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NONLINEAR CONTROLLER DESIGN VIA APPROXIMATE NORMAL FORMS*

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1. Introduction. Over the past several years a group of faculty and graduate students at UC Davis have been developing a set of tools for the design of controllers and observers for nonlinear systems. Our approach has been based on normal forms and approximate normal forms for nonlinear systems. When a nonlinear system admits a normal form the design of a controller or observer is greatly simplified and standard linear design tools can be employed. The people that have been involved in this program are Mont Hubbard, Sinan Karahan, Andrew Phelps, Yi Zhu Ruggero Frezza and myself. This work has been supported in part by AFOSR. In this paper I'll give an overview of our program.

2. Normal Forms. Following Kailath's terminology, [10], there are four normal forms for linear systems, i.e., controllable, observable, controller and observer form. The first two are relatively straightforward to obtain, provided the system is controllable or observable. However, the latter two are more useful in the design of stabilizing state feedback control laws and asymptotic state observers. If a linear system is both controllable and observable then it admits all four normal forms.

In [14] we discussed the nonlinear generalizations of the four linear normal forms. Unfortunately, even controllable and observable nonlinear systems do not admit all four nonlinear normal forms. A nonlinear system which admits controller normal form is sometimes said to be state feedback linearizable in the sense of Hunt–Su [8] and Jakubczyk–Respondek [9]. For a system in controller normal form, the design of a stabilizing state feedback control law is a straightforward task. However, most systems do not admit a controller normal form and even when one does, the transformation of a system into controller normal form involves solving a system of first order linear partial differential equations which can be quite difficult.

Similar remarks are even more appropriate for observer normal form. For a system in observer form, the design of an observer is a straightforward task. But very few systems admit such forms and the computation of observer normal form is, in general, extremely difficult.

For these reasons, we have introduced approximate versions of nonlinear controller and observer form [15, 16]. These may be thought of as finding systems nearby to the original which admit controller or observer form. The computation of such a system is relatively straightforward, and reduces to solving a set of linear equations. Unfortunately, these linear equations are not always solvable and they increase in size quite rapidly with the dimension of the system.

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We start by introducing modified versions of controller and observer normal forms of the nonlinear system.

$$\dot{\xi} = f(\xi) + g(\xi)u \tag{2.1a}$$

$$y = h(\xi) \tag{2.1b}$$

$$\xi(0) \sim \xi^\circ = 0 \tag{2.1c}$$

around the nominal operating point ξ°, which for convenience we assume to be 0. We assume $f(0) = 0$ and $h(0) = 0$. If this is not the case, then in many important cases it can be made so by a possibly time varying change of state and output coordinates. As is usual, the state ξ is n dimensional, the control u is m dimensional and the output y is p dimensional. It is relatively straightforward generalization to consider systems where y depends directly on u, as in

$$y = h(\xi) + k(\xi)u, \tag{2.1d}$$

however to simplify the exposition we shall not do so.

We are interested in studying (2.1) under the pseudogroup of state coordinate transformations around $\xi^\circ = 0$. In [14] we studied arbitrary change of coordinates and attempted to bring the systems into normal form based on prime systems. Such normal forms are closely related to Brunovsky form and its dual. In this article we shall restrict our attention to changes of state coordinates $x = x(\xi)$ whose Jacobian at $\xi^\circ = 0$ is the identity

$$\frac{\partial x}{\partial \xi}(0) = I.$$

Such transformations have two virtues. The first is that they leave invariant the first order linear approximation to (2.1),

$$\dot{z} = Az + Bu + 0(z,u)^2 \tag{2.2a}$$

$$y = Cz + 0(z)^2 \tag{2.2b}$$

$$z(t) = \xi(t) + 0(z)^2 \tag{2.2c}$$

where

$$A = \frac{\partial f}{\partial \xi}(0) \tag{2.3a}$$

$$B = g(0) \tag{2.3b}$$

$$C = \frac{\partial h}{\partial \xi}(0) \tag{2.3c}$$

The second is that the nonlinear coordinates ξ and the normal form coordinates x agree to first order,

$$\xi = x + \phi(x) \tag{2.4a}$$

where

$$\phi(0) = 0, \quad \frac{\partial \phi}{\partial x}(0) = 0. \tag{2.4b}$$

Typically, the original coordinates in which the system is described have some natural meaning and the coordinates have different dimensions, e.g., distance, velocity, mass, etc. Property (2.4) means that at least to first order the normal form coordinates have the same dimensions and intuitive meanings as the natural coordinates.

