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APPROXIMATE NORMAL FORMS OF NONLINEAR SYSTEMS*

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Abstract A first degree approximation by a linear system is the standard approach for treating most nonlinear systems. Exact transformation of certain nonlinear systems into linear systems is possible under nonlinear state feedback and coordinate change, as shown by Jakubzcyk and Respondek [13] and Hunt and Su [10]. The approximation of nonlinear systems to higher degrees by linear systems has been treated in [16] and recently in [17]. In this paper, we develop a method of solution to find such higher degree approximations by reducing the linearization problem into the solution of a set of linear equations. We suggest a solution that, in some sense, minimizes the error in the approximation.

1. Introduction

In the analysis of scientific and engineering systems, one often encounters situations which do not lend themselves to exact solutions by conventional methods. The assumption of linearity in most control system models, for example, is an oversimplification at best, and it reflects the difficulties one would rather avoid in dealing with an otherwise nonlinear model. Seldom a technique can be found to solve a given nonlinear problem exactly. Since the control system designer is equipped with powerful methods and tools for attacking linear control systems, the motivation for "linearizing" a nonlinear problem is clearly very strong.

Therefore, whenever possible, a nonlinear control problem must be suitably transformed to bring it into an appropriate form that enables the implementation of linear control design techniques. However, the systematics of such modifications by transformation are usuallly not self-evident. The simplest of these modifications is a first degree linear approximation by calculating a series expansion at a nominal operating point. The validity of this approximation depends on the relative size of the second degree terms. In systems where nonlinearities are strong, the higher degree terms cannot be neglected, and the approximation fails.

The earliest example on the question of whether a nonlinear system can be equivalent to a linear system under some group of transformations such as change of coordinates was solved by Poincaré [19]. Various researchers in [6,8,9,15,20,21] discussed the question of when a nonlinear control system can be transformed into a linear system by a change of state coordinates. Jakubczyk and Respondek [13], and Hunt and Su [10,11,12] independently considered the full state feedback and coordinate change problem. Related work also appeared in [2,22,23,24]. In [16], Krener investigated an approximate linearization considering the second and higher degree terms in the truncated series expansion of a the vector field, and proved a weakened version of the Hunt–Su linearization condition. In [17], further results in an attempt to solve for the resulting transformations were presented. An application for nonlinear observers also appeared in [4].

The objectives of this paper are to: 1) Present a solution to the approximate linearization problem, 2) Suggest a method to solve the Homological equations to minimize the error in the approximation in some sense. For further work see [14].

2. Higher Degree Approximations to Autonomous Systems: Normal Form Theorem

In this section, the normal form theorem of Poincaré will be introduced. The approximate linearization problem for a control system will be formulated later in a similar spirit. As a reminder of the connection between the two, we continue to use the term "Homological Equations" (after Arnold [1]).

Let us consider an autonomous system:

$$\dot{x} = f(x) \tag{1a}$$

$$x(0) = x^{*}. \tag{1b}$$

where $x \in \mathbb{R}^n$ and the system is assumed to be at rest at the origin, i.e. $f(0) = 0$. Without loss of generality we will assume $x^{*} = 0$. First, consider the linearization of (1) at x^{*}:

$$\dot{x} = Fx \tag{2a}$$

$$F = \frac{\partial f}{\partial x}(0) \tag{2b}$$

We will seek a coordinate change for (1) of the form identity plus higher degree terms, such that the resulting system will agree with (1) up to an error of degree $O(x)^{\rho+1}$ where ρ is the degree of approximation. The following treatment is for $\rho = 2$. The results can be easily generalized to any arbitrary degree ρ by induction.

We assume a transformation of the form:

$$z = x - \phi^{(2)}(x) \tag{3}$$

where z denotes a new set of coordinates. $\phi^{(2)}(x)$ is a polynomial of degree 2. The function $f(x)$ in (1a) is expanded in a series:

$$\begin{aligned} f(x) &= f^{(1)}(x) + f^{(2)}(x) + O(x)^3 \\ &= Fx + f^{(2)}(x) + O(x)^3 \end{aligned} \tag{4}$$

The goal of the transformation (3) is to choose $\phi^{(2)}(x)$ such that in z coordinates the dynamics of the system is represented by

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$$\dot{z} = Fz + O(z)^3 \qquad (5)$$

namely the second degree terms in the series expansion (4) vanish under the coordinate change. We take the time derivative of (3) and using (1a), (4) and (5) evaluate each side by ignoring $O(x)^3$ and higher terms:

$$F(x - \phi^{(2)}(x)) = Fx + f^{(2)}(x) - \frac{\partial\phi^{(2)}(x)}{\partial x}Fx \qquad (6)$$

