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Observation of a Rigid Body from Measurement of a Principal Axis*

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Abstract

The spacecraft attitude control has been studied before, see [3]. In this note, we give a method of observing the attitude of a freely rotating spacecraft by measuring one of its principal axis. The problem is solved in §2 and §3. In §4, a method of determining the angular velocity by the trajectory of one of its coordinates is given.

Key words: Nonlinear systems, nonlinear estimation, rigid body, nonlinear observations

1 Introduction

In the following, we consider a freely rotating rigid body with no external torques acting on it. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a set of orthonormal axis fixed in the spacecraft, with the origin at the center of mass and each axis parallel to one of the principal axis. A second frame $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ is an inertially fixed basis. In page 445 of [2], Symon describes the motion of a freely rotating body as follows. Fix an ellipsoid on the spacecraft, which can be represented as

$$\left\{\sum_{i=1}^{3} x_i \mathbf{e}_i \mid \sum_{i=1}^{3} I_i x_i^2 = 1\right\}.$$

It is called the inertia ellipsoid. There is a fixed plane, P, which is called the invariant plane. As the spacecraft is rotating freely, one can imagine that the inertia ellipsoid is fixed on the spacecraft and it is rolling on the invariant plane without slipping and its center is fixed at the origin, see Figure 1.

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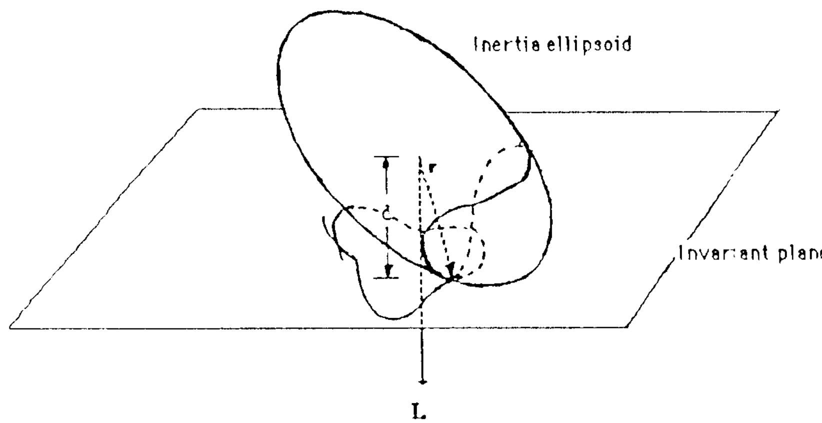

Figure 1. The inertia ellipsoid rolls on the invariant plane.

Suppose **r** is the vector from the origin to the point of contact between the ellipsoid and the invariant plane. Then, the angular velocity ω satisfies

$$\omega = b\mathbf{r}$$

where b is a constant. The following equations describe the evolution of the spacecraft's attitude

$$\begin{aligned}
&I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = 0\\
&I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = 0\\
&I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 = 0\\
&\dot{\phi} = \frac{\sin\psi}{\sin\theta}\omega_1 + \frac{\cos\psi}{\sin\theta}\omega_2\\
&\dot{\psi} = \frac{\sin\psi\cos\theta}{\sin\theta}\omega_1 - \frac{\cos\psi\cos\theta}{\sin\theta}\omega_2 + \omega_3\\
&\dot{\theta} = \cos\psi\,\omega_1 - \sin\psi\,\omega_2
\end{aligned} \tag{1.1}$$

where (ϕ, ψ, θ) are Euler angles (see [3]), and

$$\omega = \sum_{i=1}^{3} \omega_i \mathbf{e}_i$$

is the angular velocity. The intertia tensor is

$$I = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}.$$

2 Observability

Suppose the output of system (1.1) is the position of the third principal axis, $e_3(t)$, in inertial coordinates. We consider the observability of the model, i.e., whether the complete motion can be determined from the time history of $e_3(t)$ in inertial coordinates and the equations of motion (1.1). Rewrite the system as follows:

$$\begin{aligned} \dot{\omega}_1 &= \alpha\omega_2\omega_3 \\ \dot{\omega}_2 &= \beta\omega_1\omega_3 \\ \dot{\omega}_3 &= \gamma\omega_1\omega_2 \end{aligned} \tag{2.1}$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\psi} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{\sin\psi}{\sin\theta} & \frac{\cos\psi}{\sin\theta} & 0 \\ -\sin\psi\operatorname{ctg}\theta & -\cos\psi\operatorname{ctg}\theta & 1 \\ \cos\psi & -\sin\psi & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}. \tag{2.2}$$

