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EXTENDED NORMAL FORMS OF QUADRATIC SYSTEMS

Arthur J. Krener and Wei Kang

Department of Mathematics
and
Institute of Theoretical Dynamics
University of California
Davis,CA95616

Abstract

In this paper, a set of extended quadratic controller normal forms of linearly controllable nonlinear systems is given. Then we proved that any linearly controllable nonlinear system is linearizable to second degree by a dynamic feedback.

§1. Introduction

It is well known that there are four normal forms for linear systems, i.e, controllable, observable, controller and observer form. The nonlinear generalizations of these four linear normal forms were given and discussed in Krener [13], Hunt-Su [5], Jakubczyk-Respondek [8], Brocket [1] and Sommer [17] etc. Unfortunately, most controllable systems do not admit a controller normal form and even when one does, the transformation of a system into controller normal form involves solving a system of first order linear partial differential equations which can be numerically quite difficult. For these reasons, the approximate versions of nonlinear controller and observer normal forms were introduced in Krener [12], Krener-Karahan-Hubbard-Frezza [14], and Karahan [11] etc. It was proved that for certain kind of nonlinear controllable systems, we can find a nonlinear change of coordinates and nonlinear feedback that transforms the system into the linear approximation of the plant dynamics which is accurate to second or higher degree. The computation of such a change of coordinates and feedback is reduced to solving a set of linear equations. However, these linear equations are not always solvable and most of the nonlinear systems do not admit such a linear approximation.

In this paper, a set of extended quadratic controller normal forms of linearly controllable systems with single input is given (Theorem 2). We can consider these normal forms as the extension of the Brunovsky form to the nonlinear systems. Then we prove that given a nonlinear system, there exists a dynamic feedback so that the extended system has a linear approximation which is accurate to second or higher degree (Theorem 3).

In this paper, we only consider the single-input systems. The generalization to multi-input systems will be given in another paper. We only give the sketch of the proofs here and the detail proof can be found in Kang-Krener [10]

§2. Extended quadratic controller form and dynamic feedback linearization

From Brunovsky [2] and Kailath [9], we know that any controllable linear system can be transformed into a controller form by a linear change of coordinates . If, in addition, we also allow linear change of coordinates in the input space and linear state feedback, any controllable linear system can be transformed into a Brunovsky form. This result is summarized in the following theorem. The change of coordinates and feedback used in this theorem is:

$$\begin{cases} \xi = Tx \\ v = \alpha x + \beta u \end{cases} \tag{2.1}$$

where T is a constant nxn nonsingular matrix, α is a row vector and β is a nonzero real number.

Theorem 1: Consider a single input, time-invariant linear system:

$$\dot{\xi} = F\xi + Gu. \tag{2.2}$$

If it is controllable, then by a suitable change of coordinates and feedback (2.1), this linear system can be transformed into the following system of Brunovský form:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} u. \tag{2.3}$$

In the following, we are going to give a nonlinear generalization of Brunovsky form from the quadratic approximation point of view. We will study the following nonlinear systems:

$$\dot{\xi} = f(\xi) + g(\xi)u \tag{2.4}$$

where $f(\xi)$ and $g(\xi)$ are nonlinear vector fields such that

$$f(0) = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}. \tag{2.5}$$

Throughout this paper, we shall use the following notation:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{nxn} \qquad B = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}_{nx1} \tag{2.6a}$$

$$f^{[2]}(\xi) = \begin{bmatrix} f_1^{[2]}(\xi) \\ f_2^{[2]}(\xi) \\ \cdot \\ \cdot \\ f_n^{[2]}(\xi) \end{bmatrix}, \qquad g^{[1]}(\xi) = \begin{bmatrix} g_1^{[1]}(\xi) \\ g_2^{[1]}(\xi) \\ \cdot \\ \cdot \\ g_n^{[1]}(\xi) \end{bmatrix} \tag{2.6b}$$

$$F = \frac{\partial f}{\partial \xi}(0) \qquad G = g(0). \tag{2.6c}$$

The upper index of $f^{[2]}(\xi)$ and $g^{[1]}(\xi)$ means that $f_i^{[2]}(\xi)$ and $g_i^{[1]}(\xi)$ are homogeneous polynomials of second and first degree in ξ. This kind of upper index will also be applied to some other vector fields and functions (e.g. $\alpha^{[2]}(x)$ or $\beta^{[1]}(x)$).