Now we introduce some notation. The Lie bracket of two vector fields f, g is another vector field defined by:

$$[f,g] = \frac{\partial g}{\partial x}f - \frac{\partial f}{\partial x}g \qquad (7)$$

Rearranging and cancelling terms in (6), and using (7) we obtain

$$f^{(2)}(x) = [Fx, \phi^{(2)}(x)] \qquad (8)$$

Equation (8) is called the *Homological Equation* [1]. In [5], a similar derivation is also presented. The Lie bracket operation in the above defines a mapping

$$[Fx, \cdot\,] : \phi^{(2)}(x) \to [Fx, \phi^{(2)}(x)] \qquad (9)$$

Obviously, (9) represents a linear mapping from $n^2(n+1)/2$ dimensional parameter space of the coefficients of $\phi^{(2)}(x)$ to an $n^2(n+1)/2$ dimensional parameter space. The question is whether $f^{(2)}(x)$ in the range of this mapping, i.e. can we always find a $\phi^{(2)}(x)$ that will satisfy (8)? This problem was first solved by Poincaré [19]. In the following, we present a slightly modified proof that closely follows [1,5]:

Suppose F has a full set of linearly independent eigenvectors. Then we can take the right eigenvectors of F as a set of basis vectors, and the left eigenvectors as a set of coordinates, which are defined by

$$Fv^k = \lambda_k v^k \qquad (10a)$$

$$w_i F = \lambda_i w_i \qquad (10b)$$

where $v^k \in \mathbb{C}^{n \times 1}$, $w_i \in \mathbb{C}^{1 \times n}$ and $\lambda_i, \lambda_k \in \mathbb{C}$. We define a basis for n-dimensional vector valued polynomials of degree 2 as follows:

$$\phi_{ij}^k(x) = v^k(w_i x)(w_j x) \qquad \text{for } j, k = 1, \ldots, n;\ i = 1, \ldots, j. \qquad (11)$$

Using this basis for the polynomials in Eqn. (8), we evaluate the Lie bracket:

$$[Fx, \phi_{ij}^k(x)] = (\lambda_i + \lambda_j - \lambda_k)\phi_{ij}^k(x) \qquad (12)$$

The mapping (9) is onto if $(\lambda_i + \lambda_j - \lambda_k) \neq 0$ for all $j, k = 1, \ldots, n$; $i = 1, \ldots, j$. In the literature, this is called the *resonance condition.* We note that this is only a sufficient condition. A general proof for the case when F does not have a full set of independent eigenvectors may be found in [1].

The above can easily be extended to an arbitrary degree of linearization ρ. We present the final form:

$$[Fx, \phi_{i_1, \ldots, i_\rho}^k(x)] = (\lambda_{i_1} + \cdots + \lambda_{i_\rho} - \lambda_k)\phi_{i_1, \ldots, i_\rho}^k(x) \qquad (13)$$

with $(\lambda_{i_1} + \cdots + \lambda_{i_\rho} - \lambda_k) \neq 0$ the condition of no resonance.

3. Higher Degree Approximations to Control Systems

In this section we will seek a solution to the problem of linearization for control. Full state observability is implicitly assumed. Consider a nonlinear system affine in control:

$$\dot{x} = f(x) + g(x)u \qquad (14a)$$

$$x(0) = x^*. \qquad (14b)$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. The system is assumed to be at rest at the nominal operating point (x^*; $u^* = 0$). Again, we will assume $x^* = 0$. First, consider the linearization of (14) at x^*:

$$\dot{x} = Fx + Gu \qquad (15a)$$

$$F = \frac{\partial f}{\partial x}(0),\ G = g(0). \qquad (15b)$$

We want a coordinate change for (14) of the form identity plus higher degree terms, such that the resulting linear plant will agree with (14) up to an error of degree $O(x,u)^{\rho+1}$ (i.e. terms of $O(x)^{\rho+1}$ and $O(x,u)^{\rho}$) where ρ is the degree of approximation. When $\rho = 1$, the first degree approximation (15) is obtained. Similar to the previous section, the case for $\rho = 2$ will be derived first, and the results will be generalized to an arbitrary degree ρ by induction. First, the functions f and g are expanded in a series:

$$f(x) = f^{(1)}(x) + f^{(2)}(x) + O(x)^3$$
$$= Fx + f^{(2)}(x) + O(x)^3 \qquad (16)$$

$$g(x) = g^{(0)}(x) + g^{(1)}(x) + O(x)^2$$
$$= G + g^{(1)}(x) + O(x)^2 \qquad (17)$$

and the nonlinear system (14a) is rewritten as

$$\dot{x} = Fx + f^{(2)}(x) + \left(G + g^{(1)}(x)\right)u + O(x,u)^3 \qquad (18)$$