The system (2.1) admits a modified controller form if there exists a change of state coordinates (2.4) which transforms (2.1) into

$$\dot{x} = Ax + Bu + B(\alpha(x) + \beta(x)u) \tag{2.5a}$$

$$y = Cx + \gamma(x) \tag{2.5b}$$

It follows from (2.4) that the nonlinear terms are quadratic or higher in (x, u), i.e.

$$\alpha(0) = 0, \ \frac{\partial \alpha}{\partial x}(0) = 0 \tag{2.6a}$$

$$\beta(0) = 0, \tag{2.6b}$$

$$\gamma(0) = 0, \ \frac{\partial \gamma}{\partial x}(0) = 0. \tag{2.6c}$$

We require that the $m \times m$ matrix $1 + \beta(x)$ be invertible for x of interest. These conditions (2.4) and (2.6) insure that A, B, C are given by (2.3). Hence the linear part of modified controller form of (2.1) is the same as the first order approximation (2.2) to (2.1).

The system (2.1) admits a modified observer form if there exists a change of state coordinates (2.4) which transforms (2.1) into

$$\dot{x} = Ax + Bu + \alpha(\overline{y}) + \beta(\overline{y})u, \tag{2.7a}$$

$$y = \overline{y} + \gamma(\overline{y}), \tag{2.7b}$$

$$\overline{y} = Cx. \tag{2.7c}$$

It follows from (2.1) that the nonlinear terms are quadratic or higher

3. Poincaré Linearization. Henri Poincaré considered the problem of transforming a nonlinear vector field into a linear field by a change of coordinates around a critical point. We briefly describe his theory, a fuller description can be found in Guckenheimer and Holmes [6] and Arnold [1].

We are given a single vector field

$$\dot{\xi} = f(\xi) \tag{3.1c}$$

$$f(0) = 0 \tag{3.1b}$$

with a critical point at $\xi^\circ = 0$. We are interested in finding a change of coordinates (2.4) which transforms (3.1) into a linear vector field,

$$\dot{x} = Ax \tag{3.2}$$

where A is given by (2.3a).

Poincaré noted that one could develop the change of coordinates term by term in homogeneous powers of x. At degree two we seek an n dimensional vector field $\phi^{(2)}(x)$ whose entries are homogeneous polynomials of degree 2 in x such that under the change of coordinates

$$\xi = x + \phi^{(2)}(x) \tag{3.3}$$

the differential equation (3.1) is transformed to

$$\dot{x} = Ax + O(x)^3 \tag{3.4}$$

whose $O(x)^3$ denotes cubic and higher terms in x. Superscripts in parentheses will be used to indicate that the function is homogeneous of the degree of the superscript in its arguments. If we expand $f(\xi)$ in homogeneous powers of ξ,

$$f(\xi) = A\xi + f^{(2)}(\xi) + f^{(3)}(x) + \cdots \tag{3.5}$$

then (3.1) is transformed into (3.4) iff $\phi^{(2)}$ (ξ) satisfies the so called homological equation

$$[Ax, \phi^{(2)}(x)] = f^{(2)}(x) \tag{3.6a}$$

where [,] is the Lie–Jacobi bracket

$$[Ax, \phi^{(2)}(x)] = \frac{\partial \phi^{(2)}}{\partial x}\ (x)\ Ax - A\phi^{(2)}(x) \tag{3.6b}$$

It is straightforward to verify that $[Ax,\ \cdot]$ is linear map from homogeneous vector fields of degree 2 into homogeneous vector fields of degree 2. Moreover the homogeneous n dimensional vector fields of degree 2 form a linear space of dimension $n^2(n+1)/2$. Hence (3.6a) is solvable for arbitrary $f^{(2)}$ iff zero is not an eigenvalue of the linear mapping defined by $[Ax,\ \cdot]$. Poincaré noted that the eigenvalues of this mapping are related to the eigenvalues of A in a simple fashion. To see why, suppose A is semisimple, i.e., there exists a basis $v^1, \ldots, v^n$ of eigenvectors of A

$$Av^i = \lambda_i\, v^i \tag{3.7a}$$

possibly over the complex numbers.

Let $w_1, \ldots, w_n$ be a cobasis of left eigenvectors of A,

$$w_i A = \lambda_i w_i \tag{3.7b}$$

Then the space of n vector fields homogeneous of degree 2 has as a basis

$$\phi^k_{ij}(x) = v^k(w_i x)\,(w_j x) \tag{3.8}$$

when $1 \le i \le j \le n$ and $1 \le k \le n$. A straightforward calculation yields

$$[Ax, \phi_{ij}^k(x)] = (\lambda_i + \lambda_j - \lambda_k)\ \phi_{ij}^k(x).$$

Hence the eigenvalues of $[Ax,\ \cdot]$ on vector fields homogeneous of degree 2 are

$$\lambda_i + \lambda_j - \lambda_k \tag{3.9}$$

when $1 \le i \le n$ and $1 \le k \le n$.

