(t,t)+\frac{\partial^2 R^*}{\partial s^2}(t,t)\right)x\,dt^2+o(dt)^2 \\ &= (F(t)+G(t)U(t))x\,dt^2+o(dt)^2. \qquad (2.11b) \end{aligned}$$

One can also show that

$$E_{\bar{x},dx}(d^2x)=(F(t)x+G(t)v)\,dt^2+o(dt)^2. \qquad (2.11c)$$

To obtain the conditional second moments of dx and d^2x, we utilize the particular property of Gaussian random variables that the conditional variance is independent of the conditioning and so

$$E_{\bar{x}}(dx-E_{\bar{x}}(dx))^{*2}=E(dx-E_{\bar{x}}(dx))^{*2}.$$

Hence

$$\begin{aligned} E_{\bar{x}}(dx)^{*2} &= E(dx)^{*2}+(E_{\bar{x}}(dx))^{*2}-E(E_{\bar{x}}(dx))^{*2} \\ &= -\frac{1}{2}\left(\frac{\partial R}{\partial t}(t,t)-\frac{\partial R^*}{\partial s}(t,t)\right)dt+\left(\frac{1}{2}\left(\frac{\partial^2 R}{\partial t\,\partial s}(t,t)+\frac{\partial^2 R^*}{\partial t\,\partial s}(t,t)\right)\right. \\ &\quad \left.+\,U(t)(xx^*-I)U^*(t)\right)dt^2+o(dt)^2 \end{aligned}$$

$$=\tfrac{1}{2}H(t)H^*(t)\,dt+\pi(x,t)\,dt^2+o(dt)^2. \tag{2.12a}$$

In a similar fashion we conclude that

$$E_{\bar{x}}(d^2x)^{*2}=E(d^2x)^{*2}+(E_{\bar{x}}(d^2x))^{*2}-E(E_{\bar{x}}(d^2x))^{*2}$$

$$=2H(t)^{*2}\,dt+o(dt)^2 \tag{2.12b}$$

$$E_{\bar{x},dx}(d^2x)^{*2}=2H(t)^{*2}\,dt+o(dt)^2 \tag{2.12c}$$

and

$$E_{\bar{x}}(d^2x\,dx^*)=E(d^2x\,dx^*)+E_{\bar{x}}(d^2x)(E_{\bar{x}}(dx))^*$$

$$-E(E_{\bar{x}}(d^2x)(E_{\bar{x}}(dx))^*)$$

$$=-\frac{1}{2}\left(\frac{\partial^2R}{\partial t^2}(t,t)-\frac{\partial^2R^*}{\partial s^2}(t,t)\right)dt^2$$

$$+\frac{1}{2}\left(\frac{\partial^3R}{\partial s\,\partial t^2}(t,t)+\frac{\partial^3R^*}{\partial t\,\partial s^2}(t,t)\right.$$

$$\left.+(F(t)+G(t)U(t))(xx^*-I)U^*(t)\right)dt^3+o(dt)^3$$

$$=\tfrac{1}{2}G(t)H(t)H(t)^*\,dt^2+o(dt)^2. \tag{2.12d}$$

If $x(t)$ reciprocal then by utilizing the sum of partials of (2.4) and (2.6) with respect to s and t respectively we obtain

$$E_{\bar{x}}(d^2x\,dx^*)=\tfrac{1}{2}G(t)H(t)H^*(t)\,dt^2$$

$$+((F(t)+G(t)U(t))xx^*U^*(t)$$

$$-G(t)\sigma(t))\,dt^3+o(dt)^3. \tag{2.12e}$$

Of course (1.1f) follows from the Gaussian assumption. Q.E.D.

It is enlightening to apply the above formulas to particular classes of reciprocal processes. For example suppose $x(t)$ is a stationary

Gauss–Markov process satisfying (2.1) with covariance $R(t,s)=R(t-s)$. It is well-known that $R(t)=\exp(At)$ where the spectrum of A lies in the open left half of the complex plane and that $x(t)$ satisfies on $[0.\infty]$ the first order stochastic differential equation

$$d^+x=Ax\,dt+Bd^+w \tag{2.13a}$$

$$x(0)\sim N(0,I) \tag{2.13b}$$

where $w(t)$ is an n dimensional standard Wiener process independent of $x(0)$ and the fluctuation-dissipation relation is satisfied,

$$A+A^*+BB^*=0. \tag{2.13c}$$

By the above discussion this process also satisfies the second order stochastic differential equation

$$d^2x=Fx\,dt^2+G\,dx\,dt+H\,d^2w \tag{2.14a}$$

where

$$F=A^2-GA \tag{2.14b}$$

$$G=-(A^2-A^{*2})(HH^*)^{-1} \tag{2.14c}$$

$$HH^*=BB^*. \tag{2.14d}$$

Moreover

$$U=\tfrac{1}{2}(A-A^*) \tag{2.15a}$$

$$\sigma=\tfrac{1}{2}(A^2+A^{*2})+UU^*. \tag{2.15b}$$

The stationary Gaussian reciprocal one dimensional processes have been completely classified [7, 8, 11]. See also [15]. The covariance $R(t)$ must satisfy (2.2) which in the case of scalars reduces to

$$\ddot{R}=FR.$$

There are three cases $F>0$, $F=0$ and $F<0$. If $F>0$ there are after various normalizations only 3 possibilities.