Definition 1: If (F, G) is controllable, we call (2.4) a linearly controllable system. In this paper, we always assume that a nonlinear system is linearly controllable.

As mentioned in §1, most linearly controllable nonlinear systems do not admit a controller normal form, therefore they can not be transformed into the Brunovsky form (2.3). Theorem 1 implies that there exists a linear change of coordinates and linear feedback (2.1) which transforms a nonlinear system (2.4) into a system:

$$\dot{x} = Ax + Bu + f^{[2]}(x) + g^{[1]}(x)u + O(x,u)^3 \qquad (2.7)$$

where (A, B) has Brunovsky form (2.6a). So, Brunovsky form is a normal form for the linear part of linearly controllable nonlinear system (2.4). The question is what is the normal form of the quadratic terms in this system. We will answer this question in the next theorem. To solve this problem, let us consider the following nonlinear systems, the linear part of which is in Brunovsky form:

$$\dot{\xi} = A\xi + Bv + f^{[2]}(\xi) + g^{[1]}(\xi)v + O(\xi,v)^3 \qquad (2.8)$$

Since the linear part of (2.8) is already in Brunovsky form, we want to leave it invariant under a change of coordinates and feedback. Therefore, we consider the change of coordinates and feedback of the following form:

$$\begin{cases} \xi = x + \phi^{[2]}(x) \\ v = u + \alpha^{[2]}(x) + \beta^{[1]}(x)u \end{cases} \qquad (2.9)$$

Here $\phi^{[2]}(x)$ is a n-dimensional vector field whose entries are homogeneous polynomials of second degree in x, $\alpha^{[2]}(x)$ is a homogeneous polynomial of second degree and $\beta^{[1]}(x)$ is a polynomial of first degree. The transformation given by (2.9) has two virtues. The first is that they leave invariant the linear part of (2.8). The second is that the nonlinear coordinates ξ and x agree to the first degree.

Now, we can answer the question what is the normal form of the quadratic terms in (2.8) under the change of coordinates and feedback (2.9).

Theorem 2: By a change of coordinates and feedback (2.9), system (2.8) can be transformed into one and only one of the following systems:

$$\dot{x} = Ax + Bv + \tilde{f}^{[2]}(x) + O(x,v)^3 \qquad (2.10a)$$

where

$$\tilde{f}^{[2]}(x) = \begin{bmatrix} \tilde{f}_1^{[2]}(x) \\ \tilde{f}_2^{[2]}(x) \\ \cdot \\ \cdot \\ \cdot \\ \tilde{f}_n^{[2]}(x) \end{bmatrix} \qquad (2.10b)$$

$$\tilde{f}_i^{[2]}(x) = \begin{cases} \sum_{j=i+2}^{n} a_{ij}x_j^2 & 1 \le i \le n-2 \\ 0 & i = n-1 \text{ or } n \end{cases} \qquad (2.10c)$$

Definition 2: A system such as (2.10) is said to be in extended quadratic controller form.

Since most nonlinear systems (2.4) do not admit a controller form, they can not be completely linearized by a change of coordinates and a feedback. we wish to use (2.9) to transform (2.4) into a linear system plus an error of second or higher degree:

$$\dot{x} = F x + Gv + O(x, v)^3. \qquad (2.11)$$

A system with this property is said to be quadratically linearizable by (2.9). From the result of Theorem 2, we know that a system (2.8) is quadratically linearizable by (2.9) if and only if the corresponding extended quadratic controller form satisfies:

$$\tilde{f}^{[2]}(x) = 0. \qquad (2.12)$$

Therefore, most nonlinear systems are not quadratically linearizable. In the following, we are going to introduce a method of linearizing a nonlinear system to the second degree by a dynamic feedback. The concept of dynamic feedback was introduced and studied in Singh [16] and Charlet-Lévine-Marino [4].