Hence the homological equation (3.6a) is solvable if no expression of the form (3.9) is zero. Of course this is a sufficient but not necessary condition because a particular $f^{(2)}$ might well be the range of $[Ax,\ \cdot]$, e.g., $f^{(2)} = 0$.

If (3.6a) is solvable one can proceed to look for a transformation canceling the third degree terms in f,

$$\xi = x + \phi^{(3)}(x) \tag{3.10}$$

and $[Ax,\ \cdot]$ is linear mapping of these vector fields homogeneous of degree 3 into themselves. The eigenvalues of this mapping are

$$\lambda_i + \lambda_j + \lambda_k - \lambda_\ell \tag{3.12}$$

where $1 \le i \le j \le k \le n,\ 1 \le \ell \le n$.

Hence (3.11) is solvable for arbitrary $f^{(3)}$ iff none of (3.12) is zero. This generalizes to higher degree. If one of (3.9) or (3.12) or their generalization is zero then there is "resonance" and linearization is not always possible. We refer the reader to [1] and [6] for more details.

4. Approximate Controller Form. S. Karahan in his Ph.D. thesis [12] studied the application of Poincaré's method to finding controller forms and approximate controller forms. We give a brief description of his work.

One starts by expanding (2.1) into homogeneous powers of (ξ, u),

$$\dot{\xi} = A\xi + Bu + f^{(2)}(\xi) + g^{(1)}(\xi)u + \cdots \tag{4.1a}$$

$$y = C\xi + h^{(2)}(\xi) + \cdots \tag{4.1b}$$

One seeks a change of coordinates

$$\xi = x + \phi^{(2)}(x) \tag{4.2}$$

transforming (4.1) into approximate controller form

$$\dot{x} = Ax + Bu + B(\alpha^{(2)}(x) + \beta^{(1)}(x)u) + O(x,u)^3 \tag{4.3a}$$

$$y = Cx + \gamma^{(2)}(x) + O(x)^3 \tag{4.3b}$$

Following Poincaré, we see that this will happen iff

$$[Ax,\ \phi^{(2)}(x)] + B\ \alpha^{(2)}(x) = f^{(2)}(x) \tag{4.4a}$$

$$[Bu,\ \phi^{(2)}(x)] + B\beta^{(1)}(x)u = g^{(1)}(x)u \tag{4.4b}$$

where (4.4b) must hold for each constant u. We refer to these as the generalized homological equations. Like the homological equations, they are linear equations but they are generally not square. The space of unknown $\phi^{(2)}(x), \alpha^{(2)}(x)$ and $\beta^{(1)}(x)$ is $n^2(n+1)/2 + mn(n+1)/2 + m^2n$ dimensional. The constraint space of $f^{(2)}$ and $g^{(1)}$ is $n^2(n+1)/2 + n^2m$ space. These dimensions agree iff $n = 2m+1$. Generally the map $\phi^{(2)}, \alpha^{(2)}, \beta^{(1)} \longmapsto f^{(2)}, g^{(1)}$ is not of full rank so it is not always solvable even when $n = 2m+1$.

Karahan has analyzed this mapping using a basis and cobasis related to the controllability matrix $(B, AB, \dots, A^{n-1}B$. We refer the reader to [12] for details.

Since the system (4.4a) is generally not solvable one is forced to seek approximate solutions. One way of doing this is to find $\tilde{f}^{(2)}$ and $\tilde{g}^{(1)}$ in the range of the mapping (4.2) which is closest in some least squares sense to the given $f^{(2)}$ and $g^{(1)}$. Moreover one would like to choose the smallest $\phi^{(2)}, \alpha^{(2)}$ and $\beta^{(1)}$ which maps into $\tilde{f}^{(2)}$ and $\tilde{g}^{(1)}$. Again we refer the reader to [12] for more details.

Before closing this section it should be mentioned how an approximate controller form (4.3) can be used to stabilize a nonlinear system (2.1) (or equivalently (4.1)) by state feedback. The standard approach is to approximate the nonlinear system to first order by (2.2), choose a stabilizing feedback law for (2.2), $u = Fz$, transform this back into original coordinates,

$$u = F\xi. \tag{4.5}$$

Expressed in homogeneous terms the closed loop dynamics is

$$\dot{\xi} = (A+BF)\xi + f^{(2)}(\xi) + g^{(1)}(\xi)F\xi + O(\xi)^3 \tag{4.6}$$

and hence the system is locally stable around $\xi^\circ = 0$. Of course, if it is too far from $\xi^\circ = 0$, the quadratic and higher terms may drive it unstable.

