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# GAUSSIAN RECIPROCAL PROCESSES AND SELF-ADJOINT STOCHASTIC DIFFERENTIAL EQUATIONS OF SECOND ORDER

ARTHUR J. KRENER, RUGGERO FREZZA and BERNARD C. LEVY

*Institute of Theoretical Dynamics, University of California, Davis, CA 95616, USA*

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We show that under suitable conditions the covariance of a Gaussian reciprocal process satisfies a self-adjoint linear differential equation of second order. We also give a revised definition of a linear stochastic differential equation of second order and necessary and sufficient conditions for the existence of solutions of such equations with Dirichlet boundary conditions. We close with a series of examples of the theory applied to the scalar stationary Gaussian Reciprocal processes which have been completely classified.

KEY WORDS: Gaussian reciprocal processes, second order stochastic differential equations.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

In 1932 Bernstein [1] formalized the concept of a reciprocal process following ideas of Schrödinger [2, 3]. Reciprocal processes generalize Markov Processes which had been given a rigorous treatment by Kolmogorov around the same time [4]. In 1956, P. Levy [5] defined the concept of a Markov field, but perhaps he should have used the term "reciprocal field" since a Markov field with a one dimensional parameter is a reciprocal rather than a Markov process.

A stochastic process $x(t)$ taking values in $\mathbb{R}^n$ (or $\mathbb{C}^n$) with time $t$ lying in a subset of $\mathbb{R}$ or $\mathbb{Z}$ is said to be reciprocal if for any two times $t_0, t_1$, the process interior to $[t_0, t_1]$ is independent of the process exterior to $[t_0, t_1]$ given $x(t_0)$ and $x(t_1)$. For a more rigorous definition we refer the reader to [6]. References [7–22] also discuss reciprocal process and [23–27] relate then to Quantum Mechanics following Schrödinger original motivation.

Throughout this paper we restrict our attention to reciprocal processes satisfying the following assumptions.

A1. The process $x(t)$ is a zero mean Gaussian reciprocal process defined in continuous time for $t \in [0, T]$ and taking values in $\mathbb{R}^{n\times 1}$ (or $\mathbb{C}^{n\times 1}$), with continuous sample paths almost surely.

A2. The covariance $R(t,s)$ of $x(t)$ is a $C^2$ function on $0 \leq s \leq t \leq T$, i.e., continuous limits of $R$ and its first and second partials exist on the boundary of this triangle.

A3. For $0 < t_0 < t_1 < T$ and small $t_1 - t_0$, the two time covariance matrix

$$\begin{bmatrix} R(t_0,t_0) & R(t_0,t_1) \\ R(t_1,t_0) & R(t_1,t_1) \end{bmatrix} \tag{1.1}$$

of $(x(t_0), x(t_1))$ is invertible.

A4. The covariance $R(t,s)$ has a full rank jump as $t$ passes through $s$, i.e. the $C^1$ matrix

$$Q(t) = -\frac{\partial R}{\partial t}(t^+,t) + \frac{\partial R}{\partial t}(t^-,t) \tag{1.2}$$

is invertible. Because $R(t,s)$ is a covariance, $Q(t)$ is always symmetric (Hermitian) and nonnegative definite. We refer to this as the assumption that $x(t)$ has *full rank noise* for reasons that will become apparent later on.

Under these assumptions we shall prove the following results many of which can be found in [20]. Similar results for discrete time Gaussian reciprocal process can be found in [21].

Let $\mathscr{D}$ be the space of all continuous and piecewise $C^2$ functions $\phi(t)$ from $[0,T]$ into $\mathbb{R}^{n\times 1}(\mathbb{C}^{n\times 1})$ which vanish at the endpoints, $\phi(0)=\phi(T)=0$, with the standard inner product

$$\langle \phi_1, \phi_2 \rangle = \int_0^T \phi_1^*(t)\phi_2(t)\,dt,$$

where superscript * denotes conjugate transpose.

THEOREM 1 *Assuming A1–A4 there exists $C^0$ matrices $F(t)$ and $G(t)$ such that*

$$-\frac{\partial^2 R}{\partial t^2}(t,s) + F(t)R(t,s) + G(t)\frac{\partial R}{\partial t}(t,s) = Q(t)\,\delta(t-s). \tag{1.3}$$

(This was shown in [20, 16]).

The differential operator

$$\mathscr{L} = -\frac{\partial^2}{\partial t^2} + G(t)\frac{\partial}{\partial t} + F(t) \tag{1.4}$$

has no pair of conjugate points in $[0,T]$ and $Q^{-1}\,\mathscr{L}$ is self-adjoint and positive definite on $\mathscr{D}$, i.e.,

$$\int_0^T \phi_1^*(t)Q^{-1}(t)\mathscr{L}(\phi_2)(t)\,dt = \int_0^T \phi_2^*(t)Q^{-1}(t)\mathscr{L}(\phi_1)(t)\,dt. \tag{1.5}$$

and this quantity is positive whenever $\phi_1 = \phi_2 \neq 0$.

We shall frequently invoke another assumption

A5. $Q(t), F(t)$ and $G(t)$ are $C^2$.

If A5 holds then $Q^{-1}\mathscr{L}$ (1.4) is self-adjoint iff

$$GQ+QG^*=2\dot{Q} \tag{1.6a}$$

and

$$FQ-QF^*=\ddot{Q}-Q\dot{G}^*-G\dot{Q}=\tfrac{1}{2}(\dot{G}Q-Q\dot{G}^*+\dot{Q}G^*-G\dot{Q}) \tag{1.6b}$$

Suppose $M(t)$ is $n\times n$ invertible matrix which is $C^2$ on $[0,T]$. If $x(t)$ satisfies A1–A5 then so does $y(t)=M(t)\,x(t)$. It is straightforward to verify that the covariance $R_y(t,s)$ of $y$ satisfies

$$\mathscr{L}_y R_y(t,s)=Q_y(t)\,\delta(t-s) \tag{1.7a}$$

where

$$\mathscr{L}_y=M(t)\mathscr{L}M^{-1}(t)=-\frac{\partial^2}{\partial t^2}+G_y(t)\frac{\partial}{\partial t}+F_y(t) \tag{1.7b}$$

and

$$R_y(t,s)=M(t)\,R(t,s)\,M^*(s) \tag{1.8a}$$

$$Q_y(t)=M(t)\,Q(t)\,M^*(t) \tag{1.8b}$$

$$G_y(t)=(M(t)G(t)+2\dot{M}(t))\,M^{-1}(t) \tag{1.8c}$$

$$F_y(t)=((M(t)\,F(t)+\ddot{M}(t))-G_y(t)\,\dot{M}(t))\,M^{-1}(t). \tag{1.8d}$$

A particularly useful change of coordinates is to choose an $n\times n$ matrix $H(t)$ such that

$$H(0)H^*(0)=Q(0) \tag{1.9a}$$

$$H(t)=\tfrac{1}{2}G(t)H(t) \tag{1.9b}$$

then by (1.6a)

$$H(t)H^*(t)=Q(t). \tag{1.9c}$$

Let

$$y(t)=H^{-1}(t)x(t) \tag{1.10a}$$

then

$$Q_y(t)=I \tag{1.10b}$$

$$G_y(t)=0 \tag{1.10c}$$

$$F_y(t)=F_y^*(t)=H^{-1}(t)(F(t)-\tfrac{1}{2}G(t)+\tfrac{1}{4}G^2(t))H(t). \tag{1.10d}$$

THEOREM 2 *Suppose $R(t,s)$ is the covariance of a zero mean Gaussian process which satisfies A2–A4 and also a second order differential equation of the form (1.3). Then $R(t,s)$ is the covariance of a reciprocal process and hence satisfies Theorem 1.*

We employ the following notation, for small $dt>0$

$$d^0x(t,dt)=\frac{x(t+dt)+x(t-dt)}{2} \tag{1.11a}$$

$$d^1x(t,dt)=\frac{x(t+dt)-x(t-dt)}{2} \tag{1.11b}$$

$$d^2x(t,dt)=x(t+dt)-2x(t)+x(t-dt) \tag{1.11c}$$

and frequently we suppress the argument $dt$ as in $d^2x(t)$.