1.a) Ornstein–Uhlenbeck Process: $R(t)=e^{-t}$, $t>0$, which satisfies the second order equation.

$$d^2x=x\,dt^2+\sqrt{2}\,d^2w \qquad x(0)\sim N(0,1)$$

and $U=0$, $\sigma=1$. Of course this is also Markov and satisfies the first order equation

$$d^+x=-x\,dt+\sqrt{2}\,d^+w.$$

1.b) Cosh Process: $R(t)=\cosh(\frac{1}{2}-t)/\cosh\frac{1}{2}$ for $0\leqq t\leqq 1$ which satisfies

$$d^2x=x\,dt^2+\sqrt{2\tanh\tfrac{1}{2}}\,d^2w \qquad x(0)=x(1)\sim N(0,1)$$

and $U=0$, $\sigma=1$. This process is not Markov but it does have a realization by a stochastic differential equation with an independent boundary condition [15].

1.c) Sinh Process: $R(t)=\sinh(\frac{1}{2}-t)/\sinh\frac{1}{2}$ for $0\leqq t\leqq 1$ which satisfies

$$d^2x=x\,dt^2+\sqrt{2\coth\tfrac{1}{2}}\,d^2w \qquad x(0)=-x(1)\sim N(0,1)$$

and $U=0$, $\sigma=1$. Again this is not Markov but can be realized by a stochastic differential equation with an independent boundary condition [15].

2) Slepian Process [4]: $R(t)=1-2t$ for $0\leqq t\leqq 1$ which satisfies

$$d^2x=2\,d^2w$$

$$x(0)=-x(1)\sim N(0,1)$$

and $U=0$, $\sigma=0$. This is not Markov and again has a stochastic boundary realization [15].

3.a) Cosine Process: $R(t)=\cos t$ for $-\infty<t<\infty$ which satisfies

$$d^2x=-x\,dt^2 \qquad x(0)=-x(\pi)\sim N(0,1)$$

and $U=0$, $\sigma=-1$. This is not Markov but the sample paths are completely determined by $x(t_1)$, $x(t_2)$ where t_1-t_2 is not a multiple of π.

3.b) Shifted Cosine Process: $R(t)=\cos(t+\tau)/\cos\tau$ for $0\leqq t\leqq\pi-2\tau$. To be a covariance, τ must satisfy $0\leqq\tau<\pi/2$. The process $x(t)$ satisfies

$$d^2x=-x\,dt^2+\sqrt{2\tan\tau}\,d^2w \qquad x(0)=-x(\pi-2\tau)\sim N(0,1)$$

and $U=0$, $\sigma=1$. This process is not Markov and cannot be realized by a scalar first order stochastic differential equation with initial or boundary condition.

We close this section with another interesting example. The Brownian bridge or pinned Wiener process $x(t)$ is obtained from a standard Wiener process by conditioning that $x(0)=x(1)=0$. Another representation is $x(t)=w(t)-tw(1)$ where $w(t)$ is a standard Wiener process. This is a zero mean Gauss–Markov process with covariance $R(t,s)=s(1-t)$ for $0\leqq s\leqq t\leqq 1$. It satisfies the first order differential equation

$$d^+x=\frac{-x}{(1-t)}\,dt+d^+w \qquad x(0)=0$$

and also satisfies the second order differential equation

$$d^2x=d^2w \qquad x(0)=x(1)=0.$$

Note that this is essentially the same differential equation as that of the Slepian Process.

3. CHANGE OF VARIABLES

In this section we develop some formulas that we shall need in the next. Suppose $x(t)$ is a second order diffusion satisfying (1.1) and (1.2). Let $\phi(x,t),\psi(x,t)$ be C^∞ scalar valued functions and define $\phi(t)=\phi(x(t),t)$ $\psi(t)=\psi(x(t),t)$. We compute the centered mean, first difference and second difference of $\phi(t)$ using the identities

$$dx(t) = \pm(x(t \pm dt) - \bar{x}(t)) \tag{3.1a}$$

$$d^2x(t) = 2(\bar{x}(t) - x(t)). \tag{3.1b}$$

Now

$$\bar{\phi}(t) = \frac{\phi(t+dt) + \phi(t-dt)}{2}$$

$$= \phi(\bar{x}(t), t) + \tfrac{1}{2}(\phi(t+dt) - \phi(\bar{x}(t), t) + \phi(t-dt) - \phi(\bar{x}(t), t))$$

$$\bar{\phi}(t) = \phi(\bar{x}(t), t) + \frac{1}{2}\frac{\partial^2 \phi}{\partial x_i \partial x_j}(\bar{x}(t), t)\, dx_i\, dx_j + O(dx)^4 + O(dt)^2. \tag{3.2a}$$