Definition 3: A dynamic feedback is a system:

$$\begin{aligned} \dot{\omega} &= a(\xi,\omega) + b(\xi,\omega)v \qquad \omega(t) \in R^q \\ u &= c(\xi,\omega) + d(\xi,\omega)v \qquad v(t) \in R \,. \end{aligned} \qquad (2.13)$$

Where q is called the dimension of the dynamic feedback. In general, the q-dimensional vector fields $a(\xi,\omega)$ and $b(\xi,\omega)$ are nonlinear. $c(\xi,\omega)$ and $d(\xi,\omega)$ are scalar functions and $d(0,0) \neq 0$.

Consider the system (2.4) with a dynamic feedback (2.13). The extended system is:

$$\begin{bmatrix} \dot{\xi} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} f(\xi) + g(\xi)c(\xi,\omega) \\ a(\xi,\omega) \end{bmatrix} + \begin{bmatrix} g(\xi)d(\xi,\omega) \\ b(\xi,\omega) \end{bmatrix} v$$

$$= f_1(\xi,\omega) + g_1(\xi,\omega)v \qquad (2.14)$$

Let F_1 be the Jacobian matrix of $f_1(\xi,\omega)$ at (0,0), let G_1 be $g_1(0,0)$.

Definition 4: If we can find a dynamic feedback such that the extended system (2.14) is linearly controllable and it can be transformed into:

$$\dot{z} = F_1 z + G_1 v + O(z,v)^3 \qquad (2.15)$$

by a change of coordinates (in the extended state space):

$$\begin{bmatrix} \xi \\ \omega \end{bmatrix} = z + \psi^{[2]}(z), \qquad (2.16)$$

then the system (2.4) is called quadratically linearizable by a dynamic feedback.

Theorem 3: Any linearly controllable system (2.8) is quadratically linearizable by a dynamic feedback.

Corollary 1: Any linearly controllable system (2.4) is quadratically linearizable by a dynamic feedback.

In Corollary 2, we will show that finding a suitable dynamic feedback and a change of coordinates in the extended space is equivalent to solving a set of linear equations. Suppose the Taylor series of the vector fields $f(\xi)$ and $g(\xi)$ in the system (2.4) are:

$$\begin{aligned} f(\xi) &= F\xi + f^{[2]}(\xi) + O(\xi)^3 \\ g(\xi) &= G + g^{[1]}(\xi) + O(\xi)^2 \end{aligned} \qquad (2.17)$$

Corollary 2: Suppose that the dimension of the state space of the system (2.4) is n. To quadratically linearize this system by a dynamic feedback, we can use the following n-1 dimensional dynamic feedback:

$$\begin{aligned} \dot{\omega} &= A\omega + Bv \\ u &= \omega_1 + \gamma^{[1]}(\xi,\omega) + \gamma^{[2]}(\xi,\omega) \end{aligned} \qquad (2.18)$$

where (A, B) is in Brunovsky form (2.6a) of dimension n-1. The change of coordinates (2.16) in the extended state space is:

$$\begin{bmatrix}\xi\\ \omega\end{bmatrix}=\begin{bmatrix}z\\ \omega\end{bmatrix}+\begin{bmatrix}\phi^{[2]}(z,\omega_1,\ldots,\omega_{n-2})\\ 0\end{bmatrix} \qquad (2.19)$$

The homogeneous polynomials $\gamma^{[1]}(\xi,\omega)$, $\gamma^{[2]}(\xi,\omega)$ and the vector fields $\phi^{[2]}(z,\omega_1,\ldots,\omega_{n-2})$ are chosen such that the extended system is linearly controllable and:

$$[\ Fz+G(\omega_1+\gamma^{[1]}(z,\omega)),\ \phi^{[2]}(z,\omega_1,\ldots,\omega_{n-2})\ ] + \frac{\partial\phi^{[2]}}{\partial\omega}A\omega$$

$$= G\gamma^{[2]}(z,\omega) + f^{[2]}(z) + g^{[1]}(z)(\omega_1+\gamma^{[1]}(z,\omega)) \qquad (2.20)$$