THEOREM 3 [20] Suppose $x(t)$ satisfies A1 – A4 then there exists $n\times n$ matrices $F(t)$, $G(t)$, $Q(t)$, $V(t)$, $\sigma(t)$ (with $F(t)$, $G(t)$, $Q(t)$ as above) such that

$$E(d^2x(t)\,|\,d^0x(t)=x, d^1x(t)=v\,dt)=(F(t)x+G(t)v)\,dt^2+o(dt)^2 \tag{1.12a}$$

$$E(d^2x(t)\,d^2x^*(t)\,|\,d^0x(t)=x, d^1x(t)=v\,dt)=2Q(t)\,dt+o(dt)^2 \tag{1.12b}$$

$$E(d^1x(t)\,|\,d^0x(t)=x)=V(t)x\,dt+o(dt)^1 \tag{1.12c}$$

$$E(d^1x(t)\cdot d^1x^*(t)\,|\,d^0x(t)=x)=\tfrac{1}{2}Q(t)\,dt+(V(t)xx^*V^*(t)-\sigma(t))\,dt^2+o(dt)^2. \tag{1.12d}$$

We use $E(X\,|\,Y)$ to denote conditional mean of $X$ given $Y$. The symbols $o(dt)$ and $o(dt)^2$ denotes deterministic functions of $t$, $dt$, $x$ and $v$ which go to zero faster than $dt$ and $dt^2$. The symbols $O(dt)$ and $O(dt)^2$ denote similar functions going to zero like $dt$ and $dt^2$.

The matrix $Q(t)$ is given by (1.2), the other matrices are given by

$$G(t)=-\left(\frac{\partial^2 R}{\partial t^2}(t^+,t)-\frac{\partial^2 R}{\partial t^2}(t^-,t)\right)Q^{-1}(t)$$

$$F(t)=\left(\frac{\partial^2 R}{\partial t^2}(t^+,t)-G(t)\frac{\partial R}{\partial t}(t^+,t)\right)R^{-1}(t,t)$$

$$V(t)=\tfrac{1}{2}\left(\frac{\partial R}{\partial t}(t^+,t)+\frac{\partial R}{\partial t}(t^-,t)\right)R^{-1}(t,t)$$

$$\sigma(t)=-\tfrac{1}{2}\left(\frac{\partial^2 R}{\partial t\,\partial s}(t^+,t)+\frac{\partial^2 R}{\partial t\,\partial s}(t^-,t)\right)+V(t)R(t,t)V^*(t).$$

COROLLARY 4 [20] *Suppose $x(t)$ satisfies Theorem 3 then*

$$E(d^2x(t)\,|\,d^0x(t)=x)=(F(t)+G(t)V(t))x\,dt^2+o(dt)^2 \tag{1.13a}$$

$$E(d^2x(t)d^2x^*(t)\,|\,d^0x(t)=x)=2Q(t)\,dt+o(dt)^2 \tag{1.13b}$$

$$E(d^2x(t)d^1x^*(t)\,|\,d^0x(t)=x)=G(t)Q(t)\,dt^2 + ((F(t)+G(t)V(t))xx^*(t)V^*(t)-G(t)\sigma(t))\,dt^3+o(dt)^3. \tag{1.13c}$$

Suppose $x(t)$ is a reciprocal process satisfying A1–A4, we consider the residual or noise process $z(t,dt)$ defined for $dt>0$ by

$$Q(t)z(t,dt)=-d^2x(t)+F(t)d^0x(t)\,dt^2+G(t)\,d^1x(t)\,dt. \tag{1.14}$$

We shall compute the covariance $R_z(t_k,t_l)$ of $z(t)$ and the crossvariance $R_{z,x}(t_k,s)$ of $z(t)$ and $x(s)$ on discrete time steps $t_k=k\,dt$.

THEOREM 5 *Assuming A1–A5 then*

$$R_z(t_k,t_k)=2Q^{-1}(t_k)(I\,dt-K(t_k)\,dt^3)+o(dt)^3$$
$$=2(I\,dt-K^*(t_k)\,dt^3)Q^{-1}(t_k)+o(dt)^3. \tag{1.15a}$$

$$R_z(t_k,t_{k+1})=-Q^{-1}(t_k)(I\,dt-\tfrac{1}{2}G(t_k)\,dt^2-\tfrac{1}{2}F(t_k)\,dt^3-K(t_k)\,dt^3)+o(dt)^3$$
$$=-(I\,dt+\tfrac{1}{2}G^*(t_{k+1})\,dt^2-\tfrac{1}{2}F^*(t_{k+1})\,dt^3$$
$$-K^*(t_{k+1})\,dt^3)Q^{-1}(t_{k+1})+o(dt)^3 \tag{1.15b}$$

$$R_z(t_k,t_l)=o(dt)^4 \text{ if } |k-l|>1 \tag{1.15c}$$

*and*

$$R_{z,x}(t_k,t_k)=(I\,dt-K^*(t_k)\,dt^3)+o(dt)^3 \tag{1.16a}$$

$$R_{z,x}(t_k, s) = o(dt)^3 \quad \text{if } t_k \neq s \tag{1.16b}$$

*where*

$$K(t) = \tfrac{1}{3}F(t) + \tfrac{1}{12}G(t)G(t) - \tfrac{1}{6}\dot{G}(t). \tag{1.17}$$

*This theorem explains why A4 is called the full rank noise assumption.*

One can also consider the limit of $z(t, dt)/dt^2$ as $dt \to 0$, this is a generalized process which we denote by $\xi(t)$. Formally $\xi(t)$ is defined as

$$Q(t)\xi(t)\,dt^2 = -d^2x(t) + F(t)\,d^0x(t)\,dt^2 + G(t)\,d^1x(t)\,dt \tag{1.18a}$$

or

$$Q(t)\xi(t) = \mathscr{L}(x)(t) = -\frac{d^2x(t)}{dt^2} + F(t)x(t) + G(t)\frac{dx}{dt}(t). \tag{1.18b}$$

To give rigorous meaning to $\xi(t)$ as a generalized process we must define its integral against elements of $\mathscr{D}$. For $\phi \in \mathscr{D}$, we wish to define

$$\int_0^T \phi^*(t)\,\xi(t)\,dt.$$

Proceeding formally we have

$$\int_0^T \phi^*(t)\,\xi(t)\,dt = \int_0^T \phi^*(t)\,Q^{-1}(t)\,\mathscr{L}(x)(t)\,dt$$

$$= \int_0^T (\mathscr{L}(\phi)(t))^* Q^{-1}(t)\,x(t)\,dt.$$

but since $x(t)$ may not vanish at $t=0$ and $T$, the neglected boundary terms may not be zero. Let $\hat{x}(t)$ be the conditional expectation of $x(t)$ given $x(0)$ and $x(T)$ and let $\hat{R}(t,s)$ be the covariance of $\hat{x}(t)$. If the two time covariance of $x(0)$ and $x(T)$ is invertible then

$$\hat{x}(t) = [R(t,0)\;R(t,T)]\begin{pmatrix} R(0,0) & R(0,T) \\ R(T,0) & R(T,T)\end{pmatrix}^{-1}\begin{pmatrix} x(0) \\ x(T)\end{pmatrix}. \tag{1.19a}$$

and

$$\hat{R}(t,s) = [R(t,0)\;R(t,T)]\begin{pmatrix} R(0,0) & R(0,T) \\ R(T,0) & R(T,T)\end{pmatrix}^{-1}\begin{pmatrix} R(0,s) \\ R(T,s)\end{pmatrix}. \tag{1.19b}$$

If the two time covariance is not invertible then the generalized inverse must be used in the above. In any case, the sample paths of $\hat{x}(t)$ are $C^2$ almost surely and

$$\mathscr{L}(\hat{x})(t)=0 \tag{1.20a}$$

$$\mathscr{L}\hat{R}(t,s)=0. \tag{1.20b}$$

We define

$$\tilde{x}(t)=x(t)-\hat{x}(t) \tag{1.21a}$$

and

$$\tilde{R}(t,s)=R(t,s)-\hat{R}(t,s), \tag{1.21b}$$

then $\tilde{R}(t,s)$ is the covariance of the reciprocal, in fact, Markov process $\tilde{x}(t)$. It is straightforward to verify that if $x(t)$ satisfies A1–A5 then $\tilde{x}(t)$ does and $\tilde{R}(t,s)$ satisfies the same differential equation (1.3) as $R(t,s)$. Finally $\tilde{x}(t)$ vanishes at $t=0$ or $T$ so we can rigorously define $\xi(t)$ as the generalized stochastic process satisfying

$$\int_0^T \phi^*(t)\,\xi(t)\,dt=\int_0^T (\mathscr{L}(\phi)(t))^* Q^{-1}(t)\tilde{x}(t)\,dt. \tag{1.22}$$

There is an alternative way of defining $\xi(t)$ using a standard Wiener process and a factorization of the operator $Q^{-1}\,\mathscr{L}$ which we discuss at the end of Section 2. This definition leads to an infinitesimal analog of Theorem 5.