The symbols $O(dx)^4$ and $O(dt)^2$ denote quantities that go to zero as fast as $|dx|^4$ and dt^2 respectively. Next

$$d\phi = \frac{\phi(t+dt) - \phi(t-dt)}{2} + \frac{\phi(\bar{x}(t), t) - \phi(\bar{x}(t), t)}{2}.$$

By a similar Taylor expansion we obtain

$$\begin{aligned}
d\phi &= \frac{\partial \phi}{\partial x_i}(\bar{x}(t), t)\, dx_i + \frac{\partial \phi}{\partial t}(\bar{x}(t), t)\, dt \\
&\quad + \frac{1}{6}\frac{\partial^3 \phi(\bar{x}(t), t)}{\partial x_i \partial x_j \partial x_k} dx_i\, dx_j\, dx_k \\
&\quad + \frac{1}{2}\frac{\partial^3 \phi(\bar{x}(t), t)}{\partial x_i \partial x_j \partial t} dx_i\, dx_j\, dt \\
&\quad + \frac{1}{2}\frac{\partial^3 \phi(\bar{x}(t), t)}{\partial x_i \partial t^2} dx_i\, dt^2 \\
&\quad + \frac{1}{120}\frac{\partial^5 \phi(\bar{x}(t), t)}{\partial x_i \partial x_j \partial x_k \partial x_l \partial x_m} dx_i\, dx_j\, dx_k\, dx_l\, dx_m \\
&\quad + O(dx)^6 + O(dx)^4 O(dt) + O(dx)^2 O(dt)^2 + O(dt)^3
\end{aligned} \tag{3.2b}$$

and

$$d^2\phi = \frac{\partial \phi}{\partial x_i}(\bar{x}(t),t)\,d^2x_i + \frac{\partial^2 \phi}{\partial x_i \partial x_j}(\bar{x}(t),t)\left(dx_i\,dx_j - \frac{1}{4}d^2x_i\,d^2x_j\right)$$

$$+2\frac{\partial^2\phi}{\partial t\,\partial x_i}(\bar{x}(t),t)\,dx_i\,dt + \frac{\partial^2\phi}{\partial t^2}(\bar{x}(t),t)\,dt^2$$

$$+\frac{1}{24}\frac{\partial^3\phi(\bar{x}(t),t)}{\partial x_i\,\partial x_j\,\partial x_k}d^2x_i\,d^2x_j\,d^2x_k$$

$$+\frac{1}{12}\frac{\partial^4\phi(\bar{x}(t),t)}{\partial x_i\,\partial x_j\,\partial x_k\,\partial x_l}\left(dx_i\,dx_j\,dx_k\,dx_l - \frac{1}{16}d^2x_i\,d^2x_j\,d^2x_k\,d^2x_l\right)$$

$$+\frac{1}{3}\frac{\partial^4\phi(\bar{x}(t),t)}{\partial x_i\,\partial x_j\,\partial x_k\,\partial t}dx_i\,dx_j\,dx_k\,dt$$

$$+\frac{1}{1920}\frac{\partial^5\phi(\bar{x}(t),t)}{\partial x_i\,\partial x_j\,\partial x_k\,\partial x_l\,\partial x_m}d^2x_i\,d^2x_j d^2x_k\,d^2x_l\,d^2x_m$$

$$+O(d^2x)^6 + O(dx)^6 + O(dx)^4O(dt) + O(dx)^2O(dt)^2 + O(dt)^3 \quad (3.2c)$$

Hence it follows from (1.1) that

$$E_{\bar{x}}(\bar{\phi}(t)) = \phi + \frac{1}{4}\frac{\partial^2\phi}{\partial x_i\,\partial x_j}h_{ik}h_{jk}dt + o(dt) \quad (3.3a)$$

$$E_{\bar{x}}(d\phi) = \frac{\partial\phi}{\partial x_i}u_i\,dt + \frac{\partial\phi}{\partial t}dt + o(dt) \quad (3.3b)$$

$$E_{\bar{x}}(d^2\phi) = \frac{\partial\phi}{\partial x_i}(f_i + g_{ij}u_j)\,dt^2$$

$$+\frac{\partial^2\phi}{\partial x_i\,\partial x_k}\pi_{ik}\,dt^2 + 2\frac{\partial^2\phi}{\partial t\,\partial x_i}u_i\,dt^2 + \frac{\partial^2\phi}{\partial t^2}dt^2 + o(dt)^2 \quad (3.3c)$$

$$E_{\bar{x}}(d\phi\,d\psi) = \frac{\partial\phi}{\partial x_i}\frac{\partial\psi}{\partial x_j}(\tfrac{1}{2}h_{ik}h_{jk}\,dt + \pi_{ij}\,dt^2) + \frac{\partial\phi}{\partial t}\frac{\partial\psi}{\partial t}dt^2 + o(dt)^2 \quad (3.3d)$$