In the normal form approach, we typically will use the same stabilizing state feedback gain F but to apply it to the second order linearization (4.3) rather than the first order linearization (2.2). The resulting feedback is

$$u + \alpha^{(2)}(x) + \beta^{(1)}(x)u = Fx \tag{4.7}$$

which results in x coordinates the closed loop system

$$\dot{x} = (A+BF)x + O(x)^3 \tag{4.8}$$

Generally speaking, it is better to implement the feedback in the original ξ coordinates taking advantage of the fact that the inverse to (4.2) is

$$x = \xi - \phi^{(2)}(\xi) + O(\xi)^3 \tag{4.9}$$

Neglecting higher than quadratic terms we obtain from (4.7) the feedback

$$u = F\xi - (F\phi^{(2)}(x) + \alpha^{(2)}(\xi) + \beta^{(1)}(\xi)F\xi) + O(\xi)^3. \tag{4.10}$$

Note that to first degree the standard feedback (4.5) and the feedback (4.9) agree. However, the second degree terms of (4.10) cancel the second degree terms of (4.6) to obtain in x coordinates (4.8). One expects that (4.10) is asymptotically stabilizing over a larger neighborhood of ξ° then (4.5).

Of course one can also seek a higher degree approximate controller form. The dimensions of the homological and generalized homological equations grow exponentially in the degree of the approximation. Hence this approach may not be recommended. It might be more efficient and effective to find approximate controller forms of degree two around several operating points rather than an approximate controller form of degree three around a single point.

5. Approximate Observer Form. The work I'm about to describe is joint with Andrew Phelps. We seek a change of coordinates of the form (4.2) which transforms (4.1) into approximate observer form

$$\dot{x} = Ax + Bu + \alpha^{(2)}(\overline{y}) + \beta^{(1)}(\overline{y})u + O(x,u)^3 \tag{5.1a}$$

$$y = Cx + \gamma^{(2)}(\overline{y}) + O(x)^3 \tag{5.1b}$$

$$\overline{y} = Cx. \tag{5.1c}$$

As before this is possible iff we can solve another set of generalized homological equations

$$[Ax, \phi^{(2)}(x)] + \alpha^{(2)}(Cx) = f^{(2)}(x) \tag{5.2a}$$

$$[Bu, \phi^{(2)}(x)] + \beta^{(1)}(Cx)u = g^{(1)}(x)u \tag{5.2b}$$

$$\gamma^{(2)}(Cx) - C\phi^{(2)}(x) = h^{(2)}(x) \tag{5.2c}$$

As before (5.2b) must hold for each constant u.

These equations are linear mapping from the space of functions $\phi^{(2)}(x), \alpha^{(2)}(Cx), \gamma^{(2)}(Cx)$ to the space of functions $f^{(2)}(x), g^{(1)}(x), h^{(2)}(x)$. the dimension of the domain is $n^2(n+1)/2 + np(p+1)/2 + m\ 2p + p^2(p+1)/2$ and that of the range is $n^2(n+1)/2 + mn(n+1)/2 + pn(n+1)/2$. In general, these equations (5.2) are not solvable so as before one must seek a least squares solution. We shall report on that in more detail at another time.

If (2.1) (equivalently (4.1)) can be transformed to approximate observer form then it is easy to construct an observer. We choose H so that $A + HC$ is sufficiently stable. An approximation $\hat{x}(t)$ to $x(t)$ is defined to evolve according to

$$\begin{aligned} \dot{\hat{x}} = (A + HC)\hat{x} + Bu - H(y - \gamma^{(2)}(y)) \\ + \alpha^{(2)}(y - \gamma^{(2)}(y)) + \beta^{(2)}(y - \gamma^{(2)}(y))u \end{aligned} \tag{5.3}$$

then the error $\tilde{x}(t) = x(t) - \hat{x}(t)$ satisfies

$$\dot{\tilde{x}} = (A + HC)\tilde{x} + O(x, \hat{x}, u)^3. \tag{5.4}$$

Hence if the initial error is not too large and u is also not too large, we can expect $\tilde{x}(t) \to 0$ as $t \to \infty$.