THEOREM 6 *Assuming A1–A5,*

$$R_\xi(t,s)=\delta(t-s)Q^{-1}(t)\,\mathscr{L} \tag{1.23a}$$

$$R_{\xi,x}(t,s)=\delta(t-s)\,I. \tag{1.23b}$$

In other words if we define Gaussian random variables $Y_1$ and $Y_2$ using test functions $\phi_1$ and $\phi_2\in\mathscr{D}$ by

$$Y_i=\int_0^T \phi_i^*(t)\xi(t)\,dt \tag{1.24}$$

then

$$E(Y_1\,Y_2)=\int_0^T \phi_1^*(t)Q^{-1}(t)\,\mathscr{L}(\phi_2)(t)\,dt \tag{1.25a}$$

$$E(Y_1 x^*(s)) = \phi_1^*(s). \tag{1.25b}$$

An interesting consequence of (1.25b) is that if $I$ and $J$ are two disjoint subintervals of $[0, T]$ and if the support of $\phi_1(\cdot)$ is concentrated on $I$, then for $s \in J$, $E[Y_1 x^*(s)] = 0$. In other words, the random variables spanned by $\xi(\cdot)$ and $x(\cdot)$ on two disjoint subintervals are orthogonal. This shows that $\xi(t)$ is the conjugate process to $x(t)$, we refer the reader to [29–31] for more information on conjugate processes.

The process $z(t, dt)/dt^2$ converges weakly to $\xi(t)$ as $dt \to 0$ in the following sense.

THEOREM 7 *Assuming A1–A5 and $z(t, dt), \xi(t)$ as above. Let $\phi \in \mathscr{D}$ and $t_k = k\,dt$, $T = (N+1)\,dt$ then as $dt \to 0$,*

$$\sum_{k=1}^{N} \phi^*(t_k) \frac{z(t_k, dt)}{dt^2}\, dt \to \int_0^T \phi^*(t)\, \xi(t)\, dt.$$

*weakly in the Hilbert space spanned by $\{x(t): 0 \leq t \leq T\}$.*

In [20], the first author introduced the concept of a stochastic differential equation of second order. We give a modified definition here.

DEFINITION A process $x(t)$ satisfying A1 is said to satisfy the linear second order stochastic differential equation

$$-d^2x + F(t)\, d^0x\, dt^2 + G(t)\, d^1x\, dt = Q(t)\, \xi(t)\, dt^2 \tag{1.26}$$

on $[0, T]$ if $x(t)$ is reciprocal and for all $t \in (0, T)$,

$$E(d^2x(t) \mid d^0x(t) = x, d^1x(t) = v\,dt)(F(t)x + G(t)v)\, dt^2 + o(dt)^2 \tag{1.27a}$$

$$E(d^2x(t)\, d^2x^*(t) \mid d^0x(t) = x, d^1x(t) = v\,dt)$$

$$= 2Q(t)\, dt + o(dt)^2. \tag{1.27b}$$

Notice that in this definition we do not explicitly define $\xi(t)$. The reason why this is not necessary is given by the following theorem.

THEOREM 8 *Suppose $x(t)$ satisfies A1, A2 and the linear second order stochastic differential equation (1.26) on $[0, T]$ where the coefficients $F(t)$, $G(t)$ and $Q(t)$ are $C^2$ and $Q(t)$ is invertible. Then the covariance $R(t, s)$ of $x(t)$ satisfies (1.3) where $Q^{-1}\mathscr{L}$ is a self-adjoint positive definite operator on $[0, T]$ and the residual or noise process $\xi(t)$ defined by (1.18) (or more precisely (1.22)) has the covariance structure (1.23).*

The above definition is the second order analog of Feller's postulates for a Markov diffusion [33] at least for the linear-Gaussian case. Feller's postulates could be thought of as the definition of a first order stochastic differential equation but since the introduction of Ito's stochastic integration, it is standard to use the corresponding integral equation as the definition [34].

Since we don't specify the noise process $\xi(t)$, the above definition corresponds to a weak solution to (1.26). For a given noise process $\xi(t)$ with covariance satisfying (1.23a), one could define $x(t)$ to be a strong solution of (1.26) if it is weak solution and $\xi(t)$ is the residual satisfying (1.22).

In [20], the first author used different notation for and gave a different definition of a linear stochastic differential equation of second order. He used the notation

$$d^2x = F(t)x\,dt^2 + G(t)\,dx\,dt + H(t)\,d^2w. \tag{1.28}$$

We have changed notations to (1.26) because $Q(t)\xi(t)\,dt^2$ is the true noise process. If $H(t)$ and $Q(t)$ are related by (1.9) then the moments of $Q(t)\xi(t)\,dt^2$ agree with those of $H(t)\,d^2w$ to order $dt^2$ but not to higher powers of $dt$. The old notation should be thought of as a mnemonic for the conditional moments (1.27) of $d^2x$. If one has the true noise process $\xi(t)$ it is possible to integrate the Eq. (1.26) and thereby construct $x(t)$ satisfying (1.27). Given a Brownian motion $w(t)$, it is not possible to integrate (1.28) and construct a reciprocal process satisfying (1.27). On the other hand the mnemonic (1.28) does contain enough information to specify the right sides of (1.27) and hence the solutions.

The previous definition [20] essentially required that (1.12c, d) and (1.13) hold under the weaker conditions on $d^0x(t) = x$, rather than (1.27) under the stronger conditioning on $d^0x(t) = x. d^1x(t) = v\,dt$. In the Gaussian case these are equivalent, but we believe that in the non-Gaussian case, the stronger conditioning is the appropriate one [28].

Note that Theorems 1 and 3 assert that if $x(t)$ satisfies A1–A4 then $x(t)$ satisfies a linear stochastic differential equation of second order. Theorem 8 is essentially the converse to this statement.

Definition A process $x(t)$ is a solution of the linear stochastic second order boundary value problem

$$-d^2x + F(t)\,d^0x\,dt^2 + G(t)\,d^1x\,dt = Q(t)\xi(t)\,dt^2 \tag{1.29a}$$

$$x(0) = x^0,\ x(T) = x^T \tag{1.29b}$$

where $x^0, x^T$ are jointly Gaussian if $x(t)$ satisfies the second order stochastic differential equation (1.29a) and the boundary conditions (1.29b).

Theorem 9 *The linear stochastic second order boundary value problem (1.29) with $C^2$ coefficients $F(t)$, $G(t)$ and $Q(t)$ has a solution satisfying A1–A5 if the operator $Q^{-1}\mathscr{L}$ is self-adjoint and positive definite on $\mathscr{D}$. The solution is unique up to law.*

We shall show that the solution can be constructed as

$$x(t) = \Psi(t)\begin{pmatrix} x^0 \\ x^T \end{pmatrix} + \int_0^T \Gamma(t,s)\,\xi(s)\,ds \tag{1.30}$$

where the boundary value transition matrix $\Psi(t)$ is an $n \times 2n$ matrix function satisfying

$$-\frac{d^2\Psi}{dt^2}(t)+F(t)\Psi(t)+G(t)\frac{d\Psi}{dt}(t)=0 \tag{1.31a}$$

$$\Psi(0)=(I\,0),\ \Psi(T)=(0\,I). \tag{1.31b}$$

$\Gamma(t,s)$ is the Green's function of (1.29) satisfying

$$\mathscr{L}\Gamma(t,s)=Q(t)\delta(t-s) \tag{1.32a}$$

$$\Gamma(0,s)=0,\ \Gamma(T,s)=0 \tag{1.32b}$$

$$\Gamma(t,s)=\Gamma^*(s,t) \tag{1.32c}$$

and $\xi(t)$ is a generalized process constructed from a Markov solution to (1.26). The integral is well defined because for each fixed $t$, the function $s \mapsto \Gamma(t,s)$ is a test function in $\mathscr{D}$. The Green's function $\Gamma(t,s)$ equals $\tilde{R}(t,s)$ (1.21b) because $\tilde{R}(t,s)$ satisfies (1.3) and is zero when $t=0$ or $T$.