$$E_{\bar{x}}(d^2\phi\, d^2\psi) = 2\frac{\partial\phi}{\partial x_i}\frac{\partial\psi}{\partial x_j}h_{ir}h_{jr}\,dt + o(dt)^2 \tag{3.3e}$$

$$E_{\bar{x}}(d^2\phi\, d\psi) = \frac{1}{2}\frac{\partial\phi}{\partial x_i}\frac{\partial\psi}{\partial x_j}g_{ik}h_{kr}h_{jr}\,dt^2 + \frac{\partial\psi}{\partial x_j}u_k h_{ir}h_{jr}\,dt^2$$

$$+\frac{\partial^2\phi}{\partial t\,\partial x_i}\frac{\partial\psi}{\partial x_j}h_{ir}h_{jr}\,dt^2 + o(dt)^2. \tag{3.3f}$$

The right sides of the above are evaluated at $(\bar{x}(t), t) = (x, t)$. Note that these are not in the form of (1.1) in that the mean differences of $\phi(t)$ are conditioned by $\bar{x}(t)$ rather than $\bar{\phi}(t)$.

4. CONSERVATION LAWS

Suppose $x(t)$ is a Markov diffusion satisfying the first order stochastic differential equation.

$$d^+x = f\,dt + h\,d^+w. \tag{4.1}$$

The probability density $\rho(x,t)$ of $x(t)$ satisfies the Fokker–Planck equation,

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x_i}(\rho f_i) - \frac{1}{2}\frac{\partial}{\partial x_i\,\partial x_j}(\rho h_{ik}h_{jk}) = 0. \tag{4.2}$$

This is a second order parabolic partial differential equation.

Suppose $x(t)$ is a strongly reciprocal diffusion satisfying the second order stochastic differential equation.

$$d^2x = f\,dt^2 + g\,dx\,dt + h\,d^2w \tag{4.3}$$

then we shall demonstrate that the density $\rho(x,t)$, mean velocity $u(x,t)$ and momentum flux tensor $\rho\pi(x,t)$ satisfy, at least in the weak sense, a system of conservation laws, very similar to those of continuum mechanics,

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x_k}(\rho u_k) = 0 \tag{4.4a}$$

$$\frac{\partial}{\partial t}(\rho u_i)=\rho(f_i+g_{ik}u_k)-\frac{\partial}{\partial x_k}(\rho\pi_{ik}) \tag{4.4b}$$

$$\frac{\partial}{\partial t}(\rho h_{ir}h_{jr})=\frac{\rho}{2}(g_{ik}h_{kl}h_{jl}+g_{jk}h_{kl}h_{il})-\frac{\partial}{\partial x_k}(\rho u_k h_{ir}h_{jr}). \tag{4.4c}$$

In addition we shall show in Section 5 that at least for the reciprocal Gaussian processes of Section 2 an additional conservation condition must hold

$$\frac{\partial}{\partial_t}(\rho\pi_{ij})=\rho(f_iu_j+u_jf_i+g_{ik}\pi_{kj}+\pi_{ik}g_{jk})$$
$$-\frac{\partial}{\partial x_k}(\rho(u_iu_ju_k-\sigma_{ij}u_k-\sigma_{ik}u_j-\sigma_{jk}u_i)). \tag{4.4d}$$

Since $\sigma_{ij}=u_iu_j-\pi_{ij}$, (4.4a, b, d) appear to be a complete set of equations for the unknowns ρ, u, π in terms of f and g in the Gaussian case. But, we doubt that (4.4d) holds for all reciprocal diffusions.

Before we derive these equations, let's take a look at their meaning. Equation (4.4a) express the conservation of probability under the mean flow described by u. This corresponds to conservation of mass in continuum mechanics. A similar equation relates the density and current velocity of Nelson [18].

Equation [4.4b) expresses the balance of momentum ρu. If we integrate the left side over a volume in x-space we obtain the time rate of change of momentum within the volume. The integral of the right side of (4.4b) has contributions from two sources. The first integral involving $\rho(f_i+g_{ik}u_k)$ is the change of momentum due to the mean acceleration of the particles inside the volume, the random acceleration produces no net change of momentum. The integral of the second term over the volume can be converted to a surface integral over its boundary by Stokes' Theorem. The integral over the surface bounding the volume of $\rho\pi_{ik}$ contracted with outward unit normal is the total flux through the boundary. Recall the definition (1.1b) of π^{ij} as the dt^2 part of $E_{\bar{x}}(dx_i\,dx_j)$. This tensor has a deterministic and a random component, $\rho\pi_{ij}=\rho u_iu_j-\rho\sigma_{ij}$ and each contributes to the momentum flux. Notice that the order dt part of $E_x(dx_i\,dx_j)$ in (1.1b) does not contribute anything to the momentum flux. Intuitively this is because these changes are so fast and so random, they cannot transport momentum.