Of course, it is preferable to implement the observer in natural coordinates so we transform (5.3) using $\widehat{\xi} = \hat{x} + \phi^{(2)}(\hat{x})$ to obtain

$$\begin{aligned}\dot{\widehat{\xi}} = {} & A\widehat{\xi} + Bu + H(\hat{y} - y) + f^{(2)}(\widehat{\xi}) + g^{(1)}(\widehat{\xi})u \\ & + \alpha^{(2)}(y) - \alpha^{(2)}(\hat{y}) + (\beta^{(2)}(y) - \beta^{(2)}(\hat{y}))u \\ & + H(\gamma^{(2)}(y) - \gamma^{(2)}(\hat{y})) + \frac{\partial \phi^{(2)}}{\partial \widehat{\xi}}\,(\widehat{\xi})H(\hat{y} - y) \\ & + O(\xi, \widehat{\xi}, u)^3\end{aligned} \tag{5.5a}$$

$$\hat{y} = C\widehat{\xi} + h^{(2)}(\widehat{\xi}) \tag{5.5b}$$

$$\widehat{\xi}(0) = \xi^\circ = 0. \tag{5.5c}$$

Notice that the linear part of (5.5) is the observer for (2.1) one would obtain from the linear approximation (2.2), namely

$$\dot{\hat{z}} = A\hat{z} + Bu + H(\hat{y} - y) \tag{5.6a}$$

$$\hat{y} = C\hat{z} \tag{5.6b}$$

$$\hat{z}(0) = \xi^\circ = 0. \tag{5.6c}$$

The error $\tilde{z} = \xi - \hat{z}$ between (2.1) and (5.6) satisfies

$$\dot{\tilde{z}} = (A + HC)\tilde{z} + O(\xi, \widehat{\xi}, u)^2 \tag{5.7a}$$

while the error of the observer (5.5) expressed in $\tilde{x}$ coordinates satisfies (5.4). Hence one expects (5.5) to perform better as an observer for (2.1) over a larger operating range.

As with the state feedback (4.10), the second degree terms of the observer (5.5) are a correction to the standard linear observer for the quadratic nonlinearities of the original system. In implementations one would replace the state ξ in the state feedback control law (4.9) with the estimate $\widehat{\xi}$ from (5.5).

One can continue this process and look for a third degree change of coordinates which transforms the system into approximate observer form where the error terms are $O(\xi, u)^3$. One obtains in this fashion third order corrections to the state feedback (4.10) and observer (5.5). Viewed in this light, we see that the approximate normal form approach allows us to start with a standard linear design based on the linear approximation (2.2) and build in a succession of higher degree corrections to overcome the nonlineararities of (4.1). Throughout we can keep the same feedback gain K and observer gain H, and these can be chosen by standard linear design techniques applied to the linear approximation (2.2).

6. Coprime Factorizations. This work is joint with Yi Zhu [19]. Suppose we have a system in controller normal form

$$\dot{x}_c = Ax_c + Bu + B(\alpha_c(x_c) + \beta_c(x_c)u) \tag{6.1a}$$
$$y = Cx_c + \gamma_c(x_c) \tag{6.1b}$$
$$x_c(0) = 0 \tag{6.1c}$$

where the c-subscripts indicate coordinates and functions associated to controller normal form. We view (6.1) as defining an input/output map

$$G : u(\cdot) \longmapsto y(\cdot) \tag{6.2a}$$

from functions $u(t)$ to $y(t)$ for $t \geq 0$.

We seek a right factorization of G

$$G = N \circ M^{-1}$$

where N and M are input/output maps

$$M : v(\cdot) \longmapsto u(\cdot) \tag{6.2b}$$

$$N : v(\cdot) \longmapsto y(\cdot), \tag{6.2c}$$

M is invertible and $\circ$ denotes composition. There is a large and growing literature on coprime factorization of both linear and nonlinear systems. A sampling is $2 - 5, 7, 10, 11, 13, 17, 18, 20 - 26$]. In particular our approach follows [3, 4].

To describe the input/output maps M and N we shall use a state space realization. In particular we define M to be the input/output map of

$$\dot{\xi}_c = (A + BF)\xi_c + Bv \tag{6.3a}$$
$$\alpha_c(\xi_c) + (1 + \beta_c(\xi_c))u = F\xi_c + v \tag{6.3b}$$
$$\xi_c(0) = 0 \tag{6.3c}$$

where (6.3b) defines u as a function of ξ_c and v.

We consider the composition $N = G \circ M$, this is realized by the $2n$ dimensional system (6.1, 3) described in ξ_c, x_c coordinates. Let $e = x_c - \xi_c$ then

$$\begin{aligned} \dot{e} = {} & Ae + B(-F\xi_c - v + \alpha_c(x_c) \\ & + (1 + \beta_c(x_c))(1 + \beta_c(\xi_c))^{-1}(F\xi_c + v - \alpha_c(\xi_c))) \end{aligned} \tag{6.4}$$

If $e(t) = 0$ then $\dot{e}(t) = 0$. Since $e(0) = 0$ we conclude that $e(t) = 0$ then $\dot{e}(t) = 0$ is unaffected by the input $v(t)$.