## 2. PROOFS

*Proof of Theorem 1* Following [20, 16] we start by showing that $R(t,s)$ satisfies a linear differential equation of second order. Given $t \in [t_0, t_1] \subseteq [0,T]$ and $s \in [0,t_0] \cup [t_1,T]$, the reciprocal property implies that

$$R(t,s)=[R(t,t_0)\,R(t,t_1)]\begin{bmatrix} R(t_0,t_0) & R(t_0,t_1) \\ R(t_1,t_0) & R(t_1,t_1)\end{bmatrix}^{-1}\begin{bmatrix} R(t_0,s) \\ R(t_1,s)\end{bmatrix} \tag{2.1}$$

whenever the inverse exists. If the inverse does not exist, the (2.1) must hold with the generalized inverse replacing the inverse. Let $t_0=t-\sigma, t_1=t+\sigma$ for $|t-s|>\sigma>0$ then by A3 for small $\sigma$ the inverse exists and

$$R(t,s)=[K_0(t,\sigma)\,K_1(t,\sigma)]\begin{bmatrix} R(t-\sigma,s) \\ R(t+\sigma,s)\end{bmatrix} \tag{2.2}$$

where $K_0, K_1$ are defined by

$$[K_0(t,\sigma)K_1(t,\sigma)]\begin{bmatrix} R(t-\sigma,t-\sigma) & R(t-\sigma,t+\sigma) \\ R(t+\sigma,t-\sigma) & R(t+\sigma,t-\sigma)\end{bmatrix}=[R(t,t-\sigma)\,R(t,t+\sigma)] \tag{2.3}$$

By A2, $K_i(t,\sigma)$ are $C^2$ for $\sigma>0$. In [16] it is shown using A4 that continuous limits of $K_i$ and its first and second derivatives exist as $\sigma \to 0$. Clearly $K_0(t,0)+K_1(t,0)=I$. Therefore we can differentiate (2.2) twice with respect to $\sigma$ and evaluate at $\sigma=0$ to obtain

$$0=\frac{\partial^2 R}{\partial t^2}(t,s)+2\left(\frac{\partial K_0}{\partial \sigma}(t,0)-\frac{\partial K_1}{\partial \sigma}(t,0)\right)\frac{\partial R}{\partial t}(t,s)$$

$$+\left(\frac{\partial^2 K_0}{\partial \sigma^2}(t,0)+\frac{\partial^2 K_1}{\partial \sigma^2}(t,0)\right)R(t,s).$$

Hence $R(t,s)$ satisfies a linear differential equation of second order for all $t \neq s$,

$$-\frac{\partial^2 R}{\partial t^2}(t,s)+F(t)R(t,s)+G(t)\frac{\partial R}{\partial t}(t,s)=0. \tag{2.4}$$

This and Assumption A4 are equivalent to (1.3).

Since $R(t,s)$ is a covariance

$$R(\tau,\sigma)=R^*(\sigma,\tau) \tag{2.5a}$$

$$\frac{\partial R}{\partial t}(\tau,\sigma)=\frac{\partial R^*}{\partial s}(\sigma,\tau) \tag{2.5b}$$

$$\frac{\partial^2 R}{\partial t^2}(\tau,\sigma)=\frac{\partial^2 R^*}{\partial s^2}(\tau,\sigma) \tag{2.5c}$$

so we also have shown for all $t \neq s$,

$$-\frac{\partial^2 R^*}{\partial s^2}(t,s)+F(s)R^*(t,s)+G(s)\frac{\partial R^*}{\partial s}(t,s)=0. \tag{2.6}$$

We subtract (2.6) from (2.4) and take the limit as $s \to t^-$ to obtain

$$G(t)=\left(\frac{\partial^2 R}{\partial t^2}(t^+,t)-\frac{\partial^2 R^*}{\partial s^2}(t^+,t)\right)\left(\frac{\partial R}{\partial t}(t^+,t)-\frac{\partial R^*}{\partial s}(t^+,t)\right)^{-1}$$

$$=-\left(\frac{\partial^2 R}{\partial t^2}(t^+,t)-\frac{\partial^2 R^*}{\partial s^2}(t^+,t)\right)Q^{-1}(t)$$

$$=-\left(\frac{\partial^2 R}{\partial t^2}(t^+,t)-\frac{\partial^2 R}{\partial t^2}(t^+,t)\right)Q^{-1}(t) \tag{2.7a}$$

and

$$F(t)=\left(\frac{\partial^2 R}{\partial t^2}(t^+,t)-G(t)\frac{\partial R}{\partial t}(t^+,t)\right)R(t,t)^{-1}. \tag{2.7b}$$

Note that $F(t)$ and $G(t)$ are $C^0$ by A2.

Now we show that differential operator $Q^{-1}\mathscr{L}$ is self-adjoint. Recall the process $\tilde{x}(t)$ defined by (1.21a). As we noted its covariance $\tilde{R}(t,s)$ satisfies

$$Q^{-1}(t)\mathscr{L}\tilde{R}(t,s)=\delta(t-s) \tag{2.8a}$$

$$\tilde{R}(0,s)=\tilde{R}(T,s)=0. \tag{2.8b}$$

This implies that $\tilde{R}(t,s)$ is the unique Green's function of the boundary value problem (1.31) ([35, Theorem 6–4.1]).

But $\tilde{R}(t,s)$ is a covariance hence it is self-adjoint $\tilde{R}(t,s)=\tilde{R}^*(s,t)$ and nonnegative definite,

$$\int_0^T\int_0^T \phi^*(t)\tilde{R}(t,s)\phi^*(s)\,dt\,ds \geqq 0.$$

This implies that the differential operator $Q^{-1}\mathscr{L}$ is self-adjoint and nonnegative definite.

The existence of a Green's function for (1.31) implies that this problem is well-posed, in other words, zero is not an eigenvalue of $Q^{-1}\mathscr{L}$. Therefore $Q^{-1}\mathscr{L}$ is positive definite. The smallest eigenvalue of $Q^{-1}\mathscr{L}$ on $[t_0,t_1]$ is the infimum of

$$\int_{t_0}^{t_1} \phi^*(t)Q^{-1}(t)\mathscr{L}(\phi)(t)\,dt$$

over all all $\phi\in\mathscr{D}$ satisfying $\phi(t_0)=\phi(t_1)=0$. Hence the smallest eigenvalue of $Q^{-1}\mathscr{L}$ on $[t_0,t_1]\subseteq[0,T]$ is greater than or equal to the smallest eigenvalue of $Q^{-1}\mathscr{L}$ on $[0,T]$. It follows that $Q^{-1}\mathscr{L}$ is positive definite on any $[t_0,t_1]\subseteq[0,T]$ hence $\mathscr{L}$ has no pair of conjugate points in $[0,T]$.

*Proof of Theorem 2* Suppose $x(t)$ is a zero mean Gaussian process satisfying A2–A4 with covariance $R(t,s)$ satisfying the differential equation (1.3). We define $x(t)$ and $\tilde{R}(t,s)$ as before and note that $\tilde{x}(t)$ satisfies A2–A4 and $\tilde{R}(t,s)$ also satisfies (1.3) with zero boundary conditions, (2.8). Hence $\tilde{R}(t,s)$ is the Green's function of (1.29). Following the proof of Theorem 1 we see that $Q^{-1}\mathscr{L}$ is self-adjoint operator on $[0,T]$ and $\mathscr{L}$ has no conjugate points in $[0,T]$.

A process satisfying A2–A4 is reciprocal iff for all $t\in[t_0,t_1]\in[0,T]$ and $s\in[0,t_0]\cup[t_1,T]$

$$R(t,s)=[R(t,t_0)\,R(t,t_1)]\begin{bmatrix}R(t_0,t_0) & R(t_0,t_1)\\ R(t_1,t_0) & R(t_1,t_1)\end{bmatrix}^{-1}\begin{bmatrix}R(t_0,s)\\ R(t_1,s)\end{bmatrix} \tag{2.9}$$

If the above inverse does not exist, it should be replaced by the generalized inverse. For fixed $t_0$, $t_1$ and $s$, let $X(t)$ denote the right side of (2.9). It is straightforward to verify that $X(t)$ satisfies the two point boundary value problem

$$-\frac{d^2X}{dt^2}(t)+F(t)X(t)+G(t)\frac{dX}{dt}(t)=0 \tag{2.10a}$$

$$X(t_0)=R(t_0,s) \qquad X(t_1)=R(t_1,s). \tag{2.10b}$$

Since $R(t,s)$ satisfies (1.3) it also satisfies (2.10). $\mathscr{L}$ has no conjugate points in $[0,T]$ so the solution of (2.10) is unique and (2.9) holds. □