In continuum mechanics the contraction $\frac{1}{2}\rho\pi_{ii}$ describes the density of kinetic energy. This has two parts, the first $\frac{1}{2}\rho u_i u_i$ due to deterministic part of the velocity and the other $-(\rho/2)\sigma_{ii}$ due to the random part. The latter is frequently called the internal energy density.

The tensor $\frac{1}{2}\rho\pi_{ij}$ describes the density of kinetic energy in every component of the x process. If λ_i is a constant n vector then the scalar valued process $z(t)=\lambda_i x_i(x)$ has kinetic energy density given by $\frac{1}{2}\rho\pi_{ij}\lambda_i\lambda_j$. For this reason we call $\frac{1}{2}\rho\pi_{ij}$ the tensor kinetic energy.

There is an alternative definition of π_{ij} as $\frac{1}{2}E_x(d^+x_i\,d^-x_j + d^+x_j d^-x_i) = \pi_{ij}d^2 + o(dt)^2$ which reduces to the standard one for smooth process. Based on this and Eq. (4.4c) which we discuss in a moment, we define the kinetic energy density to be $(\rho/2)\pi_{ii}$. This definition of kinetic energy is similar in spirit but different from that of Guerra-Morato [19]. It is interesting to note that some of the examples of Section 2 have negative or zero kinetic energy. For the Ornstein–Uhlenbeck, Cosh and Sinh process, $\pi=-1$, and for the Slepian process $\pi=0$. The Brownian bridge has both negative and positive kinetic energy depending on x and t.

Equation (4.4d), which may hold only for Gaussian processes, is a tensor form of the balance of kinetic energy and work. In other words Eq. (4.4d) expresses the balance of kinetic energy and work for every scalar process $z(t)=\lambda_i x_i(t)$. The momentum flux or *tensor kinetic energy* is $(\rho/2)\pi_{ij}$, the dt^2 part of $(\rho/2)E_x(dx_i\,dx_j)$. The tensor part of the rate of work done or power is the dt^2 part of $(\rho/2)E_x(d^2x_i\,dx_j + d^2x_i\,dx_j)$ which explains the first term on the right side of (4.4c). The second, flux term represents the flow of tensor kinetic energy across the boundary of the volume under consideration. This flux has contributions both from the deterministic and random parts of the motion. The first term $u_i u_j u_k$ represents the flux due to strictly deterministic motion, the others due to a mix of deterministic and stochastic motion. In continuum mechanics, $(\rho/2)\sigma_{ii}u_k$ is the flux of internal energy and $\rho\sigma_{ik}u_i$ is the flux of energy due to viscosity or stress. In our stochastic model, $(\rho/2)\sigma_{ii}u_k$ is the flux of random kinetic energy transported by the mean velocity and

$\rho\sigma_{ik}u_i$ is the flux of energy due to the random motion of particles between regions of differing mean velocity.

But (4.4d) only expresses a balance between tensor kinetic energy and tensor work terms of size dt^2. The quantities involved also have terms of size dt and the balance of these is expressed by (4.4d). We view $\rho h_{i_r}h_{j_r}/4$ as the tensor form of the *hyperkinetic energy* due to *hypervelocity* part of dx, namely, $\widetilde{dx}=0(dt)^{1/2}$. The tensor $(\rho/2)(g_{ik}h_{k_r}h_{j_r}+g_{jk}h_{k_r}h_{i_r})$ is the *hyperpower* and $(\rho/2)h_{i_r}h_{j_r}u_k$ is the flux tensor of hyperkinetic energy. Of course this extra equation leads to an overdetermined system of equations for ρ, u, and π but if we consider $h_{i_r}h_{j_r}$ as an unknown also, this problem disappears. An interesting question which we don't address is that of boundary and/or initial conditions for (4.4).

In particular, (4.4c) implies that we cannot find processes satisfying the second order stochastic differential equation (1.2) for arbitrary choices of f, g and h. Notice that if h is constant in x and t, (4.4c) and (4.4a) imply that the tensor field $g(x,t)$ is skew-symmetric relative to the symmetric tensor field $h^{*2}(x,t)$, i.e.,

$$ghh^* + hh^*g^* = 0. \tag{4.5}$$

We shall derive Eqs. (4.4a, b, c) using no *a priori* assumptions of conserved quantities. Rather they shall follow from basic mathematical facts. We warn the reader that our derivation is somewhat formal, we shall interchange limiting operations, neglect small quantities, etc. In the next section we shall verify that the reciprocal Gaussian diffusions treated in Section 2 satisfy (4.4a, b, c) and also (4.4d).