A controllable realization of N is

$$\dot{\zeta}_c = (A + BF)\zeta_c + Bv \tag{6.5a}$$
$$y = C\zeta_c + \gamma_c(\zeta_c) \tag{6.5b}$$
$$\zeta_c(0) = 0 \tag{6.5c}$$

Hence we conclude that $G = N \circ M^{-1}$ where N and M are realized by (6.5) and (6.3). Notice that M is invertible since $(1+\beta_c)$ is invertible by assumption.

Notice also that if (A, B) is a controllable pair then we can choose F so that (6.3) and (6.5) are stable systems. Hence we have factored G over the ring of stable nonlinear systems. We are being deliberately vague about the precise definition of a stable nonlinear system. It is clear that (5.3, 5) are "stable" under any reasonable definition.

Of course, we are interested in *coprime* factorizations over the ring of stable nonlinear systems. Again we should not try to make this concept precise but following Hammer [7] and others we shall say that $G = N \circ M^{-1}$ is a coprime factorization if there exists $\widetilde{P}$, the input/output map of a stable system,

$$\widetilde{P} : \begin{pmatrix} u \\ y \end{pmatrix} \longmapsto w \tag{6.6a}$$

such that the composition

$$\widetilde{P} \circ \begin{pmatrix} M \\ N \end{pmatrix} : v \longmapsto \begin{pmatrix} u \\ y \end{pmatrix} \longmapsto w \tag{6.6b}$$

is the identity, $w = v$.

The input/output map of $\binom{M}{N}$ can be realized by an n dimensional system

$$\dot{\xi}_c = (A + BF)\xi_c + Bv \tag{6.7a}$$

$$\alpha_c(\xi_c) + (1 + \beta_c(\xi_c))u = F\xi_c + v \tag{6.7b}$$

$$y = C\xi_c + \gamma_c(\xi_c) \tag{6.7c}$$

$$\xi_c(0) = 0 \tag{6.7d}$$

A left inverse of (6.7) is

$$\dot{z}_c = Az_c + Bu + B(\alpha_c(z_c) + \beta_c(z_c)u) \tag{6.8a}$$

$$w = \alpha_c(z_c) + (1 + \beta_c(z_c))u - Fz_c \tag{6.8b}$$

$$z_c(0) = 0 \tag{6.8c}$$

If $e = \xi_c - z_c$ then

$$\dot{e} = Ae + B(\alpha_c(\xi_c) - \alpha_c(z_c) + (\beta_c(\xi_c) - \beta_c(z_c))u)$$

If $e(t) = 0$ the $\dot{e}(t) = 0$ and since $e(0) = 0$ it follows that $e(t) = 0$ for all $t \geq 0$. If $e(t) = \xi_c(t) - z_c(t) = 0$ then $w(t) = v(t)$ so (6.8) inverts (6.7).

However we do not know that (6.8) is stable. To insure the stability of (6.8), we must add to (6.8a) an extra term. This term must stabilize (6.8) and also must be zero when $\xi_c = z_c$ so that (6.8) remains a left inverse of (6.7). How do we find such a term?

Notice that the dynamics (6.8a) is the same as the dynamics of the original system (6.1a) and notice that the other output y of (6.7) does not appear in (6.8). Perhaps we can inject y into (6.8a) to stabilize it? This is more or less equivalent to asking whether output injection can be used to stabilize the original system (6.1). This is always possible for systems in observer form, hence we assume that there exists a change of coordinates

(6.9) $$x_c = x_0 + \phi(x_0)$$

satisfying (2.4b) transforming (6.1) into observer form

(6.10a) $$\dot{x}_0 = Ax_0 + Bu + \alpha_0(Cx_0) + \beta_0(Cx_0)u$$

(6.10b) $$y = Cx_0 + \gamma_0(Cx_0)$$

(6.10c) $$x_0(0) = 0$$

Suppose we consider a similar change of coordinates for (6.8)

(6.11) $$z_c = z_0 + \phi(z_0)$$

to obtain

(6.12a) $$\dot{z}_0 = Az_0 + Bu + \alpha_0(Cz_0) + \beta_0(Cz_0)u$$

(6.12b) $$w = \alpha_0(z_0 + \phi(z_0)) + (1 + \beta_c(z_0 + \phi(z_0)))u - F(z_0 + \phi(z_0))$$

(6.12c) $$z_0(0) = 0.$$

We add to (6.12a) the term

(6.13a) $$\alpha_0(\overline{y}) - \alpha_0(Cz_0) + (\beta_0(\overline{y}) - \beta_0(Cz_0))u + H(Cz_0 - \overline{y})$$

to obtain

(6.12aa) $$\dot{z}_0 = (A + HC)z_0 + Bu + \alpha_0(\overline{y}) + \beta_0(\overline{y})u - Hy$$

where $\overline{y}$ is a function of y of (6.7)c) defined by

(6.13b) $$y = \overline{y} + \gamma_0(\overline{y}) = C\xi_0 + \gamma_0(C\xi_0)$$

and ξ_0 is the state of (6.7) in observer coordinates

(6.13c) $$\xi_c = \xi_0 + \phi(\xi_0)$$

Notice that (6.13a) is zero whenever $\xi_0 = z_0$, hence the input/output map $\widetilde{P}$ of the (6.12aa, b, c) is stable.