*Proofs of Theorem 3 and Corollary 4* We modify the proof of [20]. We define partial difference operators

$$\partial_t^0 R(t,s)=\frac{R(t+dt,s)+R(t-dt,s)}{2} \tag{2.11a}$$

$$\partial_t^1 R(t,s)=\frac{R(t+dt,s)-R(t-dt,s)}{2} \tag{2.11b}$$

$$\partial_t^2 R(t,s)=R(t+dt,s)-2R(t,s)+R(t-dt,s). \tag{2.11c}$$

Suppose

$$E(d^2x(t)\,|\,d^0x(t)=x,d^1x(t)=v\,dt)=K_0(t,dt)x+K_1(t,dt)v\,dt \tag{2.12a}$$

where $K_0, K_1$ are uniquely characterized by the pair of equations

$$E(d^2x(t)x^*(t\pm dt))=K_0(t,dt)E(d^0x(t)x^*(t\pm dt))+K_1(t,dt)E(d^1x(t)x^*(t\pm dt)) \tag{2.12b}$$

or equivalently

$$\partial_t^2 R(t,t\pm dt)=K_0(t,dt)\partial_t^0 R(t,t\pm dt)+K_1(t,dt)\partial_t^1 R(t,t\pm dt). \tag{2.12c}$$

We expand both sides in Taylor series at $(t,t\pm dt)$

$$\frac{\partial^2R}{\partial t^2}(t,t\pm dt)\,dt^2+O(dt)^4=(F(t)R(t,t\pm dt)+G(t)\frac{\partial R}{\partial t}(t,t\pm dt))\,dt+O(dt)^4$$

$$=K_0(t,dt)(R(t,t\pm dt)+O(dt)^2)$$

$$+K_1(t,dt)\left(\frac{\partial R}{\partial t}(t,t\pm dt)\,dt+O(dt)^3\right). \tag{2.13}$$

By A4 these equations have a unique solution.

$$K_0(t,dt)=F(t)\,dt^2+O(dt)^4 \tag{2.14a}$$

$$K_1(t,dt)=G(t)\,dt+O(dt)^3, \tag{2.14b}$$

and (1.12a) is proven.

Let

$$d^2\tilde{x}(t\,|\,t\pm dt)=d^2x(t)-E(d^2x(t)\,|\,d^0x(t)=x,d^1x(t)=v\,dt) \tag{2.15a}$$

then by (1.12a),

$$\begin{aligned}&E(d^2x(t)\,d^2x^*(t)\,|\,d^0x(t)=x,d^1x(t)=v\,dt)\\&\quad=E(d^2\tilde{x}(t|t\pm dt)d^1\tilde{x}^*(t|t\pm dt)\,|\,d^0x(t)=x,d^1x(t)=v\,dt)+o(dt)^2.\end{aligned} \tag{2.15b}$$

Since the variables are Gaussian, the first term on the right side is independent of the conditioned values of $d^0x(t)$ and $d^1x(t)$, so

$$\begin{aligned}&E(d^2x(t)\,d^2x^*(t)\,|\,d^0x(t)=x,d^1x(t)=v\,dt)=E(d^2x(t)\,d^2x^*(t))+o(dt)^2\\
&=\partial_t^2R(t,t+dt)-2\partial_t^2R(t,t)+\partial_t^2R(t,t-dt)+o(dt)^2\\
&=\left(\frac{\partial^2R}{\partial t^2}(t,t+dt)+\frac{\partial^2R}{\partial t^2}(t,t-dt)\right)dt^2\\
&\quad-2((R(t+dt,t)-R(t,t))+(R(t-dt,t)-R(t,t)))+o(dt)^2\\
&=-2\left(\frac{\partial R}{\partial t}(t^+,t)-\frac{\partial R}{\partial t}(t^-,t)\right)dt\\
&\quad+\left(\frac{\partial^2R}{\partial t^2}(t,t+dt)+\frac{\partial^2R}{\partial t^2}(t,t-dt)-\frac{\partial^2R}{\partial t^2}(t^+,t)-\frac{\partial^2R}{\partial t^2}(t^-,t)\right)dt^2+o(dt)^2\\
&=2Q(t)\,dt+o(dt)^2.\end{aligned} \tag{2.16}$$

We have proved (1.12b). The other two assertions are proven in a similar fashion.

The corollary follows immediately from (1.12) using the nested property of conditional expectations,

$$E(\cdot\,|\,d^0x(t)=x)=E(E(\cdot\,|\,d^0x(t)=x,d^1x(t)=v\,dt)\,|\,d^0x(t)=x). \quad \square$$

*Proof of Theorem 5* If $t_k \neq s$ then

$$Q(t_k)E(z(t_k)x^*(s))=\partial_t^2 R(t_k,s)-F(t_k)\,\partial_t^0 R(t_k,s)\,dt^2-G(t_k)\partial_t^1 R(t_k,s)\,dt.$$

We expand in Taylor series around $t_k, s$ to obtain

$$Q(t_k)E(z(t_k)x^*(s))=\left(\tfrac{1}{12}\,\frac{\partial^4 R}{\partial t^4}(t_k,s)-\tfrac{1}{2}F(t_k)\,\frac{\partial^2 R}{\partial t^2}(t_k,s)+\tfrac{1}{6}G(t_k)\,\frac{\partial^3 R}{\partial t^3}(t_k,s)\right)dt^4+o(dt)^4.$$

If $t_k = s$

$$Q(t_k)E(z(t_k)x^*(t_k))=-\partial_t^2 R(t_k,t_k)+F(t_k)\,\partial_t^0 R(t_k,t_k)+G(t_k)\partial_t^1 R(t_k,t_k)$$

$$=(-R(t_{k+1},t_k)+R(t_k,t_k)+\tfrac{1}{2}(F(t_k)+G(t_k))\,R(t_{k+1},t_k))$$

$$+(-R(t_{k-1},t_k)+R(t_k,t_k)+\tfrac{1}{2}(F(t_k)-G(t_k))\,R(t_{k-1},t_k)).$$

We expand in Taylor series around $(t_k^+,t_k)$ and $(t_k^-,t_k)$ to obtain

$$Q(t_k)\,E(z(t_k)x^*(t_k))=Q(t_k)\,dt+K(t_k)\,Q(t_k)\,dt^3+o(dt)^3.$$

The self-adjoint relations (1.6) imply

$$K(t_k)Q(t_k)=Q(t_k)K^*(t_k).$$

Hence we have proven (1.16). The other equations follow immediately from this. $\square$

*Proof of Theorem 6* Let $Y_1$, $Y_2$ be defined by (1.23), then

$$E(Y_1Y_2)=E\left[\int_0^T(\mathscr{L}(\phi_1)(t))^*Q^{-1}(t)\tilde{x}(t)\,dt\left(\int_0^T(\mathscr{L}(\phi_2(s))^*Q^{-1}(s)\tilde{x}(s)\,ds\right)^*\right]$$

$$=\int_0^T\int_0^T(\mathscr{L}(\phi_1)(t))^*Q^{-1}(t)\,\tilde{R}(t,s)\,Q^{-1}(s)\,\mathscr{L}(\phi_2)(s)\,dt\,ds$$

$$\times\int_0^T\int_0^T\phi_1^*(t)Q^{-1}(t)\mathscr{L}_t\tilde{R}(t,s)Q^{-1}(s)\mathscr{L}(\phi_2)(s)\,dt\,ds$$

$$\times \int_0^T \int_0^T \phi_1^*(t)\,\delta(t-s)\,Q^{-1}(s)\,\mathscr{L}(\phi_2)(s)\,dt\,ds$$

and we have proven (1.24a). Next

$$E(Y_1 x^*(s)) = E\int_0^T (\mathscr{L}(\phi_1)(t))^* Q^{-1}(t)\,\tilde{x}(t) x^*(s)\,dt$$

$$= E\int_0^T (\mathscr{L}(\phi)(t))^* Q^{-1}(t)\,\tilde{x}(t)\tilde{x}^*(s)\,dt$$

because $x(t) = \hat{x}(t) + \tilde{x}(t)$ and $E(\tilde{x}(t)\hat{x}^*(s)) = 0$, then (1.24b) follows. □

*Proof of Theorem 7* Consider the Hilbert space $\mathscr{H}$ of Gaussian random variables which is the closed span of $\{x_i(t): i = 1, \ldots, n, t \in [0, T]\}$.