The inertial coordinates of $e_3(t)$ are the observation and are given by

$$y = \begin{bmatrix} \sqrt{1-\cos^2\theta}\sin\phi \\ -\sqrt{1-\cos^2\theta}\cos\phi \\ \cos\theta \end{bmatrix} = \begin{bmatrix} h_1(\theta,\phi) \\ h_2(\theta,\phi) \\ h_3(\theta,\phi) \end{bmatrix}. \tag{2.3}$$

Here

$$\alpha = \frac{I_2 - I_3}{I_1}, \quad \beta = \frac{I_3 - I_1}{I_2}, \quad \gamma = \frac{I_1 - I_2}{I_3}.$$

Theorem 2.1 *The system (2.1-2.3) is observable iff* $\alpha + \beta \neq 0$ *and* $(\beta + 1)\omega_1^2 + (\alpha - 1)\omega_2^2 \neq 0$.

Proof. To prove this system is observable, using the method of [1], we need to find the dimension of the distribution $\mathcal{C}(r)$ generated by $\{dy, L_F dy, \ldots, L_F^{r-1} dy\}$. Because

$$h_3 = \pm\sqrt{1 - h_1^2 + h_2^2}$$

the dimension of $\mathcal{C}(r)$ can be determined by:

$$dh_1, L_F dh_1, \ldots, L_F^{r-1} dh_1$$

$$dh_2, L_F dh_2, \ldots, L_F^{r-1} dh_2.$$

But $L_F^k dh_i$, $i = 1, 2$, are complicated. To determine the dimension, we make a change of coordinates in the output space. Let

$$A = \begin{bmatrix} \frac{\partial h_1}{\partial \phi} & \frac{\partial h_1}{\partial \theta} \\ \frac{\partial h_2}{\partial \phi} & \frac{\partial h_2}{\partial \theta} \end{bmatrix}.$$

Then, $\det(A) = \cos\theta \sin\theta$; and A is nonsingular whenever $\theta \neq \frac{k\pi}{2}$. The angle θ depends on the choice of the inertially fixed basis. We can chose suitable basis to avoid the case $\theta = \frac{k\pi}{2}$ in a local neighborhood. Therefore, by change of coordinates in the output space we can take θ and ϕ the output. The observability of (2.1–2.3) is equivalent to the observability with respect to the output

$$y_1 = \begin{bmatrix} \theta \\ \phi \end{bmatrix}.$$

By calculation, we know that

$$\begin{aligned} L_F\theta &= \omega_1 \cos\psi - \omega_2 \sin\psi \\ L_F\phi &= (\omega_1 \sin\psi + \omega_2 \cos\psi)\frac{1}{\sin\theta} \\ L_F^2\theta &= \sin\theta\cos\theta(L_F\phi)^2 - \omega_3 \sin\theta(L_F\phi) \\ &\quad + \alpha\omega_2\omega_3\cos\psi - \beta\omega_1\omega_3\sin\psi \\ L_F^2\phi &= \operatorname{ctg}\theta(L_F\phi)(L_F\theta) + \frac{\cos\theta}{\sin\theta}(L_F\theta L_F\phi) + \omega_3\frac{L_F\theta}{\sin\theta} \\ &\quad + \frac{\sin\psi}{\sin\theta}\alpha\omega_2\omega_3 + \frac{\cos\psi}{\sin\theta}\beta\omega_1\omega_3. \end{aligned}$$