In summary, we have shown that if a nonlinear system admits both controller and observer form than its input/output map G can be factored into the composition $N \circ M^{-1}$ of input/output maps of stable systems N and M. Moreover this composition is coprime in the sense that the input/output map $\binom{M}{N}$ has a left inverse $\widetilde{P}$ which is realized by a stable system.

We have not presented this as a theorem because we are reluctant at this point in time to give formal definitions of coprimeness and stability for nonlinear systems. However the above development is very analogous to the linear theory [3, 4]. See also Hammer [7]

Unfortunately the analogy is not so straightforward for left coprime factorizations. The theory of left coprime factorizations for nonlinear systems has some substantial differences with the linear theory.

We start with a system in observer form (6.10) realizing an input/output map G. We define another input output map

$$\widetilde{M} : \begin{pmatrix} u \\ y \end{pmatrix} \longmapsto w, \tag{6.14}$$

by

$$\dot{\xi}_0 = (A + HC)\xi_0 - H\overline{y} + \alpha_0(\overline{y}) + \beta_0(\overline{y})u \tag{6.15a}$$

where $\overline{y}$ is an invertible function of the input y defined by

$$y = \overline{y} + \gamma_0(\overline{y}) \tag{6.15b}$$

and the output is

$$w = -C\xi_0 + \overline{y} \tag{6.15c}$$

$$\xi_0(0) = 0 \tag{6.15d}$$

Consider the serial connection of (6.10) and (6.15), this is not a realization of the $\widetilde{M} \circ G$ but it is a realization of $\widetilde{N} = \widetilde{M} \circ \binom{I}{G}$. (This is the first important difference with the linear theory). If we define $\xi_0 = x_0 - \xi_0$ then $\widetilde{N}$ is realized by

$$\dot{\zeta}_0 = (A + HC)\zeta_0 + Bu \tag{6.16a}$$

$$w = C\zeta_0 \tag{6.16b}$$

$$\zeta_0(0) = 0 \tag{6.16c}$$

because in ξ_0, x_0 coordinates for (6.10, 15) only the ξ_0 coordinates are observable from the output w. We consider $\widetilde{N}, \widetilde{M}$ as a left factorization of G, although it is really a left factorization of $\binom{I}{G}$ in the sense that

$$\widetilde{M} \circ \begin{pmatrix} I \\ G \end{pmatrix} = \widetilde{N} \tag{6.17}$$

Notice that we cannot compose this on the left with $\widetilde{M}^{-1}$ since $\widetilde{M}$ is not invertible as a mapping from $\binom{u}{y}$ to w.

Perhaps the best way of viewing the situation is

$$\begin{pmatrix} I & 0 \\ 0 & \widetilde{M} \end{pmatrix} \circ \begin{pmatrix} I \\ G \end{pmatrix} = \begin{pmatrix} I \\ \tilde{N} \end{pmatrix} \tag{6.18a}$$

or

$$\begin{pmatrix} I \\ G \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & M \end{pmatrix}^{-1} \circ \begin{pmatrix} I \\ N \end{pmatrix} \tag{6.18b}$$

The matrix notation is somewhat misleading because $\widetilde{M}$ depends on both u and y.

In any case, if (C, A) is an observable pair then both (6.15) and (6.16) can be made stable by proper choice of H. In particular, the nonlinearities in (6.15a) are the memoryless functions of the inputs u and y hence (6.15) is BIBO stable.

Next we address the coprimeness of the above factorization.

We consider the input/output map

$$[-\widetilde{N}, \widetilde{M}] : \begin{pmatrix} u \\ y \end{pmatrix} \longmapsto w \tag{6.19a}$$

where again the matrix notation is somewhat misleading since both u and y are inputs to $\widetilde{M}$, i.e.,

$$w = -\widetilde{N}(u) + \widetilde{M}\begin{pmatrix} u \\ y \end{pmatrix}. \tag{6.19b}$$