For any test function $\phi(t) \in \mathscr{D}$, clearly

$$\sum_{k=1}^{N} \phi^*(t_k)\,\frac{z(t_k, dt)}{dt^2}\,dt$$

and

$$\int_0^T \phi^*(t)\,\xi(t)\,dt$$

are in $\mathscr{H}$. To show that the former converges to the latter in $\mathscr{H}$, it suffices to show that for any $s \in [0,T]$

$$E\left(\left(\sum_{k=1}^{N} \phi^*(t_k)\,\frac{z(t_k, dt)}{dt^2}\,dt\right) x(s)\right)$$

$$\to E\left(\int_0^T \phi^*(t)\,\xi(t)\,dt\,x(s)\right)$$

as $N \to \infty$. This follows immediately from Theorems 5 and 6. □

*Proof of Theorem 8* By the reciprocal property for $|t-s| > dt$, we have

$$E(d^2 x(t) x^*(s)) = E(E(d^2 x(t) \mid d^0 x(t), d^1 x(t)) x^*(s)).$$

By (1.27a) this yields

$$\partial_t^2 R(t,s) = F(t)\,\partial_t^0 R(t,s)\,dt^2 + G(t)\,\partial_t^1 R(t,s)\,dt + o(dt)^2$$

and we divide by $dt^2$ and let $dt \to 0$ to obtain for $t \neq s$

$$\frac{\partial^2 R}{\partial t^2}(t,s) = F(t)R(t,s) + G(t)\frac{\partial R}{\partial t}(t,s).$$

Let $d^2\tilde{x}(t \mid t \pm dt)$ be as in (2.15a), then $d^2\tilde{x}(t \mid t \pm dt)$ is orthogonal to $d^0x(t)$ and $d^1x(t)$.

Now

$$\begin{aligned} d^2x(t \mid t \pm dt) - \partial_t^2 R(t,t) - F(t)\,\partial_t^0 R(t,t) - G(t)\,\partial_t^1 R(t,t) \\ = -E(d^2\tilde{x}(t)x^*(t)) + o(dt)^2 \\ = \tfrac{1}{2}E(d^2\tilde{x}(t)\,d^2x^*(t)) + o(dt)^2 \\ = \tfrac{1}{2}E(d^2\tilde{x}(t \mid t \pm dt)\,d^2\tilde{x}^*(t \mid t \pm dt) \mid d^0x = x, d^1x = v\,dt) + o(dt)^2 \\ = \tfrac{1}{2}E(d^2x(t)d^2x^*(t) \mid d^0x = x, d^1x = v\,dt) + o(dt)^2 \\ = Q(t)\,dt + o(dt)^2, \end{aligned}$$

so $R(t,s)$ is a reciprocal covariance satisfying the differential equation (1.3) and hence A1–A5 hold. The rest follows from the previous theorem.

*Proof of Theorem 9* Suppose $Q^{-1}\mathscr{L}$ is self-adjoint and positive definite on $\mathscr{D}$ so that the Green's function $\Gamma(t,s)$ and $\Psi(t)$ exist. We would like to construct the solution $x(t)$ using (1.31) but to do so we need a generalized noise process $\xi(t)$ which is independent of $x^0, x^T$ and whose covariance satisfies (1.23a). To construct such a $\xi(t)$, we shall find a Markov process $\zeta(t)$ which also satisfies (1.26).

Suppose $\zeta(t)$ satisfies the first order equation

$$d\zeta(t) = A(t)\zeta(t)\,dt + B(t)\,dw(t) \tag{2.17a}$$

$$\zeta(0) = \zeta^0 \tag{2.17b}$$

for some $A(t)$ and $B(t)$. Then the covariance $R_\zeta(t,s)$ of $\zeta(t)$ satisfies

$$\frac{\partial R_\zeta}{\partial t}(t,s) = A(t)R_\zeta(t,s) \qquad \text{if } t > s \tag{2.18a}$$

$$\frac{\partial R_\zeta}{\partial t}(t,s) = A(t)R_\zeta(t,s) + B(t)B^*(t)\Phi^*(s,t) \quad \text{if } t < s \tag{2.18b}$$

where $\Phi(t,s)$ satisfies

$$\frac{\partial \Phi}{\partial t}(t, s) = A(t)\Phi(t, s) \tag{2.19a}$$

$$\Phi(t, t) = I. \tag{2.19b}$$

Since $\zeta(t)$ also satisfies (1.26), its covariance satisfies (1.3) and it follows that $A(t)$ and $B(t)$ are related to $F(t)$, $G(t)$ and $Q(t)$ as follows,

$$\dot{A} + A^2 = F + GA \tag{2.20a}$$

$$AQ - QA^* = GQ - \dot{Q} \tag{2.20b}$$

$$BB^* = Q. \tag{2.20c}$$

Consider the first order differential operator $M$ associated (2.17),

$$\mathcal{M}(x)(t) = \frac{dx}{dt}(t) - A(t)x(t). \tag{2.21a}$$

Its adjoint operator is $\mathcal{M}^*$,

$$\mathcal{M}^*(x)(t) = -\frac{dx}{dt}(t) - A^*(t)x(t). \tag{2.21b}$$

The Eqs. (2.20a, b) are equivalent to the factorization,

$$\mathcal{M}^* Q^{-1} \mathcal{M} = Q^{-1} \mathcal{L}. \tag{2.21c}$$

Hence to construct a Markov process $\zeta(t)$ via (2.17) that also satisfies (1.26) we must find $A$ and $B$ satisfying (2.20). It is not all apparent that such $A$ and $B$ exist of $[0, T]$ since (2.20a) is a Ricatti differential equation for $A$ and (2.20b) is a side constraint.

Actually it is not too hard to show that a solution exists, certainly there are $B$ satisfying (2.20c). Suppose $X(t)$ and $Y(t)$ are $n \times n$ matrix solutions of

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} = \begin{bmatrix} 0 & I \\ F & G \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} \tag{2.22a}$$

then it is straightforward to verify that

$$A(t) = Y(t)X^{-1}(t) \tag{2.22b}$$

is a solution of (2.20a). In particular suppose $A(t_0)$ satisfies (2.20b) for some $t_0 \in [0, T]$ then it follows from (2.20a), the self-adjointness relations (1.6) and the uniqueness of solutions to ordinary differential equations that $A(t)$ satisfies (2.20b) for all $t$.

Of course $A(t)$ defined by (2.22b) blows up when $X(t)$ is not invertible. Consider the solution of (2.22a) satisfying the terminal conditions

$$X(T)=0 \qquad Y(T)=I. \tag{2.23a}$$

Since the operator $\mathscr{L}$ (1.4) corresponding to (2.22a) has no pair of conjugate points in $[0,T]$ this solution must have $X(t)$ invertible for all $t\in[0,T)$. Moreover by perturbing the terminal condition a little bit

$$X(T)=-\varepsilon I \qquad Y(T)=I-\frac{\varepsilon}{2}G \tag{2.23b}$$

we can ensure that $X(t)$ is invertible for all $t\in[0,T]$. Hence $A(t)$ exists satisfying (2.20) and we can construct a Gauss Markov process $\zeta(t)$ satisfying (2.17) and (1.26). We assume that $\zeta(0)$ and $w(t)$ are independent of $x^0$ and $x^T$.

From this we can construct the generalized noise process $\xi(t)$ which is independent of $x^0$ and $x^T$ by specifying

$$Q(t)\xi(t)\,dt^2=-d^2\zeta(t)+F(t)\,d^0\,\zeta(t)\,dt^2+G(t)\,d^1\zeta(t)\,dt. \tag{2.24a}$$

More precisely for every $\phi\in\mathscr{D}$

$$\int_0^T \phi^*(t)\xi(t)\,dt=\int_0^T (\mathscr{L}(\phi)(t))^*\,Q^{-1}(t)\tilde{\zeta}(t)\,dt \tag{2.24b}$$

where $\tilde{\zeta}(t)=\zeta(t)-\hat{\zeta}(t)$ and $\hat{\zeta}(t)$ is the conditional mean of $\zeta(t)$ given $\zeta(0)$ and $\zeta(T)$. In fact,

$$\hat{\zeta}(t)=\Psi(t)\begin{bmatrix}\zeta(0)\\ \zeta(T)\end{bmatrix} \tag{2.15}$$

where $\Psi(t)$ satisfies (1.32).