In $dL_F^2\phi$ and $dL_F^2\theta$, all the terms containing $d\theta$, $d\phi$, $dL_F\phi$ or $dL_F\theta$ can be cancelled. The dimension of $\mathcal{C}(3)$ is the same as the rank of the following matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\omega_1\sin\psi - \omega_2\cos\psi & -\cos\psi & -\sin\psi & 0 \\ 0 & 0 & \omega_1\cos\psi - \omega_2\sin\psi & \sin\psi & \cos\psi & 0 \\ 0 & 0 & -\alpha\omega_2\omega_3\sin\psi - \beta\omega_1\omega_3\cos\psi & -\beta\omega_3\sin\psi & \alpha\omega_3\cos\psi & \rho_1 \\ 0 & 0 & \alpha\omega_2\omega_3\cos\psi - \beta\omega_1\omega_3\sin\psi & \beta\omega_3\cos\psi & \alpha\omega_3\sin\psi & \rho_2 \end{bmatrix}.$$

where $\rho_1 = (\alpha-1)\omega_2\cos\psi - (\beta+1)\omega_1\sin\psi$ and $\rho_2 = (\alpha-1)\omega_2\sin\psi + (\beta+1)\omega_1\cos\psi$. The determinant of this matrix is

$$[(\beta+1)\omega_1^2 + (\alpha-1)\omega_2^2](\alpha+\beta).$$

Therefore, this is not zero if and only if $\mathcal{C}(3)$ has full dimension.

Remark 1 $(\beta+1)\omega_1^2 + (\alpha-1)\omega_2^2 \neq 0$ implies that ω_1 and ω_2 can not be zero at the same time. If $\omega_1 = \omega_2 = 0$, then the spacecraft turns around

$\mathbf{e}_3$, the output is a constant vector and it is impossible to determine $\mathbf{e}_1$, $\mathbf{e}_2$ from the trajectory of $\mathbf{e}_3(t)$.

Remark 2 The condition $\alpha + \beta \neq 0$ implies $I_1 \neq I_2$. If $I_1 = I_2$, then the spacecraft is symmetric with respect to $\mathbf{e}_3$. We can not tell the difference between $\mathbf{e}_1$ and $\mathbf{e}_2$. Moreover, $\mathbf{e}_1$ and $\mathbf{e}_2$ are not uniquely defined. So it is impossible to determine the position of $\mathbf{e}_1$, $\mathbf{e}_2$.

3 Attitude Determination

In this section, we assume that $I_1 > I_2 > I_3$ and $\omega_3 \neq 0$. If $\omega_3 = 0$, then $\omega_1 \neq 0$ (see [3]). So, from the similar method in this section, we can determine the attitude by measuring $\mathbf{e}_1$.

From [3] and the introduction, we know that the motion of the spacecraft is totally determined by the following three constants.

(1) The direction of $\mathbf{L}$, which is the normal vector of the invariant plane P.

(2) The distance, d, from the origin to the tangent plane P.

(3) The energy T.

As the inertia ellipsoid rolls on P, the vector $\mathbf{e}_3$ turns around the axis $\mathbf{L}$. Suppose that the coordinates of $\mathbf{e}_3(t)$ in the inertially fixed basis is $(x(t), y(t), z(t))$, it is expressed by Euler angles in (2.1–2.3). Imagine that the curve described by $\mathbf{e}_3(t)$ has mass with the density at constant 1. Then $\mathbf{L}$ is a vector passing through the center of this mass. So the coordinates of $\mathbf{L}$ in the inertially fixed basis are

$$x_L = \frac{\int_0^{s_0} x(s)ds}{s_0}, \quad y_L = \frac{\int_0^{s_0} y(s)ds}{s_0}, \quad z_L = \frac{\int_0^{s_0} z(s)ds}{s_0}. \tag{3.1}$$

Where s is the length of the curve described by $\mathbf{e}_3(t)$ at time t. The number s_0 is the length of the smallest closed curve described by $\mathbf{e}_3(t)$ if $\mathbf{e}_3(t)$ moves in periodic. If it is not periodic, we must take the limit of these integrals as the length s_0 goes to ∞.