Before we start we list some basic formlulas about centered differences that will be useful. Let $x(t)$, $y(t)$ be n-dimensional processes defined on $[0,T]$. Suppose $0<\tau_0<\tau_1<T$ and $t_r=\tau_0+(r-\frac{1}{2})\,dt$, $\tau_1=t_N+\frac{1}{2}\,dt$. Then

$$\sum_{r=1}^{N} dx(t_r;dt)=\bar{x}(\tau_1;dt/2)-\bar{x}(\tau_0;dt/2) \tag{4.6a}$$

$$\sum_{r=1}^{N} d^2x(t_r;dt)=2(dx(\tau_1;dt/2)-dx(\tau_0;dt/2)) \tag{4.6b}$$

$$d(xy^*)(t;dt) = \bar{x}(t;dt)\,dy^*(t;dt) + dx(t;dt)\,\bar{y}^*(t;dt) \tag{4.6c}$$

$$d\bar{x}(t;dt) = \overline{dx}(t;dt) = \tfrac{1}{2}\,dx(t;2\,dt) \tag{4.6d}$$

$$d(dx)(t,dt) = \tfrac{1}{4}\,d^2x(t;2\,dt) \tag{4.6e}$$

$$d(dx\,dy^*)(t;dt) = \tfrac{1}{8}(dx(t;2\,dt)\,d^2y^*(t;2\,dt) + d^2x(t;2\,dt)\,dy^*(t;2\,dt)). \tag{4.6f}$$

Let $x(t)$ be a second order diffusion satisfying (1.1a–f) with density $\rho(x,t)$, mean velocity $u(x,t)$, momentum flux $\rho(x,t)\ \pi(x,t)$ and stress $\rho(x,t)\ \sigma(x,t)$. We assume that as $|x|\to\infty$, ρ goes to zero faster than every rational function of $|x|$ uniformly for all $t\in[0,T]$. We also assume that $|u|$, $|\pi|$ and $|\sigma|$ are bounded above by some polynomial in $|x|$ for all $t\in[0,T]$. Let $\phi(x,t)$ be a smooth scalar valued function also bounded in norm by a polynomial in $|x|$ and suppose $\phi(t,x)$ vanishes off some closed subinterval of (τ_0,τ_1). Finally we assume that density $\bar{\rho}(x,t,dt)$ of $\bar{x}(t;dt)$ converges to the density $\rho(x,t)$ of $x(t)$ as $dt\to 0$.

Using (4.6a) we have

$$0 = E\sum_{r=1}^{N} d\phi(t_r;dt) = E\sum_{1}^{N} E_{\bar{x}}(d\phi(t_r;dt))$$

We employ (3.3b) and let $dt\to 0$ to obtain

$$0 = \int\int\left(\frac{\partial\phi}{\partial x_k}u_k + \frac{\partial\phi}{\partial t}\right)\rho\,dt\,dx$$

Integration by parts yields a weak form of (4.4b),

$$0 = \int\int\phi\left(\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_k}\rho u_k\right)dt\,dx.$$

In similar fashion (4.6b) yields

$$0 = \frac{1}{dt}E\sum_{r=1}^{N} d^2\phi(t_r;dt) = \frac{1}{dt}E\sum E_{\bar{x}}(d^2\phi(t_r;dt))$$

We employ (3.3c) and let $dt \to 0$ to obtain

$$0=\int\int\left(\frac{\partial\phi}{\partial x_i}(f_i+g_{ij}u_j)+\frac{\partial^2\phi}{\partial x_i x_k}\pi_{ik}+2\frac{\partial^2\phi}{\partial t\,\partial x_i}u_i+\frac{\partial^2\phi}{\partial t^2}\right)\rho\,dt\,dx.$$

Integrating by parts yields

$$0=\int\int\frac{\partial\phi}{\partial x_i}\left(\rho(f_i+g_{ij}u_j)-\frac{\partial}{\partial x_k}(\rho\pi_{ik})-\frac{\partial}{\partial t}(\rho u)\right)-\frac{\partial\phi}{\partial t}\left(\frac{\partial}{\partial x_i}\rho u_u+\frac{\partial\rho}{\partial t}\right)dt\,dx.$$

By (4.4a) this reduces to a weak form of (4.4b),

$$0=\int\int\frac{\partial\phi}{\partial x_i}\left(\rho(f_i+g_{ij}u_j)-\frac{\partial}{\partial x_k}(\rho\pi_{ik})-\frac{\partial}{\partial t}(\rho u)\right)dt\,dx.$$

Finally we start with (4.6f) applied to ϕ, which we sum and divide by dt to obtain

$$0=\frac{1}{8\,dt}E\sum_{r=1}^{N}d^2\phi(\tau_r;2\,dt)\,d\phi(\tau_r;2\,dt)$$

$$=\frac{1}{8\,dt}E\sum_{1}^{N}E_{\bar{x}}(d^2\phi(\tau_r;2\,dt)\,d\phi(\tau_r;2\,dt)).$$

So by (3.3f)

$$0=E\sum_{1}^{N}\frac{1}{2}\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_j}g_{ik}h_{kl}h_{jl}dt+\frac{\partial^2\phi}{\partial x_i\,\partial x_k}\frac{\partial\phi}{\partial x_j}u_k h_{ir}h_{jr}\,dt$$

$$+\frac{\partial^2\phi}{\partial t\,\partial x_i}\frac{\partial\phi}{\partial x_j}h_{ir}h_{jr}\,dt+o(dt). \tag{4.7}$$