We define $x(t)$ by (1.31) which by (2.24b) becomes

$$x(t)=\Psi(t)\begin{bmatrix}x^0\\ x^T\end{bmatrix}+\int_0^T \mathscr{L}_S(\Gamma(t,s))Q^{-1}(s)\tilde{\zeta}(s)\,ds \tag{2.26b}$$

where the subscript $s$ in $\mathscr{L}_s$ denote partial differentiation with respect to $s$ rather than $t$ as in (1.4). Since $\Gamma(t,s)$ is the Green's function of a self-adjoint boundary value problem (1.29)

$$\mathscr{L}_S\Gamma(t,s)=\delta(t-s)Q(s) \tag{2.27}$$

and so

$$x(t)=\Psi(t)\begin{bmatrix} x^0 \\ x^T \end{bmatrix}+\tilde{\zeta}(t) \tag{2.28a}$$

or

$$x(t)=\Psi(t)\begin{bmatrix} x^0-\zeta(0) \\ x^T-\zeta(T) \end{bmatrix}+\zeta(t) \tag{2.28b}$$

Since $\tilde{\zeta}(t)$ is independent of $x^0$ and $x^T$, the covariance $R(t,s)$ of $x(t)$ is easily computed,

$$R(t,s)=\Psi(t)E\left(\begin{bmatrix} x^0 \\ x^T \end{bmatrix}\begin{bmatrix} x^0 \\ x^T \end{bmatrix}^*\right)\Psi^*(s)+\Gamma(t,s) \tag{2.29a}$$

where

$$\Gamma(t,s)=R_{\tilde{\zeta}}(t,s) \tag{2.29b}$$

Clearly $R(t,s)$ satisfies (1.3). By Theorem 2 $x(t)$ is reciprocal process and a desired solution of the linear second order stochastic differential equation (1.26).

To show that the solution of (1.26) is unique up to law, suppose $R_y(t,s)$ is the covariance of another solution $y(t)$ to (1.26). Since $x(t)$ and $y(t)$ are equal at $t=0,T$ we have

$$R(t,s)=R_y(t,s) \tag{2.30}$$

if ($t=0$ or $T$) and ($s=0$ or $T$). Both $R$ and $R_y$ satisfy the well-posed differential equation (1.3) so (2.30) holds for $t\in[0,T]$ and ($s=0$ or $T$). Moreover by (2.5), $R^*(t,s)$ and $R_y^*(t,s)$ also satisfy (1.3) hence (2.30) holds for all $t,s\in[0,T]$. $\square$

There is an alternative way of constructing the noise process $\xi(t)$ directly from a standard Wiener process $W(t)$ and a factorization (2.21c) of the operator $\mathscr{L}$. Formally $\xi(t)$ is the generalized process given by

$$\xi(t)=\mathscr{M}^*\left(B(t)\frac{dw}{dt}\right). \tag{2.31}$$

The precise meaning of this is obtained by integrating against a test function $\phi\in\mathscr{D}$,

$$\int_0^t \phi^*(t)\xi(t)\,dt = \int_0^T (\mathcal{M}(\phi)(t))^* B(t)\,dw \tag{2.32}$$

where the right side is a Wiener integral. Given such integrals $Y_1, Y_2$ (1.24) we see immediately that

$$E(Y_1 Y_2) = \int_0^T (\mathcal{M}(\phi_1)(t))^* B(t) B^*(t) \mathcal{M}(\phi_2)(t)\,dt$$

$$= \int_0^T \phi_1^*(t) \mathcal{M}^*(Q^{-1}(t)\mathcal{M}(\phi_2))(t)\,dt. \tag{2.33}$$

which agrees with (1.25a).

## 3. EXAMPLES

The one dimensional stationary Gaussian reciprocal processes have been completely classified [8,9,12]. We discuss the stochastic differential equations that they satisfy. A similar discussion can be found in [20] employing the prior notation and definition of a stochastic differential equation of second order.

The classification is based on the fact that covariance $R(t,s)=R(t-s)$ must satisfy (1.3) which in the scalar stationary case reduces to

$$-\ddot{R}(t) + FR(t) = Q\delta(t) \tag{3.1}$$

where $F$ and $Q$ are constants and $G=0$. Hence there are three cases (i) $F>0$, (ii) $F=0$ and (iii) $F<0$. By change of $x$ coordinates we normalize so that $R(0)=1$.

(ia) *Ornstein–Uhlenbeck Processes* There are Markov processes $x(t)$ defined for $t\in(-\infty,\infty)$ whose covariances are of the form

$$R(t) = \exp A\,|t/B| \tag{3.2}$$

where $A<0$. They are solutions of the first order stochastic differential equations

$$dx = Ax\,dt + B\,dw \tag{3.3a}$$

$$x(0) = x^0 \sim N(0,1). \tag{3.3b}$$

These processes also satisfy the second order stochastic differential equation

$$-d^2x + Fd^0x\,dt^2 = Q\xi\,dt^2 \tag{3.4}$$

where

$$F = A^2 \tag{3.5a}$$

$$Q = B^2. \tag{3.5b}$$

Given a solution $x(t)$ of (3.3) we can define the generalized noise process $\xi(t)$ by (3.4) or more precisely (1.22) and we can use $\xi(t)$ to construct other non-Markov solutions to (3.4), as described in (ib) and (ic).

(ib) *Hyperbolic Cosine Processes* These are reciprocal but not Markov processes $y(t)$ defined for $t \in [0, T]$ whose covariances are of the form

$$R_y(t) = \cosh\left(A\left(\frac{T}{2} - t\right)\right) \Big/ \cosh\left(\frac{AT}{2}\right). \tag{3.6}$$

Since they are not Markov they do not satisfy any first order stochastic differential equations but they do satisfy the second order stochastic boundary value problems

$$-d^2 y + F d^0 y\, dt^2 = Q\xi\, dt^2 \tag{3.7a}$$

$$y(0) = y(T) = y^0 \sim N(0, 1) \tag{3.7b}$$

where

$$F = A^2 > 0 \tag{3.8a}$$

$$Q = 2A \tanh \frac{AT}{2}. \tag{3.8b}$$

The Green's function $\Gamma(t, s)$ and boundary value transition matrix $\Psi(t)$ of the deterministic boundary value problem

$$-\frac{d^2 y}{dt^2} + Fy = Qu \tag{3.9a}$$

$$y(0) = y^0 \qquad y(T) = y^T \tag{3.9b}$$

are given by

$$\Gamma(t, s) = \begin{cases} \dfrac{-Q \sinh A(T-t) \sinh As}{A \sinh AT} & \text{if} \quad t \geqq s \\[2ex] \dfrac{-Q \sinh At \sinh A(T-s)}{A \sinh AT} & \text{if} \quad t \leqq s \end{cases} \tag{3.10a}$$

$$\Psi(t)=\left[\frac{\sinh A(T-t)}{\sinh AT}\ \frac{\sinh At}{\sinh AT}\right]. \tag{3.10b}$$

The solution $y(t)$ to (3.7) is given by

$$y(t)=\Psi(t)\begin{bmatrix} y^0 \\ y^0 \end{bmatrix}+\int_0^T \Gamma(t,s)\,\xi(s)\,ds \tag{3.11}$$

where $\xi(s)$ is the residual defined by (3.4).

Alternatively, if $x(t)$ is a solution (3.3) and $\hat{x}(t)$ is the conditional means of $x(t)$ given $x(0)$ and $x(T)$ then the covariance $\tilde{R}(t,s)$ of $\tilde{x}(t)=x(t)-\hat{x}(t)$ equals $\Gamma(t,s)$ (3.10a). The solution $y(t)$ to (3.7) is given by

$$y(t)=\Psi(t)\begin{bmatrix} y^0 \\ y^0 \end{bmatrix}+\tilde{x}(t). \tag{3.11b}$$

(1c) *Hyperbolic Sine Processes* These are reciprocal but not Markov processes $y(t)$ defined for $t\in[0,T]$ whose covariances are given by

$$R_y(t)=\sinh\left(A\left(\frac{T}{2}-t\right)\right)\sinh\left(\frac{AT}{2}\right). \tag{3.12}$$

Since they are not Markov, they do not satisfy any first order stochastic differential equation but they do satisfy the second order stochastic boundary value problems (3.7a) and

$$y(0)=-y(T)=y^0\sim N(0,1) \tag{3.7c}$$

where $F$ satisfies (3.8a) and

$$Q=2A\coth\frac{AT}{2}. \tag{3.8c}$$

The solutions to (3.7a, c) are given by

$$y(t)=\Psi(t)\begin{bmatrix} y^0 \\ -y^0 \end{bmatrix}+\int_0^T \Gamma(t,s)\,\xi(s)\,ds \tag{3.13}$$

or

$$y(t) = \Psi(t)\begin{bmatrix} y^0 \\ -y^0 \end{bmatrix} + \bar{x}(t). \tag{3.13b}$$

where $\Psi$, $\Gamma$ and $\xi$ are the same as those of the hyperbolic cosine process.