We assume a transformation similar to the one proposed in Sec. 2:

$$z = x - \phi^{(2)}(x) \qquad (19)$$

In addition, a new input v is chosen as

$$v = \alpha^{(2)}(x) + (I + \beta^{(1)}(x))u \qquad (20)$$

where $\alpha^{(2)}(x)$ is an $n \times 1$ vector of second degree polynomials, and $I + \beta^{(1)}(x)$ is an $m \times m$ identity matrix plus first degree terms with nonsingular $\beta(x)$. Now we want the system to become, in z coordinates,

$$\dot{z} = Fz + Gv + O(z,v)^3 \qquad (21)$$

We take the time derivative of (19), and introducing (18), (19), (20) and (21) we obtain:

$$f^{(2)}(x) = [Fx, \phi^{(2)}(x)] + G\alpha^{(2)}(x) \qquad (22a)$$

$$g^{(1)}(x)u = [Gu, \phi^{(2)}(x)] + G\beta^{(1)}(x)u \qquad \forall \text{ constant } u. \qquad (22b)$$

Because of its similarity to the homological equations derived in the previous section, we call (22) the *Generalized Homological Equations*. For a detailed derivation of (22), see [17]. In the same reference, the above approximation is extended to an arbitrary degree as

$$f^{(\rho)}(x) = [Fx, \phi^{(\rho)}(x)] + G\alpha^{(\rho)}(x) \qquad (23a)$$

$$g^{(\rho-1)}(x)u = [Gu, \phi^{(\rho)}(x)] + G\beta^{(\rho-1)}(x)u \qquad \forall \text{ constant } u. \qquad (23b)$$

The resulting system is accurate up to degree ρ:

$$\dot{z} = Fz + Gv + O(z,v)^{\rho+1} \qquad (24)$$

Once a higher degree linear approximation is obtained, one of the important issues is the stability of the closed loop system. Thus one may choose, for instance, a linear state feedback for the approximate model

$$\dot{z} = Fz + Gv \tag{25}$$

by setting $v = Kz$. The gain matrix K is chosen such that in the closed loop the system gives the desired performance. If we assume that the model has been linearized up to second degree, using the feedback $v = Kz$ and Eqn. (19) we evaluate (20):

$$Kx - K\phi^{(2)}(x) = \alpha^{(2)}(x) + \left(I + \beta^{(1)}(x)\right)u \tag{26}$$

and calculate the feedback u as:

$$u = \left(I + \beta^{(1)}(x)\right)^{-1}\left(Kx - K\phi^{(2)}(x) - \alpha^{(2)}(x)\right)$$
$$= Kx - \left\{\beta^{(1)}(x)Kx + K\phi^{(2)}(x) + \alpha^{(2)}(x)\right\} + O(x,u)^3. \tag{27}$$

In the above, purpose of the feedback u becomes immediately clear. In addition to the linear feedback, there are *second degree correction terms* (inside curly brackets in (27)). While the purpose of the feedback $u = Kx$ is to achieve stability, pole placement, etc. for the first degree approximation (15a) to get

$$\dot{x} = (F + GK)x, \tag{28}$$

the feedback (27) cancels certain second degree terms to achieve a second degree approximation (accurate to second degree compared with a linear model) toward the same feedback design goals:

$$\dot{x} = (F + GK)x + f^{(2)}(x) + g^{(1)}(x)Kx - G\left\{\beta^{(1)}(x)Kx + K\phi^{(2)}(x) + \alpha^{(2)}(x)\right\} + O(x,u)^3. \tag{29}$$

An important feature of the feedback (27) and the resulting closed loop system (29) is that one need not transform the state variables into the new coordinates z that was introduced for the sake of calculations. Feedback design can be performed in the natural coordinates in which the system is originally presented. If some of the states are not observable, one can estimate the unavailable state variables by means of an observer, and apply the same procedure. For further work on this problem, see [18].