Furthermore, by (2.18) and (2.19) the system (2.4) will be transformed into

$$\begin{bmatrix}\dot z\\ \dot\omega\end{bmatrix}=\begin{bmatrix}Fz+G(\omega_1+\gamma^{[1]}(z,\omega))\\ A\omega\end{bmatrix}+\begin{bmatrix}0\\ B\end{bmatrix}v+O(z,\omega,v)^3 \qquad (2.21)$$

Remark 1: In Charlet - Lévine - Marino [4], it was proved that if a single input system is not exactly linearizable, then this system is not linearizable by a dynamic feedback. The result of Corollary 1 means that in the problem of finding the quadratic linearization, the opposite result is true, i.e, any single input linearly controllable system is quadratically linearizable by a dynamic feedback.

The theorems 2, 3 and the corollaries in this section will be proved in §5.

§3. Quadratic equivalence

In this section, we will define the family of all the systems such as (2.8) of certain dimension to be a linear space. An equivalent relation on this linear space will be introduced. Then, several theorems about this equivalent relation and the associated classification will be given. All these results will be used in the proof of the theorems in §2.

Definition 5: Consider two systems:

$$\dot\xi = A\xi + Bu + f_1^{[2]}(\xi) + g_1^{[1]}(\xi)u + O(\xi,u)^3 \qquad (3.1a)$$

$$\dot x = Ax + Bv + f_2^{[2]}(x) + g_2^{[1]}(x)v + O(x,v)^3 \qquad (3.1b)$$

System (3.1a) is called to be quadratically equivalent to system (3.1b) if and only if there exists a change of coordinates and feedback (2.9) such that the system (3.1a) is transformed into

$$\dot x = Ax + Bv + f_2^{[2]}(x) + g_2^{[1]}(x)v + O(x,v)^3 \qquad (3.2)$$

i.e, system (3.1a) is transformed into a system which agree with (3.1b) up to an error of third degree.

As we know that the family of linear change of coordinate and feedback (2.1) is a group under the operation of composition. The family of nonlinear change of coordinate and feedback is a pseudogroup but not a group because the change of coordinates is a local transformation and different transformations are well defined on different domains around the origin. Let us consider the quotient group of this pseudogroup by $O(x,u)^3$. What we get is a family of transformations which only contains linear and quadratic part. This is also a pseudogroup which consist of all transformations (2.9). It is denoted by **G**. Two systems (3.1a) and (3.1b) are quadratically equivalent means that there is an element of **G** so that it transforms (3.1a) to a system which agree with (3.1b) up to an error of third degree. So, it is easy to show that quadratic equivalence is an equivalent relation(see [6]). We can define a classification on the family of all the systems of the form (2.8) by this equivalent relation. Each class of this classification contains all systems which are quadratically equivalent to each other. In §5, we will prove the Theorem 2 by showing that the extended quadratic controller forms are the representatives of all the equivalent classes.

Theorem 4: Consider two nonlinear systems

$$\dot\xi_1 = A\xi_1 + Bu_1 + f_1^{[2]}(\xi_1) + g_1^{[1]}(\xi_1)u_1 + O(\xi_1,u_1)^3 \qquad (3.3a)$$

$$\dot\xi_2 = A\xi_2 + Bu_2 + f_2^{[2]}(\xi_2) + g_2^{[1]}(\xi_2)u_2 + O(\xi_2,u_2)^3 \qquad (3.3b)$$

They are quadratically equivalent to each other if and only if there exist functions $\alpha^{[2]}(\xi_2)$, $\beta^{[1]}(\xi_2)$ and a vector field $\phi^{[2]}(\xi_2)$ such that

$$[\ A\xi_2,\ \phi^{[2]}(\xi_2)\ ] + B\alpha^{[2]}(\xi_2) = f_1^{[2]}(\xi_2) - f_2^{[2]}(\xi_2) \qquad (3.4a)$$

$$[\ B,\ \phi^{[2]}(\xi_2)\ ] + B\beta^{[1]}(\xi_2) = g_1^{[1]}(\xi_2) - g_2^{[1]}(\xi_2). \qquad (3.4b)$$

Sketch of the proof: The proof is similar to that given in Krener-Karahan-Hubbard-Frezza [14]. Q.E.D.