*Note*: The hyperbolic cosine processes are cyclic, $y(0) = y(T)$, and the hyperbolic sine processes are anticyclic, $y(\theta) = -y(T)$. They have a natural finite lifetime $[0, T]$ as their covariance cannot be analytically continued in a stationary fashion to a larger interval without violating the Cauchy–Schwartz inequality, $R_y(0) \geqq |R_y(t)|$.

In fact, every solution of the second order stochastic differential equation (3.7a) for $F > 0$ is an Ornstein–Uhlenbeck, Cosh or Sinh process when considered on its natural lifetime $(-\infty, \infty)$, $[0, T]$ or $[0, T]$.

Next we consider the case where $F = 0$.

ii) *Slepian Processes* These reciprocal but not Markov processes $y(t)$ live on $[0, T]$ and have covariances

$$R_y(t) = 1 - \frac{2t}{T}. \tag{3.14}$$

Again they do not satisfy any first order stochastic differential equation but do satisfy the second order stochastic boundary value problem

$$-d^2 y = Q\xi\, dt^2 \tag{3.15a}$$

$$y(0) = -y(T) = y^0 \sim N(0, 1). \tag{3.15b}$$

Suppose $B^2 = Q$ and $w(t)$ is a standard Wiener process then $Bw(t)$ is a nonstationary solution of the second order stochastic differential equation (3.15a) so we can realize the generalized noise process of (3.15a) as

$$Q\xi(t)\, dt^2 = -B d^2 w. \tag{3.16}$$

The Green's function $\Gamma(t, s)$ and boundary value transition matrix $\Psi(t)$ of the deterministic boundary value problem corresponding to (3.15) are given by

$$\Gamma(t, s) = \begin{cases} -\dfrac{Q}{T}(T-t)s & t \geqq s \\ -\dfrac{Q}{T}(T-s)t & t \geqq s \end{cases} \tag{3.17a}$$

$$\Psi(t) = \begin{bmatrix} \dfrac{T-t}{T} & \dfrac{t}{T} \end{bmatrix}. \tag{3.17b}$$

The solution of (3.15) is given by

$$y(t)=\Psi(t)\begin{bmatrix} y^0 \\ -y^0 \end{bmatrix}+\int_0^T \Gamma(t,s)\xi(s)\,ds \tag{3.18a}$$

where $\xi$ is defined by (3.16). Alternatively the Brownian Bridge

$$B\tilde{w}(t)=B\left(w(t)-\frac{t}{T}w(T)\right) \tag{3.19}$$

is a solution of (3.15a) satisfying zero boundary conditions so

$$y(t)=\Psi(t)\begin{bmatrix} y^0 \\ -y^0 \end{bmatrix}+B\tilde{w}(t). \tag{3.18b}$$

Note again that Slepian processes have natural finite lifetimes which are finite and they satisfy anticyclic boundary conditions.

The last class of one dimensional stationary reciprocal processes occurs when $F<0$.

iii) *Shifted Cosine Processes* These are reciprocal and non-Markov processes whose covariance are of the form

$$R_y(t)=\frac{\cos K(t+\tau)}{\cos K\tau} \tag{3.19a}$$

where

$$0<K \tag{3.19b}$$

$$0<T<\tfrac{1}{2}\left(\frac{\pi}{K}-\tau\right). \tag{3.19c}$$

Again these processes don't satisfy any first order stochastic differential equation but they do satisfy the second order stochastic boundary value problems,

$$-d^2y+F\,d^0y\,dt^2=Q\xi\,dt^2 \tag{3.20a}$$

$$y(0)=-y(T)\sim N(0,1) \tag{3.20b}$$

where

$$K^2=-F \qquad ((3.21a)$$

$$Q = 2K \tan K\tau. \tag{3.21b}$$

To realize the generalized noise process $\xi(t)$ of (3.20a) we must turn to a first order stochastic differential equation (2.17) where the coefficients $A(t)$ and $B(t)$ are related to $F$ and $Q$ by (2.20). One solution to (2.20) is

$$A(t) = K \cot K(t+\tau) \tag{3.22a}$$

$$B(t) = Q^{1/2} \tag{3.22b}$$

Suppose $x(t)$ is the Markov process defined by

$$dx = A(t)x\, dt + B\, dw \tag{3.23a}$$

$$x(0) = x^0 \tag{3.23b}$$

then $x(t)$ is also a solution of (3.20a).

The residual $\xi(t)$ defined by

$$Q(t)\xi(t)\, dt^2 = -d^2x + F\, d^0x\, dt^2 \tag{3.24}$$

can be used to construct the solution $y(t)$ of (3.20),

$$y(t) = \Psi(t)\begin{bmatrix} y^0 \\ -y^0 \end{bmatrix} + \int_0^T \Gamma(t,s)\xi(s)\, ds \tag{3.25a}$$

where

$$\Gamma(t,s) = \begin{cases} -\dfrac{Q \sin K(T-t) \sin Ks}{K \sin KT} & \text{if } t \geqq s \\[2ex] -\dfrac{Q \sin Kt \sin K(T-s)}{K \sin KT} & \text{if } t \leqq s \end{cases} \tag{3.26a}$$

$$\Psi(t) = \begin{bmatrix} \dfrac{\sin K(T-t)}{KT} & \dfrac{\sin Kt}{\sin KT} \end{bmatrix}. \tag{3.26b}$$

We can also construct $\tilde{x}(t)$, the pinned version of $x(t)$ of (3.23) and express the solution $y(t)$ as

$$y(t) = \Psi(t)\begin{bmatrix} y^0 \\ -y^0 \end{bmatrix} + \tilde{x}(t). \tag{3.25b}$$

Again notice that the shifted cosine processes have natural finite lifetimes and are anticyclic.

Before closing we take a closer look at the pinned processes $\tilde{x}(t)$ and generalized noise processes $\xi(t)$ that were used in the above constructions. We denote these by $\tilde{x}_+(t)$, $\tilde{x}_0(t)$, $\tilde{x}_-(t)$ and $\tilde{\xi}_+(t), \tilde{\xi}_0(t)$ and $\xi_-(t)$ for the three cases $F>0$, $F=0$, $F<0$.

In the first case $\tilde{x}_+(t)$ is a pinned version of the Ornstein Uhlenbeck process. $x(t)$ given by (3.3) where $A$ and $B$ satisfy (3.5). In the second case $\tilde{x}_0(t)$ is a pinned Wiener process given by (3.19). In either case there are no restrictions on the intensity of the noise (parameterized by $Q$) nor on the lifetime of the process (parameterized by $T$). The process $\tilde{x}_0(t)$ can be thought as the motion of a random particle in a free force field which is constrained to start at the origin at $t=0$ and return to the origin at $t=T$. The process $\tilde{x}_+(t)$ can be thought of as the motion of a random particle in repulsive force field around the origin pinned to start and return to the origin. The covariance of $\tilde{x}_+(t)$ and $\tilde{x}_0(t)$ are given by the Green function $\Gamma_+(t,s)=\Gamma(t,s)$ of (3.10a) and $\Gamma_0(t,s)=\Gamma(t,s)$ of (3.17a). The generalized noise processes $\xi_+(t)$ and $\xi_0(t)$ are constructed from $\tilde{x}_+(t)$ and $\tilde{x}_0(t)$ according to (1.22).

The process $\tilde{x}_-(t)$ is a pinned version of the Gauss Markov Process $x(t)$ given by (3.23) where $A, B$ satisfy (3.22). For $F<0$ the differential operator

$$\mathscr{L} = -\frac{d^2}{dt^2} + F$$

has a pair of conjugate points at 0 and $\pi/K$. Hence the pinning time $T$ of $\tilde{x}_-(t)$ must satisfy

$$0 < T < \pi/K. \tag{3.26a}$$

The noise intensity $Q$ is also limited by (3.19c) and (3.21b) to satisfy

$$0 < Q < 2K \tan \tfrac{1}{2}(\pi - KT). \tag{3.26b}$$

Hence the greater the noise, the shorter the pinning time and the longer the pinning time the lesser the noise.

The process $\tilde{x}_-(t)$ can be thought of as random particles moving in an attracting force field around the origin constrained to start at and return to the origin at $t=0$ and $T$. Its covariance is given by $\Gamma_-(t,s)=\Gamma(t,s)$ of (3.26a). The generalized noise process $\xi_-(t)$ is constructed from $\tilde{x}_-(t)$ via (1.22).

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