4. Analysis of the Linear Mapping

In the homological equations (22) of Sec. 3, the second degree terms $f^{(2)}(x)$ and $g^{(1)}(x)u$ can be cancelled out under certain solvability conditions by proper choice of $\phi^{(2)}(x)$, $\alpha^{(2)}(x)$, and $\beta^{(1)}(x)$. When the *coefficients* of the like terms in (22) are set equal, a linear mapping is obtained as

$$\begin{Bmatrix} \phi^{(2)}(x) \\ \alpha^{(2)}(x) \\ \beta^{(1)}(x) \end{Bmatrix} \longrightarrow \begin{Bmatrix} f^{(2)}(x) \\ g^{(1)}(x) \end{Bmatrix} \tag{30}$$

A simple count yields the dimensions of the domain and the range:

$$\frac{n^2(n+1)}{2} + \frac{mn(n+1)}{2} + m^2 n \longrightarrow \frac{n^2(n+1)}{2} + n^2 m \tag{31}$$

To analyze the mapping, we make a table for the dimensions:

For $m = 1$: State Space	Domain	Range	For $m = 2$: State Space	Domain	Range
$n = 2$	11	10	$n = 2$	20	14
$n = 3$	27	27	$n = 3$	42	36
$n = 4$	54	56	$n = 4$	76	72
.	.	.	$n = 5$	125	125
.	.	.	.	.	.

Dimensions of the domain and the range become equal whenever $n = 2m + 1$. However, this does not imply that the mapping is of full rank. For example, when $m = 1$, $n = 3$ the rank is 26, not 27. In general, for a single input system, the rank of the mapping is always one less than the dimension of the domain for $n \geq 3$.

We will restrict our analysis to the second degree linearization problem with a single input u, i.e. $m = 1$. We will start with the analysis of the linear mapping.

A necessary condition for finding a coordinate change-feedback pair for a nonlinear control system is the local controllability condition at the nominal point. For the system (18) with a scalar input, this implies

$$\text{rank}\,\{G \;\; FG \;\; \ldots \;\; F^{n-1}G\} = n. \tag{32}$$

On the other hand, we define a $1 \times n$ matrix K such that

$$KF^{i-1}G^{j} = \begin{cases} 0 & 1 \le i < n \\ 1 & i = n \end{cases} \tag{33}$$

Then, the following collection of one forms is of full rank:

$$\text{rank}\,\{K \;\; KF \;\ldots\; KF^{n-1}\} = n. \tag{34}$$

(32) and (34) together imply that we can define a basis for the second and first degree monomials as follows. Define as a basis

$$v^k = F^{k-1}G \tag{35a}$$

and a co–basis

$$w_i = KF^{i-1} \tag{35b}$$

Now we define a basis for second degree monomials as

$$\phi^k_{ij}(x) = v^k(w_i x)(w_j x) \quad \text{for } j, k = 1, \ldots, n\,;\; i = 1, \ldots, j. \tag{36}$$

and a basis for first degree monomials as

$$\phi^k_i(x) = v^k(w_i x) \qquad \text{for } k = 1, \ldots, n\,;\; i = 1, \ldots, n. \tag{37}$$

Using the definitions (36) and (37) is a great convenience for calculating the Lie brackets that appear in the generalized homological equations (22). Calculation of (22a) gives

$$[Fx, \phi^k_{ij}(x)] = \begin{cases} \phi^k_{i+1\,j} + \phi^k_{i\,j+1} - \phi^{k+1}_{ij} & 1 \le i \le j < n;\; 1 \le k < n \\ \phi^k_{i+1\,j} - \phi^{k+1}_{ij} & 1 \le i < j = n;\; 1 \le k < n \\ -\phi^{k+1}_{ij} & i = j = n\;;\; 1 \le k < n \end{cases} \tag{38}$$

In the evaluation of (38), when $k = n$, the expressions become slightly more complicated. However, transforming the control system into a Brunovsky canonical form [3] prior to the linearization helps simplify the expressions [14].

Next, we calculate (22b)

$$[G, \phi^k_{ij}(x)] = \begin{cases} 0 & i, j < n \\ \phi^n_i & i < j = n \\ 2\phi^n_n & i = j = n \end{cases} \tag{39}$$

These two formulas are used to compute the kernel and co-kernel of the mapping

$$\begin{Bmatrix} \phi^{(2)}(x) \\ \alpha^{(2)}(x) \\ \beta^{(1)}(x) \end{Bmatrix} \longrightarrow \begin{Bmatrix} f^{(2)}(x) \\ g^{(1)}(x) \end{Bmatrix} \tag{30}$$

and we now obtain a set of linear equations expressed in matrix form:

$$L\begin{bmatrix}\phi^{(2)}\\ \alpha^{(2)}\\ \beta^{(1)}\end{bmatrix}=\begin{bmatrix}f^{(2)}\\ g^{(1)}\end{bmatrix} \tag{40}$$