Since the set of all the homogeneous polynomials of $(x_1,x_2,\ldots,x_n)$ is a linear space of finite dimension, we can consider $(\ \phi^{[2]}(x), \alpha^{[2]}(x), \beta^{[1]}(x)\ )$ of (2.9) as an element of a linear space W and $(f^{[2]}(\xi), g^{[1]}(\xi))$ of (2.8) as an element of a linear space V. In this way, we can consider the family of transformation (2.9) and the family of nonlinear system (2.8) as linear spaces W and V. Since the linear part of (2.8) is always in Brunovsky form, sometimes we use $(f^{[2]}, g^{[1]})$ to represent a system (2.8). Define a linear map $\mathfrak{A}$ from W to V by the following Lie bracket:

$$\mathfrak{A}(\phi^{[2]}(\xi), \alpha^{[2]}(\xi), \beta^{[1]}(\xi))$$

$$= ([A\xi, \phi^{[2]}] + B\alpha^{[2]}, [B, \phi^{[2]}] + B\beta^{[1]}). \qquad (3.5)$$

Denote

$$V_0 = \mathfrak{A}(W) = \text{the image of W under } \mathfrak{A}.$$

By using these notations, we can rewrite the theorem 4 as follows:

Theorem 4': A system (3.3a) is quadratically equivalent to system (3.3b) if and only if

$$(\ f_1^{[2]}, g_1^{[1]}\ ) \in (\ f_2^{[2]}, g_2^{[1]}\ ) + V_0, \qquad (3.6)$$

i.e, $(\ f_1^{[2]}, g_1^{[1]}\ )$ and $(\ f_2^{[2]}, g_2^{[1]}\ )$ represent the same element in the quotient space V/V_0.

Remark 2: Theorem 4' means that there is a one-to-one correspondence between V/V_0 and the family of all equivalent classes.

Remark 3: A special case of Theorem 4' is that the system (2.8) is quadratically equivalent to a linear system if and only if

$$(\ f^{[2]}, g^{[1]}\ ) \in V_0.$$

Therefore, the elements of V_0 represents all the systems of the form (2.8) which are quadratically linearizable by (2.9).

The following theorem gives us a geometric necessary and sufficient condition for a system to be quadratically linearizable by the change of coordinates and feedback (2.9).

Theorem 5: Consider a system (2.8), let

$$X_r = ad^{\,r}_{A\xi+f^{[2]}(\xi)}(B+g^{[1]}(\xi)) \qquad 0 \le r \le n, \qquad (3.7a)$$

$$D^k = C^{\infty}\mathrm{Span}\,\{X_r, 0 \le r < k\}. \qquad (3.7b)$$

The system (2.8) is quadratically equivalent to the linear system

$$\dot{\xi} = A\xi + Bu \qquad (3.8)$$

if and only if D^k is first degree involutive for k=1,2,...,n-1, i.e, for any X and Y in D^k, we have

$$[\,X, Y\,] = \sum_{r=0}^{k-1} c_r X_r + O(\xi)^1. \qquad (3.9)$$

Proof: This theorem is a particular case of the theorem in Krener [12].

§4. Characteristic numbers

In §3, we defined an equivalent relation by the change of coordinates and feedback (2.9). In this section, we will answer the following question: How to determine whether or not two systems are quadratically equivalent without trying to solve the system of equations (3.4)? We will find a set of numbers associated to the system (2.8), called characteristic numbers, so that these numbers are invariant under the transformation (2.9) and two systems are quadratically equivalent if and only if they have the same characteristic numbers.

Let C be a row vector:

$$C=[1,0,0,\dots,0]. \qquad (4.1)$$

Definition 6: The characteristic numbers of the system (2.8) are:

$$a^{tr} = CA^{t-1}[\,X_{r-1}, X_{r-2}\,]\big|_{\xi=0} \qquad (4.2a)$$

where

$$\begin{array}{l} 2 \le r \le n-1 \\ 1 \le t \le n-r \end{array} \qquad (4.2b)$$

Lemma 1: The characteristic number a^{tr} is a linear map from V to R, i.e, a^{tr} is a linear function of $f^{[2]}(\xi)$ and $g^{[1]}(\xi)$.