As $dt \to 0$ we obtain

$$0=\int\int\left(\frac{1}{2}\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_j}g_{ik}h_{kr}h_{jr}+\frac{\partial^2\phi}{\partial x_i\,\partial x_k}\frac{\partial\phi}{\partial x_j}u_k h_{ir}h_{jr}+\frac{\partial^2\phi}{\partial t\,\partial x_i}\frac{\partial\phi}{\partial x_j}h_{ir}h_{jr}\right)\rho\,dt\,dx.$$

We symmetrize this with respect to i and j,

$$0=\frac{1}{2}\int\int\bigg(\frac{1}{2}\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_j}(g_{ik}h_{kr}h_{jr}+g_{jk}h_{kr}h_{ir})$$

$$+\frac{\partial}{\partial x_k}\left(\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_j}\right)u_k h_{ir}h_{jr}+\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_j}\right)h_{ir}h_{jr}\bigg)\rho\,dt\,dx$$

and integrate by parts

$$0=\frac{1}{2}\int\int\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_j}\bigg(\frac{\rho}{2}(g_{ik}h_{kr}h_{jr}+g_{jk}h_{kr}h_{ir})$$

$$-\frac{\partial}{\partial x_k}(\rho u_k h_{ir}h_{jr})-\frac{\partial}{\partial t}(\rho h_{ir}h_{jr})\bigg)\,dt\,dx$$

which we recognize as a weak form of (4.4c).

5. RECIPROCAL AND GAUSSIAN PROCESSES REVISITED

In this section we verify that the Gaussian (reciprocal Gaussian) processes discussed in Section 2 satisfy the three (four) conservation laws of Section 4. Let $x(t)$ be a Gaussian process with smooth covariance $R(t,s)$ satisfying (2.1), (2.4) and (2.6) where $F(t)$, $G(t)$ and $H(t)$ are defined by (2.7) and $u(x,t)$, $\pi(x,t)$ and $\sigma(x,t)$ by (2.8). Since $R(t,t)=I$ (2.1a),

$$\rho(x,t)=(2\pi)^{-n/2}\exp-\tfrac{1}{2}|x|^2$$

which satisfies

$$\frac{\partial\rho}{\partial t}=0 \tag{5.1a}$$

$$\frac{\partial\rho}{\partial x_k}=-\rho x_k \tag{5.1b}$$

consider the conservation of probability (4.4a)

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_k}(\rho U_{kj}x_j)=0. \tag{5.2}$$

By (5.1) and (2.8) this reduces to

$$\rho(x_k U_{kj} x_j + U_{kk}) = 0$$

which holds since U is skew symmetric.

Next we return to the balance of momentum

$$\frac{\partial}{\partial t}\rho U_{ij}x_j=\rho(F_i+G_{ir}U_{rj})x_j-\frac{\partial}{\partial x_k}(\rho \pi_{ik}). \tag{5.3}$$

The left hand side is

$$\rho\frac{d}{dt}(U_{ij})x_j=\frac{\rho}{2}\left(\frac{\partial^2 R_{ij}}{\partial t^2}+\frac{\partial^2 R_{ij}}{\partial t\,\partial s}+\frac{\partial^2 R_{ji}}{\partial t\,\partial s}+\frac{\partial^2 R_{ji}}{\partial s^2}\right)x_j.$$

All evaluations of R and its partials are always at $t=s$. It is straightforward to verify that

$$\frac{d}{dt}U_{ij}=[F+GU]_{ij}-\sigma_{ij}+[UU^*]_{ij} \tag{5.4}$$

where $[\cdot]_{ij}$ denotes the i-j entry of the enclosed matrix. Hence the left side of (5.3) equals

$$\rho[F+GU-\sigma+UU^*]_{ij}x_j.$$

The right side equals

$$\rho[F+GU]_{ij}x_j+\rho x_k([Uxx^*U^*-\sigma]_{ik})-\rho\frac{\partial}{\partial x_k}[Uxx^*U^*]_{ik}$$

$$=\rho[F+GU]_{ij}x_j-\rho G_{ik}x_k$$

$$+\rho[Ux]_i(x^*U^*x)-\rho[Ux]_iU_{kk}-\rho[UU]_{ij}x_j$$

which equals the left side by the skew symmetry of U.

Next we verify (4.4d) for reciprocal Gaussian processes. In this case (4.4d) becomes

$$\frac{\partial}{\partial t}\rho\pi_{ij}=\rho[Fxx^*U^*+Uxx^*F^*+G\pi+\pi G^*]_{ij}$$
$$-\frac{\partial}{\partial x_k}\rho(U_{ir}U_{jm}U_{kl}x_r x_m x_l-\sigma_{ij}U_{kr}x_r$$
$$-\sigma_{ik}U_{jr}x_r-\sigma_{jk}U_{ir}x_r). \tag{5.5}$$

It is convenient to break up each side of this equation into terms that are time varying multiples of ρ and terms that are time varying multiples of ρxx^*, there are no others. We refer to these as constant and quadratic terms.