In (40), L is a constant coefficient matrix of $n^2(n+1)/2 + n^2$ rows by $n^2(n+1)/2 + n(n+1)/2 + n$ columns that is found from the above evaluation of the Lie brackets of the mapping. In $\begin{bmatrix}\phi^{(2)}\\ \alpha^{(2)}\\ \beta^{(1)}\end{bmatrix}$ and $\begin{bmatrix}f^{(2)}\\ g^{(1)}\end{bmatrix}$ the constant coefficients of their corresponding second degree terms are stacked in a consistent lexicographic ordering. For the single input linearization problem, the column rank of L is $(n^2(n+1)/2+n(n+1)/2+n-1)$.

A solution to the linearization problem is developed as follows. First, we note that since the mapping (40) is deficient in rank for $n > 2$, a control system with nonlinear terms $f^{(2)}(x)$ and $g^{(1)}(x)u$ will not, in general, have an exact solution to yield a second degree linearization. In fact, the Hunt-Su result [9] (or Krener's extension of the same to the approximate case in [16]) is a test for precisely this condition. Consequently, Eqn. (40) will not usually have an exact solution. For a system with $n=3$, $m=1$:

$$2f_3^{33} - g_3^2 + g_2^3 = 0 \tag{41}$$

is the condition for exact linearizability up to second degree. In (41), f_k^{ij} represents the coefficient of an element of $f^{(2)}$ in the basis $\phi_{ij}^k(x)$ for second degree monomials obtained when the system is in Brunovsky canonical form. Similarly, g_k^i is the coefficient of the corresponding element of $g^{(1)}$ in the basis ϕ_i^k for first degree monomials. Eqn. (41) is called the *co-kernel equation.*

When an exact solution does not exist, it is reasonable to seek an approximate solution which will minimize the error in the linearization with respect to some norm. In order to give a precise meaning to this problem, first assume that we have adequate knowledge about the operating regime of the control system and the desired accuracy as determined by

$\rho(x,u)$: A probability density function; typically uniform over some compact set, or Gaussian.

Q: A sensitivity matrix, positive definite.

And define the "error"

$$\left|\left|\begin{pmatrix}f^{(2)}\\ g^{(1)}\end{pmatrix}-\begin{pmatrix}\tilde{f}^{(2)}\\ \tilde{g}^{(1)}\end{pmatrix}\right|\right|^2$$

$$\stackrel{\text{def}}{=}\iint\left|f^{(2)}-\tilde{f}^{(2)}+(g^{(1)}-\tilde{g}^{(1)})u\right|_Q^2\rho(x,u)\mathrm{d}x\mathrm{d}u \tag{42}$$

We want to choose $\begin{pmatrix}\tilde{f}^{(2)}\\ \tilde{g}^{(1)}\end{pmatrix}$ such that the above error is minimized. Note that this term is in the range of the mapping, i.e. it satisfies the generalized homological equations

$$\tilde{f}^{(2)}(x) = [Fx,\phi^{(2)}(x)] + G\alpha^{(2)}(x) \tag{43a}$$

$$\tilde{g}^{(1)}(x)u = [Gu,\phi^{(2)}(x)] + G\beta^{(1)}(x)u \qquad \forall \text{ constant } u. \tag{43b}$$

Furthermore, we wish to choose the smallest $\begin{pmatrix}\phi^{(2)}\\ \alpha^{(2)}\\ \beta^{(1)}\end{pmatrix}$ that will achieve the above. Again, we choose positive definite matrices S, R and minimize

$$\iint|\phi^{(2)}|_S^2 + \left|\alpha^{(2)}(x)+\beta^{(1)}(x)u\right|_R^2\rho(x,u)\mathrm{d}x\mathrm{d}u \tag{44}$$

or one can take a weighted combination of the above. In fact, S can be taken to be equal to Q of (42), but the choice is not limited to this case. We illustrate the minimization in Figs. (1) and (2). Fig. 1 represents the $\frac{n^2(n+1)}{2}+n^2$ dimensional parameter space for the range of the mapping. The coefficients of the second degree terms in the control system define a point in this space, denoted by $\begin{pmatrix}f^{(2)}\\ g^{(1)}\end{pmatrix}$. The range of L is represented by a straight line going through the origin. Those points in the range space of L that exactly satisfy (40) will lie on this line. Among these infinitely many points we want to find the one (shown as $\begin{pmatrix}\tilde{f}^{(2)}\\ \tilde{g}^{(1)}\end{pmatrix}$ on the figure) which will minimize, with respect to a norm as defined earlier, the *error between the actual system that is being approximately linearized and a model which is exactly linearizable* (up to degree 2) by the coordinate change and feedback. Fig. 2 shows the $n^2(n+1)/2 + n(n+1)/2 + n$ dimensional domain space of the mapping , and the minimization done in the domain space.