To find d, we consider $\mathbf{L} \cdot \mathbf{e}_3$. Let's take the inertia ellipsoid as

$$I_1 r_1^2 + I_2 r_2^2 + I_3 r_3^2 = 1. \tag{3.2}$$

So, the vector $\mathbf{L}$ is parallel to

$$I_1 r_1 \mathbf{e}_1 + I_2 r_2 \mathbf{e}_2 + I_3 r_3 \mathbf{e}_3$$

where $r = \sum_{i=1}^{3} r_i \mathbf{e}_i$ is the vector from the origin to the point of contact between the ellipsoid and the invariant plane. Therefore

$$\mathbf{L} = \frac{\sum_{i=1}^{3} I_i r_i \mathbf{e}_i}{\sqrt{\sum_{i=1}^{3} I_i^2 r_i^2}} \tag{3.3}$$

and

$$d = \mathbf{L} \cdot \mathbf{r} = \frac{I_1 r_1^2 + I_2 r_2^2 + I_3 r_3^2}{\sqrt{\sum_{i=1}^{3} I_i^2 r_i^2}} = \frac{1}{\sqrt{\sum_{i=1}^{3} I_i^2 r_i^2}}.$$

So

$$\mathbf{e}_i \cdot \mathbf{L} = \frac{I_i r_i}{\sqrt{\sum_{i=1}^{3} I_i^2 r_i^2}} = r_i I_i d. \tag{3.4}$$

The function $|\mathbf{e}_3 \cdot \mathbf{L}|$ has its maximum value if and only if $|r_3|$ has its maximum value. From the first three equations of system (2.1–2.3), we can easily prove

$$\frac{\omega_2^2}{\beta} - \frac{\omega_3^2}{\gamma} = \text{constant}.$$

So $(\omega_2(t), \omega_3(t))$ describes an ellipse. In [2], it was proved that $\mathbf{r} = b\omega$, therefore (r_2, r_3) is also on an ellipse. The function $|r_3(t)|$ has the maximum value implies $r_2 = 0$. The equation (3.4) implies that $|r_3(t)|$ has its maximum value if and only if $|\mathbf{L} \cdot \mathbf{e}_3|$ has its maximum value. Denote this maximum value of $|\mathbf{L} \cdot \mathbf{e}_3|$ by A. So $|\mathbf{L} \cdot \mathbf{e}_3| = A$ implies $r_2 = 0$. From (3.2) and (3.3), we obtain

$$I_1 r_1^2 + I_3 r_3^2 = 1 \tag{3.5}$$

$$\mathbf{L} = \frac{I_1 r_1 \mathbf{e}_1 + I_3 r_3 \mathbf{e}_3}{\sqrt{I_1^2 r_1^2 + I_3^2 r_3^2}} \tag{3.6}$$

so, $\mathbf{L}$ is in the $\mathbf{e}_1$, $\mathbf{e}_3$ plane

$$\mathbf{L} \cdot \mathbf{e}_1 = \pm\sqrt{1 - (\mathbf{L}\mathbf{e}_3)^2} = \pm\sqrt{1 - A^2}. \tag{3.7}$$

The equations (3.4) and (3.7) imply

$$\begin{aligned} r_1 I_1 d &= \pm\sqrt{1 - A^2} \\ r_3 I_3 d &= A. \end{aligned}$$

Therefore

$$r_1 = \pm\frac{\sqrt{1 - A^2}}{I_1 d}, \qquad r_2 = \frac{A}{I_3 d}. \tag{3.8}$$

Substitute (3.8) to (3.5), we have

$$\frac{1 - A^2}{I_1 d^2} + \frac{A^2}{I_2 d^2} = 1.$$

From this, we obtain

$$d = \sqrt{\frac{1 - A^2}{I_1} + \frac{A^2}{I_3}}. \tag{3.9}$$

Now, we try to determine the energy T by the frequency of $\mathbf{e}_3$. Suppose

$$X = (x_1(t), x_2(t), x_3(t))$$

is the solution of

$$\begin{aligned} \dot{x}_1 &= \alpha x_2 x_3 \\ \dot{x}_2 &= \beta x_1 x_3 \\ \dot{x}_3 &= \gamma x_1 x_2 \end{aligned}$$

such that the initial condition is on the inertia ellipsoid and

$$d = \frac{1}{\sqrt{\sum_{i=1}^{3} x_i^2 I_i^2}}.$$

Its period is a_0. Consider

$$\omega_i = \lambda x_i(\lambda t).$$

It can be proved that ω_1, ω_2 and ω_3 satisfy the first three equations in (2.2). The energy

$$T = \frac{1}{2}\omega \mathbf{I} \omega = \frac{\lambda^2}{2}(I_1 x_1^2 + I_2 x_2^2 + I_3 x_3^2) = \frac{\lambda^2}{2}.$$