Sketch of the proof: By computation, we can prove the following identity:

$$a^{tr} = L_{(-1)^{r-1}A^{r-1}B}\Big(\sum_{k=0}^{r-3} CA^{t-1}ad^{\,r-k-3}_{A\xi}[f^{[2]}(\xi), (-1)^k A^k B]$$

$$+CA^{t-1}ad^{\,r-2}_{A\xi}(g^{[1]}(\xi))\Big)$$

$$-L_{(-1)^{r-2}A^{r-2}B}\Big(\sum_{k=0}^{r-2} CA^{t-1}ad^{\,r-k-1}_{A\xi}[f^{[2]}(\xi), (-1)^k A^k B]+$$

$$CA^{t-1}ad^{\,r-1}_{A\xi}(g^{[1]}(\xi))\Big).$$

This implies that a^{tr} is a linear function of $f^{[2]}(\xi)$ and $g^{[1]}(\xi)$.

Lemma 2: A system of the form (2.8) is quadratically linearizable if and only if all the characteristic numbers are zero.

Sketch of the proof: Suppose a system of the form (2.8) is quadratically linearizable. It can be proved that the constant part of the vector fields in D^r is linearly generated by

$$\{\,B, AB, A^2B, \dots, A^{r-1}B\,\}.$$

From Theorem 5, we know that D^k is first degree involutive for k = 1,2,...,n-1. Therefore,

$$[\,X_{r-1}, X_{r-2}\,] = \sum_{i=1}^{r} c_i A^{i-1}B + O(\xi)^1. \qquad (4.3)$$

So,

$$a^{tr} = CA^{t-1}[\,X_{r-1}, X_{r-2}\,] = 0 \qquad \begin{array}{l} 2 \le r \le n-1 \\ 1 \le t \le n-r \end{array} \qquad (4.4)$$

because

$$CA^{t-1}A^{k-1}B = 0 \qquad \begin{array}{l} 1 \le k \le r \\ 1 \le t \le n-r. \end{array} \qquad (4.5)$$

On the other hand, suppose

$$a^{tr} = 0 \qquad \begin{array}{l} 2 \le r \le n-1 \\ 1 \le t \le n-r \end{array} \qquad (4.6)$$

i.e,

$$CA^{t-1}[\,X_{r-1}, X_{r-2}\,] = 0 \qquad \begin{array}{l} 2 \le r \le n-1 \\ 1 \le t \le n-r. \end{array} \qquad (4.7)$$

So,

$$[\,X_{r-1}, X_{r-2}\,] = \sum_{i=1}^{r} c_i A^{i-1}B + O(\xi)^1 \qquad (4.8)$$

This implies that the distribution D^k is first degree involutive for any $1 \le k \le n-1$. So, the system is quadratically linearizable. Q.E.D.

Theorem 6: Two systems of the form (2.8) are quadratically equivalent if and only if the corresponding characteristic numbers are equal.

Sketch of the proof: Consider two systems

$$\dot{\xi}_1 = A\xi_1 + Bu_1 + f_1^{[2]}(\xi_1) + g_1^{[1]}(\xi_1)u_1 + O(\xi_1, u_1)^3 \qquad (4.9a)$$

$$\dot{\xi}_2 = A\xi_2 + Bu_2 + f_2^{[2]}(\xi_2) + g_2^{[1]}(\xi_2)u_2 + O(\xi_2, u_2)^3 \qquad (4.9b)$$

Let a_1^{tr} and a_2^{tr} be the characteristic numbers of (4.9a) and (4.9b), respectively.