The numerical solution to (40) is found by linear algebraic methods. For illustration purposes consider a mapping

$$A : \mathbb{R}^N \longrightarrow \mathbb{R}^M \tag{45}$$

and solve

$$Ax = b. \tag{46}$$

If the mapping is not of full rank, it can be expanded as follows

$$\begin{pmatrix}I\\ A\end{pmatrix} : x \longrightarrow \begin{pmatrix}x\\ Ax\end{pmatrix} \in \mathbb{R}^{N+M} \tag{47}$$

where I is the identity matrix of appropriate dimension. The mapping (47) is always of full column rank. Now we solve for

$$\begin{bmatrix}I\\ A\end{bmatrix}x = \begin{bmatrix}0\\ b\end{bmatrix} \tag{48}$$

Then one can choose a metric G on $\mathbb{R}^{N+M}$

$$G = \begin{bmatrix}G_{11} & 0\\ 0 & G_{22}\end{bmatrix} \tag{49}$$

and find a solution to

$$\min_{x\in\mathbb{R}^N} \left|\left|_{Ax-b}^{\ \ x}\right|\right|_G^2 . \tag{50}$$

The well-known solution of (50) is

$$x = \left([I\ \ A^{\mathrm{T}}]G\begin{bmatrix}I\\ A\end{bmatrix}\right)^{-1}[I\ A^{\mathrm{T}}]G\begin{bmatrix}0\\ b\end{bmatrix} \tag{51}$$

Finally, we note the following correspondence between the dimensions and variables in (51) and in the linearization problem:

$M : \frac{n^2(n+1)}{2} + n^2$; $N : n^2(n+1)/2 + n(n+1)/2 + n$

$x : \begin{bmatrix} \phi^{(2)} \\ \alpha^{(2)} \\ \beta^{(1)} \end{bmatrix}$; $b : \begin{bmatrix} f^{(2)} \\ g^{(1)} \end{bmatrix}$

$A : L$; $G : \begin{bmatrix} S & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & Q \end{bmatrix}$

5. An Example

In this section we linearize the following nonlinear plant using the method outlined in Sec. 3.:

$$\dot{x}_1 = x_2 + 0.5x_1^2 + x_1 u \quad (52a)$$

$$\dot{x}_2 = x_3 - x_1 x_3 + x_2 u \quad (52b)$$

$$\dot{x}_3 = u + 0.5x_2^2 + x_2 u \quad (52c)$$

Calculation of the coordinate transformation and feedback gives:

$$z_1 = x_1 - x_1 x_3 \quad (53a)$$

$$z_2 = x_2 + 0.5x_1^2 - x_2 x_3 \quad (53b)$$

$$z_3 = x_3 + x_1 x_2 - x_1 x_3 - x_3^2 \quad (53c)$$

$$v = x_1 x_3 + 1.5x_2^2 - x_2 x_3 + (1 - x_1 + x_2 - 2x_3)u. \quad (54)$$

With the above, we obtain the exact linearization (implying that the system (52) satisfies the Hunt–Su condition):

$$\dot{z}_1 = z_2 \quad (54a)$$

$$\dot{z}_2 = z_3 \quad (54b)$$

$$\dot{z}_3 = v. \quad (54c)$$

A simple feedback design $v = Kz$ that places the closed loop poles at locations -1, $-0.707 \pm 0.707j$ yields the gains as $k_1 = -1$, $k_2 = -2.4142$, $k_3 = -2.4142$. Using $v = K(x - \phi^{(2)}(x))$, (54), and (27), the feedback u is evaluated. The nonlinear input is then applied to the system (52). Note that this will introduce $O(x,u)^3$ terms into (52). In Figs. 3 through 8, we present simulation results and comparisons for the above linearization and feedback problem. In all plots, continuous lines represent the time response curves for a fictitious linear system equal to the linear part of (52) with feedback $u = Kx$ applied. The higher order linearization method of this paper for the nonlinear system (52) is compared against this exact linear model (dashed lines). A feedback design based on a first order approximation (i.e. the feedback $u = Kx$), and applied to the nonlinear model (52) is plotted with dotted lines. Figs. 3, 4, and 5 show the responses to a step input $x_2 = 0.4$. In Figs. 6, 7, and 8 the response curves for a step input $x_2 = -0.4$ is shown. The simulations demonstrate the advantage of the proposed nonlinear feedback. The time response of the nonlinear system with nonlinear controller is closer to the response of a linear system (specifically, the linear system obtained from the first order part of the vector field) than a control design based on a first order approximation. The effect and the improvement of this nonlinear control on the stability bounds of a nonlinear system is being investigated.