On the left side, the constant terms are

$$\frac{\rho}{2}\left[\frac{\partial^3 R}{\partial s\,\partial t^2}+\frac{\partial^3 R}{\partial t\,\partial s^2}+\frac{\partial^3 R^*}{\partial s\,\partial t^2}+\frac{\partial^3 R^*}{\partial t\,\partial s^2}\right]_{ij}-\rho\left[\frac{d}{dt}UU^*\right]_{ij}.$$

By (2.4) (2.6) this equals

$$\frac{\rho}{2}\left[G\left(\frac{\partial^2 R}{\partial t\,\partial s}+\frac{\partial^2 R^*}{\partial t\,\partial s}\right)+\left(\frac{\partial^2 R}{\partial t\,\partial s}+\frac{\partial^2 R^*}{\partial t\,\partial s}\right)G^*\right.$$
$$\left.+F\left(\frac{\partial R^*}{\partial t}+\frac{\partial R}{\partial s}\right)+\left(\frac{\partial R}{\partial t}+\frac{\partial R^*}{\partial s}\right)F^*\right]_{ij}$$
$$-\rho\left[\frac{d}{dt}UU^*\right]_{ij}$$

By (2.8b, d) and (5.4) this equals

$$\rho[G[UU^*-\sigma)+(UU^*-\sigma)G^*+FU^*+UF^*]_{ij}$$
$$-\rho[(F+GU+UU^*-\sigma)U^*+U(F+GU+UU^*-\sigma)^*]_{ij}$$
$$=\rho\,[(U-G)\sigma+\sigma(U-G)^*]_{ij}$$

which equals the constant terms on the right side of (5.5).

The quadratic terms on the left side of (5.5) are

$$\rho\frac{d}{dt}(Uxx^*U^*)_{ij}=\rho[(F+GU-\sigma+UU^*)xx^*U^*$$
$$+Uxx^*(F+GU-\sigma+UU^*)]_{ij}$$

On the right side (5.5) the quadratic terms are

$$\rho[(F+GU)xx^*U^*+Uxx^*(F+GU)^*]_{ij}$$
$$+\rho x_k U_{kk}x_r[Uxx^*U^*]_{ij}-\rho x_k(\sigma_{ik}U_{jr}x_r+\sigma_{jk}U_{ir}x_r)$$
$$-\rho[UUxx^*U^*+Uxx^*U^*U^*]_{ij}$$
$$-\rho U_{kk}[Uxx^*U^*]_{ij}$$
$$=\rho[(F+GU-\sigma+UU^*)xx^*U^*$$
$$+Uxx^*(F+GU-\sigma+UU^*)^*]_{ij}$$

as desired to prove (5.5).

Finally we verify (4.4c) for Gaussian processes which reduces to

$$\frac{\partial}{\partial t}(\rho HH^*)=\frac{\rho}{2}(GHH^*+HH^*G^*)-\frac{\partial}{\partial x_k}(\rho U_{kr}x_r)HH^*. \tag{5.6}$$

By (5.2) this becomes

$$\frac{d}{dt}HH^*=\tfrac{1}{2}(GHH^*+HH^*G^*).$$

By (2.7c) the left side equals

$$-\left(\frac{\partial^2 R}{\partial t^2}+\frac{\partial^2 R}{\partial t\,\partial s}-\frac{\partial^2 R^*}{\partial t\,\partial s}-\frac{\partial^2 R^*}{\partial s^2}\right).$$

Since $R(t,t)=I, d^2/dt^2R=0$ and so

$$\frac{1}{2}\frac{\partial^2 R}{\partial t^2}+\frac{\partial^2 R}{\partial t\,\partial s}=-\frac{1}{2}\frac{\partial^2 R}{\partial s^2}.$$

This reduces the left side of the above to

$$-\frac{1}{2}\left(\frac{\partial^2 R}{\partial t^2}-\frac{\partial^2 R}{\partial s^2}-\frac{\partial^2 R^*}{\partial s^2}+\frac{\partial^2 R^*}{\partial t^2}\right)$$

which equals the right by (2.7, a, c).

6. CONCLUSION

We have described a theory of stochastic differential equations of second order and have demonstrated that the theory is not vacuous, it includes the reciprocal Gaussian processes which satisfy some mild assumptions. We have also demonstrated that the density, mean velocity and momentum flux obey a system of nonlinear conservation laws similar to those of continuum mechanics.

Obviously considerable work remains to be done including the following.

1) A theory of stochastic integration for second order stochastic differential equations.
2) Further study of nonlinear second order stochastic differential equations and non-Gaussian reciprocal processes.
3) Possible applications in statistical mechanics, continuum and fluid mechanics and quantum stochastic mechanics.

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