Suppose that (4.9a) and (4.9b) are quadratically equivalent. From Theorem 4' we know that

$$(\,f_1^{[2]}, g_1^{[1]}\,) \in (\,f_2^{[2]}, g_2^{[1]}\,) + V_0, \qquad (4.10)$$

i.e,

$$(\,f_1^{[2]}, g_1^{[1]}\,) = (\,f_2^{[2]}, g_2^{[1]}\,) + (\,f^{[2]}, g^{[1]}\,) \qquad (4.11a)$$

and

$$(\,f^{[2]}, g^{[1]}\,) \in V_0. \qquad (4.11b)$$

Let a^{tr} be the characteristic numbers of ($f^{[2]}$, $g^{[1]}$). Since the characteristic numbers are linear functions of $f^{[2]}$ and $g^{[1]}$ (Lemma 1), we have

$$a_1^{tr} = a_2^{tr} + a^{tr} \quad (4.12)$$

From Lemma 2 and (4.11b), we know that

$$a^{tr} = 0.$$

So,

$$a_1^{tr} = a_2^{tr} \qquad \begin{matrix} 2 \le r \le n \\ 1 \le t \le n-r. \end{matrix} \quad (4.13)$$

On the other hand, suppose that all the corresponding characteristic numbers are the same. Then

$$a_1^{tr} - a_2^{tr} = 0 \qquad \begin{matrix} 2 \le r \le n \\ 1 \le t \le n-r. \end{matrix} \quad (4.14)$$

So

$$(f_2^{[2]} - f_1^{[2]}, g_2^{[1]} - g_1^{[1]}) \in V_0 \quad (4.15)$$

Theorem 4' and (4.15) implies that the systems (4.9a) and (4.9b) are quadratically equivalent.

§5. The proofs of the theorems in §2.

The sketch of the proof of Theorem 2: Consider a special kind of $\tilde{f}^{[2]}(x)$:

$$\tilde{f}^{[2]}(x) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \tilde{f}_i(x) \\ \vdots \\ 0 \end{bmatrix} \quad \text{for some } 1 \le i \le n, \quad (5.1a)$$

here

$$\tilde{f}_i(x) = a_{ij} x_j^2 \qquad \text{for some } j \ge i+2. \quad (5.1b)$$

By a long computation we have

$$a^{tr} = CA^{t-1}[\, ad^{r-1}_{A\xi+\tilde{f}^{[2]}(\xi)}(B), ad^{r-2}_{A\xi+\tilde{f}^{[2]}(\xi)}(B) \,]$$

$$= \begin{cases} 2a_{ij} & r=n-j+2 \text{ and } t=i \\ 0 & \text{others} \end{cases} \quad . \quad (5.2)$$

As we know that any $\tilde{f}^{[2]}(x)$ of the form (2.10) is a linear combination of the vector fields given in (5.1). Given any system in the extended quadratic controller form (2.10), From (5.2) and Lemma 1, we can find the characteristic numbers:

$$a^{tr} = 2a_{t\ n-r+2}. \quad (5.3)$$

This implies that given a set of characteristic numbers, there exists one and only one system in the extended quadratic controller form which has the given characteristic numbers. Theorem 2 follows this fact and Theorem 6.

The sketch of the proof of Theorem 3. According to Theorem 2, it is sufficient to prove the result for the systems in the extended quadratic controller form. Let the dynamic feedback be

$$\begin{aligned} \dot{\omega}_1 &= \omega_2 \\ \dot{\omega}_2 &= \omega_3 \\ &\vdots \\ \dot{\omega}_{n-1} &= \tilde{v} \\ v &= \omega_1 + \gamma^{[2]}(x,\omega) \end{aligned} \quad (5.4)$$

where $\gamma^{[2]}$ is a homogeneous polynomial of second degree in (x,ω). The extended system is

$$\begin{bmatrix} \dot{x} \\ \dot{\omega} \end{bmatrix} = A_1 \begin{bmatrix} x \\ \omega \end{bmatrix} + B_1 \tilde{v} + \begin{bmatrix} \tilde{f}^{[2]}(x) \\ 0 \end{bmatrix} + \begin{bmatrix} B\gamma^{[2]}(x,\omega) \\ 0 \end{bmatrix} \tilde{v}. \quad (5.5)$$

Here, (A_1, B_1) is in the form of (2.6a) of dimension 2n-1. We define the change of coordinates as follows

$$\begin{cases} z_1 = x_1 & & (5.6a) \\ z_k = \text{linear and quadratic part of } \dot{z}_{k-1} & 2 \le k \le n \ . & (5.6b) \\ z_{n+p} = \omega_p & 1 \le p \le n-1 & (5.6c) \end{cases}$$