Conclusion

In this paper, we presented a method to solve the approximate linearization problem of nonlinear control systems. The problem is reduced to the solution of a set of linear equations as follows: First, the generalized homological equations are derived. By introducing an appropriate basis for expressing higher degree monomials in the vector field, a set of equations linear in the coefficients of the monomials are found. An exact solution to these set of equations is not always possible. A least square solution is proposed that minimizes, in a statistical sense as defined above, the error in the approximation.

We note that in the equivalent linear map, the case when the nonlinear terms to be cancelled are not in the range of the mapping exactly correspond to the violation of the integrability conditions in the Hunt–Su linearization theorem. In other words, the given nonlinear system in this case is not exactly linearizable. In the method developed here, we still find a "partially" linearizing solution to this problem. The least square solution minimizes precisely the error in such an approximation.

Especially for systems with higher dimensions and higher degrees of approximation, the dimension of the system of linear equations may become extremely large and difficult to solve. A computer program that automates the solutions is under development by the authors.

The multi–input case for the generalized homological equations is slightly more complicated to derive. Research is continuing for the description and solution of these equations in the most general input–output setting, and for an arbitrary degree of linearization.

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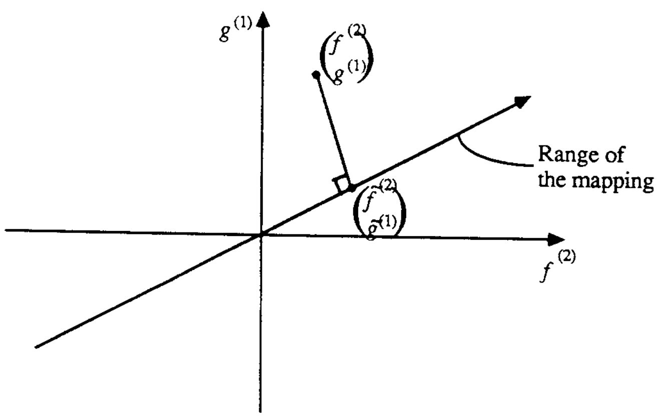

Fig.1 The range space of the mapping.

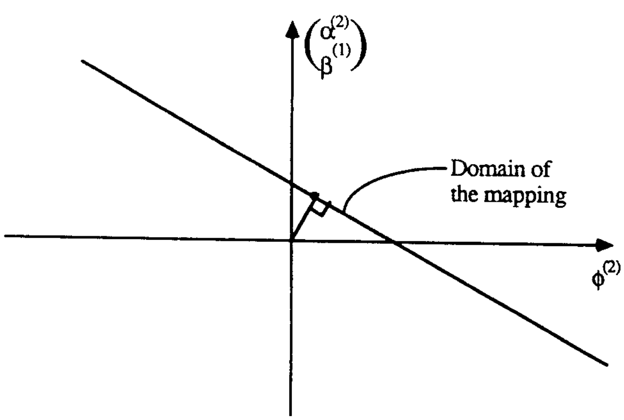

Fig. 2 Domain space of the mapping

LEGEND FOR FIGURES 3 TO 8:

(1) : ———— Fictitious linear model with linear feedbackcontrol.

(2) : — — — — Nonlinear system with nonlinear control (based on the method of this paper)

(3) : ·············· Nonlinear system with control based on first order approximation (same control as the fictitious linear model).

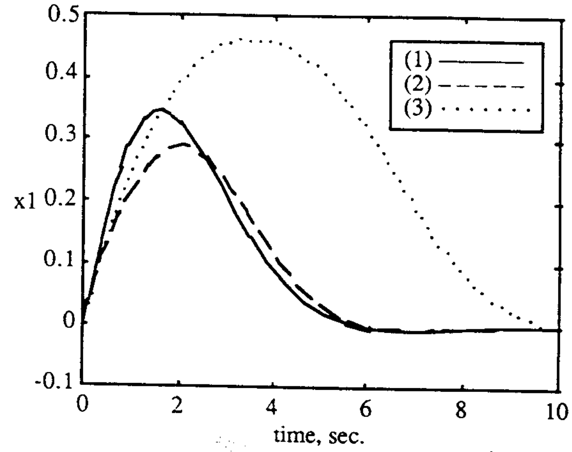

Fig. 3 Time domain response of state x_1; Simulation 1.