This input/output map can be realized by an n dimensional system

$$\dot{\xi}_0 = (A + HC)\xi_0 + Bu + \alpha(\overline{y}) + \beta_0(\overline{y})u - H\overline{y} \tag{6.20a}$$

where $\overline{y}$ is an invertible function of the input y defined by

$$y = \overline{y} + \gamma_0(\overline{y}) \tag{6.20b}$$

and the output w is given by

$$w = -C\xi_0 + \overline{y} \tag{6.20c}$$

We wish to find a input/output map P realized by a stable system so that P is a right inverse of $[-\widetilde{N}, \widetilde{M}]$,

$$P : v \longmapsto \begin{pmatrix} u \\ y \end{pmatrix} \tag{6.21a}$$

$$[-\widetilde{N}, \widetilde{M}] \circ P : v \longmapsto w = v. \tag{6.21b}$$

We start by constructing an inverse for (6.20),

$$\dot{z}_0 = Az_0 - Hv + Bu + \alpha_0(\overline{y}) + \beta_0(\overline{y})u \tag{6.22a}$$

$$\overline{y} = Cz_0 + v \tag{6.22b}$$

$$y = \overline{y} + \gamma_0(\overline{y}) \tag{6.22c}$$

$$u = ? \tag{6.22d}$$

$$z_0(0) = 0 \tag{6.22e}$$

We leave unspecified for the moment the output u which also appears in the dynamics (6.22a). Notice that if $e = \xi_0 - z_0$ is the error between the states of (6.20) and (6.22) then $\dot{e} = 0$ whenever $e = 0$. Since $e(0) = 0$ we conclude that $e(t) = 0$ for all $t \geq 0$ and so by (6.20c) and (6.22b) we have $w(t) = v(t)$. In other words (6.22) is a right inverse of (6.20).

What about the stability of (6.22)? We would like to choose the output u in such a way that (6.22a) is stable in some sense. If we ignore the –Hv term of (6.22a) this looks like the original system is observer form. This is not exactly true because $\overline{y}$ is defined by (6.22b) with v present. Suppose the original system can be transformed into controller form (6.1) by a change of coordinates (6.9). If we apply a similar change of coordinates (6.11) to (6.22) we obtain

$$\begin{aligned}\dot{z}_c = {} & Az_c + Bu + B(\alpha_c(z_c) + \beta_c(z_c)u) - Hv \\ & - \frac{\partial\phi}{\partial z_0}(Hv) + (1 + \frac{\partial\phi}{\partial z_0})(\alpha_0(Cz_0 + v) - \alpha_0(Cz_0) \\ & + (\beta_0(Cz_0 + v) - \beta_0(Cz_0))u)\end{aligned} \tag{6.23a}$$

Suppose we choose an F such that $(A + BF)$ is stable and define u by

$$\alpha_c(z_c) + \beta_c(z_c)u = Fz_c. \tag{6.22dd}$$

When the input $v = 0$, (6.23a) becomes

$$\dot{z}_c = (A + BF)z_c. \tag{6.23b}$$

Unfortunately we cannot conclude that (6.23a) is BIBO stable since the input v is multiplied by a function of the state.

We conclude by noting that a "nonlinear Bezout identity" holds for the above. In other words beside $\widetilde{P}$ being a left inverse (6.6b) for $\binom{M}{N}$ and P a right inverse (6.21b) for $[-\widetilde{N}, \widetilde{M}]$, it is also true that

$$[-\widetilde{N}, \widetilde{M}] \circ \begin{pmatrix} M \\ N \end{pmatrix} : v \longmapsto w = 0 \tag{6.24a}$$

and

$$\widetilde{P} \circ P : v \longmapsto \begin{pmatrix} u \\ y \end{pmatrix} \longmapsto w = 0 \tag{6.24b}$$

In abuse of notation we summarize these equations by

$$\begin{pmatrix} \widetilde{P} \\ [-\widetilde{N}, \quad \widetilde{M}] \end{pmatrix} \circ \left(\begin{pmatrix} M \\ N \end{pmatrix} P \right) = \begin{pmatrix} 1 & 0 \\ 0 & I \end{pmatrix} \tag{6.25}$$

The verification of (6.24) is straightforward.

From the work of Doyle [3] Francis [4] and others the, the existence of a nonlinear Bezout identity suggests that it might be possible to develop a nonlinear version of Youla's Q parameterization of all stable and stabilizing controller of a linear system. This generalization would apply to those nonlinear systems which admit both controller and observer form. This class is very thin, but perhaps such a result could be extended approximately to those systems that approximately admit controller and observer form. Work in these areas is continuing.

7. Concluding Remarks. We have briefly described an approach to nonlinear compensator design based on nonlinear normal forms and approximately normal forms. This approach is being pursued by a group of researchers at U.C. Davis with support from AFOSR. The principal advantage of the normal forms approach is that to a large extent it reduces problems in nonlinear design to problems in linear design. We are developing software tools which utilize this approach as a compliment to existing linear design software so that these linear design packages can be used for nonlinear systems that admit at least approximate normal forms.

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