We can prove that

$$z_k = x_k + \psi_k(x,\omega_1,\ldots,\omega_{k-2}) \qquad 2 \le k \le n \quad (5.7)$$

where $\psi_k(x,\omega_1,\ldots,\omega_{k-2})$ is a homogeneous polynomial of second degree. Then

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 + O(z,\tilde{v})^3 \\ &\vdots \\ \dot{z}_{n-1} &= z_n + O(z,\tilde{v})^3 \end{aligned} \quad (5.8a)$$

and

$$\dot{z}_n = \dot{x}_n + \dot{\psi}_n(x,\omega_1,\ldots,\omega_{n-2})$$

$$= \omega_1 + \gamma^{[2]}(x,\omega) + \dot{\psi}_n(x,\omega_1,\ldots,\omega_{n-2}). \quad (5.8b)$$

Let

$$\gamma^{[2]} = \text{the quadratic part of } -\dot{\psi}_n(x,\omega_1,\ldots,\omega_{n-2}) \quad (5.9)$$

then

$$\dot{z}_n = \omega_1 = z_{n+1} + O(z,\tilde{v})^3 \quad (5.10)$$

Therefore, by the change of coordinates (5.6) and (5.9), the system (5.5) is transformed into

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 + O(z,\tilde{v})^3 \\ &\vdots \\ \dot{z}_{n-1} &= z_n + O(z,\tilde{v})^3 \\ \dot{z}_n &= z_{n+1} + O(z,\tilde{v})^3 \\ \dot{z}_{n+1} &= z_{n+2} \\ &\vdots \\ \dot{z}_{2n-1} &= \tilde{v} \end{aligned} \quad (5.11)$$

It is a linearly controllable system without quadratic terms. Theorem 3 is proved.

Remark 4: Sometimes, the dimension of the dynamic feedback used in theorem 2 can be less than n-1. Suppose a system is in extended quadratic controller form (2.10). Let

$$q = \max\{ j-i \,;\, a_{ij} \neq 0, j \geq i+2, 1 \leq i \leq n-2 \}. \quad (5.12)$$

To quadratically linearize the system, a q-dimensional dynamic feedback is good enough. The proof is almost the same as above except (5.7) is changed to

$$z_k = x_k + \psi_k(x,\omega_1,\ldots,\omega_{k-1-n+q}) \quad n-q+1\leq k\leq n. \quad (5.13)$$

Remark 5: From this proof, we can find that the dynamic feedback is chosen to be in Brunovsky form:

$$\begin{cases} \dot{\omega} = A\omega + Bv \\ u = \omega_1 + \gamma^{[2]}(x,\omega) \end{cases} \quad (5.14)$$

Furthermore, (5.6) and (5.7) implies that the change of coordinates in the extended state space is:

$$\begin{bmatrix} x \\ \omega \end{bmatrix} = \begin{bmatrix} z \\ \omega \end{bmatrix} + \begin{bmatrix} \phi^{[2]}(z,\omega_1,\ldots,\omega_{n-2}) \\ 0 \end{bmatrix} \quad (5.15)$$

Here, ω is not changed and the quadratic part is independent on ω_{n-1}.

The sketch of the proof of Corollary 1: By Theorem 1, there exists a linear change of coordinates and feedback which transforms (2.4) into a system such that the linear part of which is in Brunovsky form. By Theorem 3, we can find a dynamic feedback such that the extended system can be linearized to the second degree by a change of coordinates. This implies that the system:

$$\dot{\xi} = f(\xi) + g(\xi)u \quad (5.16)$$

is quadratically linearizable by a dynamic feedback. Corollary 1 is proved.

The sketch of the proof of Corollary 2: it can be proved by Remark 5, equation (3.4) and some computation.

In this paper, all the results are restricted to the single input nonlinear systems. In fact, similar results in the multi-input case are also correct and they will be given in another paper. The idea of finding quadratic normal forms and extending the state space were also successfully used in the problem of finding nonlinear observers.

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