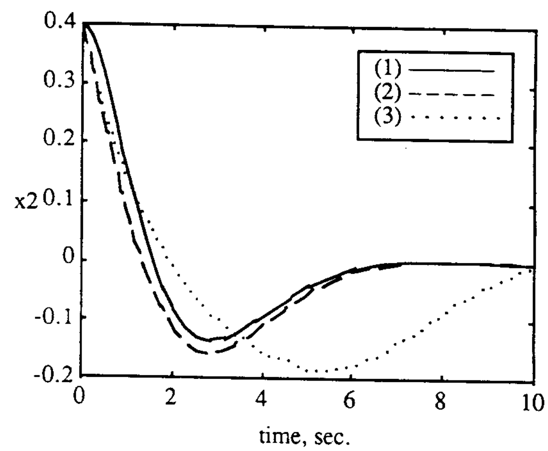

Fig. 4 Time domain response of state x_2; Simulation 1.

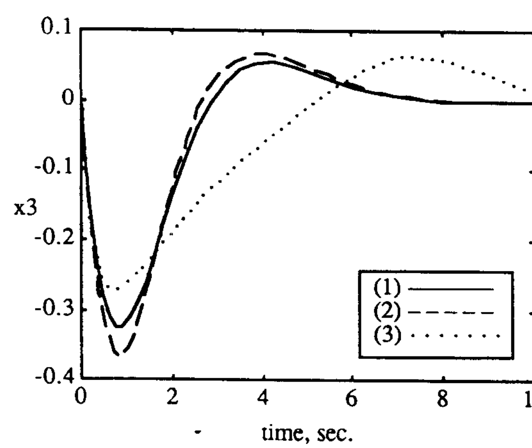

Fig. 5 Time domain response of state x_3; Simulation 1.

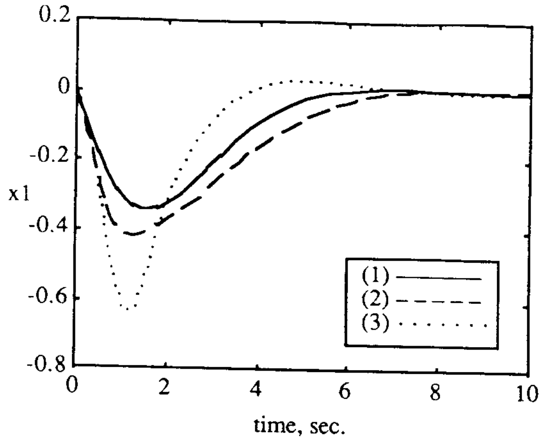

Fig. 6 Time domain response of state x_1; Simulation 2.

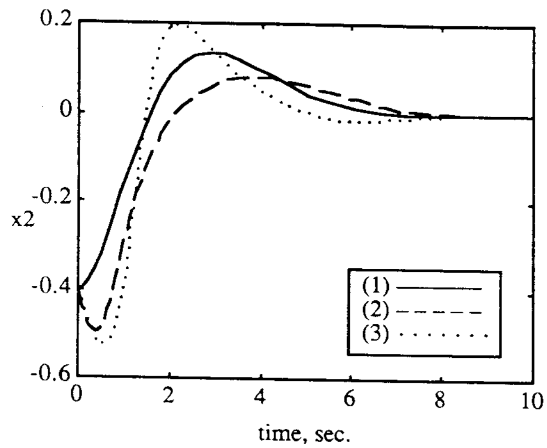

Fig. 7 Time domain response of state x_2; Simulation 2.

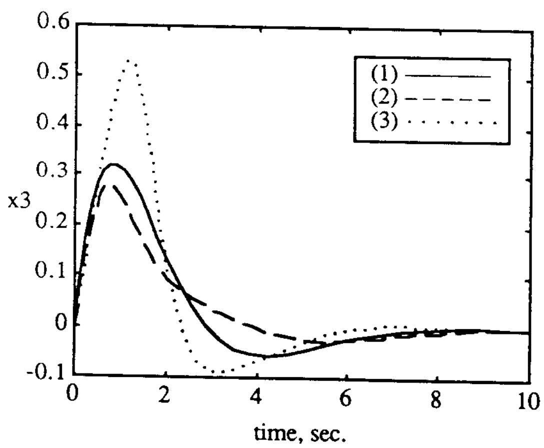

Fig. 8 Time domain response of state x_3; Simulation 2.