The period of ω is $\frac{a_0}{\lambda}$. Suppose the period of ω_3 is a, then

$$2a = \frac{a_0}{\lambda} = \frac{a_0}{\sqrt{2T}}.$$

So

$$T = \frac{a_0^2}{8a^2}. \tag{3.10}$$

Here, a is the period of ω_3, which is unknown. But we proved that

$$\begin{aligned} \mathbf{e}_3 \cdot \mathbf{L} &= r_3 I_3 d \\ r_3 &= b\omega_3 \end{aligned}$$

where b is some constant. So, a is also the period of $\mathbf{e}_3 \cdot \mathbf{L}$.

Therefore, we can use (3.1) to determine $\mathbf{L}$, (3.9) to determine d and (3.10) to determine T. From the proof, we could see that the center of the curve described by $\mathbf{e}_3(t)$ is $\mathbf{L}$, the amplitude of $|\mathbf{e}_3 \cdot \mathbf{L}|$ determines d and the frequency of $|\mathbf{e}_3 \cdot \mathbf{L}|$ determines the energy T.

4 Angular Velocity Observation and the Observer Normal Form

In this section, we study the observability of the following system:

$$\begin{aligned} \dot{\omega}_1 &= \alpha\omega_2\omega_3 \\ \dot{\omega}_2 &= \beta\omega_1\omega_3 \\ \dot{\omega}_3 &= \gamma\omega_1\omega_2 \\ y &= \omega_1. \end{aligned} \tag{4.1}$$

This is a subsystem in the spacecraft attitude problem which is related to the angular velocity and the energy. In this section, we assume $I_1 > I_2 > I_3$ or $I_1 < I_2 < I_3$.

Theorem 4.1 *If $I_3 \neq I_2$ ($\alpha \neq 0$), $\omega_1 \neq 0$, $\omega_2 + \omega_3 \neq 0$, then system (4.1) is observable.*

Proof. The following relations can be easily proved.

$$\begin{aligned} L_F d\omega_1 &= \alpha\omega_3 d\omega_2 + \alpha\omega_2 d\omega_3 \\ L_F^2 d\omega_1 &= \alpha\beta\omega_3^2 d\omega_1 + 2\alpha\beta\omega_1\omega_3 d\omega_3 + \alpha\gamma\omega_2^2 d\omega_1 + 2\alpha\gamma\omega_1\omega_2 d\omega_2. \end{aligned}$$

Therefore, the dimension of the distribution generated by $d\omega_1$, $L_F d\omega_1$, $L_F^2 d\omega_1$ is the same as the rank of the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha\omega_3 & \alpha\omega_2 \\ 0 & 2\alpha\gamma\omega_1\omega_2 & 2\alpha\beta\omega_1\omega_3 \end{bmatrix}.$$

Its determinant is

$$2\alpha^2\omega_1(\beta\omega_3^2 - \gamma\omega_2^2).$$

Because $I_1 > I_2 > I_3$ or $I_1 < I_2 < I_3$, we know that β and γ have different signs. So, the distribution has dimension 3 whenever $\alpha \neq 0$, $\omega_1 \neq 0$, $\omega_2^2 + \omega_3^2 \neq 0$. The theorem follows.

Remark In the remarks after Theorem 2.1, we explained why the condition $I_1 \neq I_2$ and $\omega_1^2 + \omega_2^2 \neq 0$ arise. This can also be used to explain the condition on I_2, I_3, ω_2 and ω_3 in Theorem 4.1. The condition $\omega_1 \neq 0$ is necessary. From the following discussion, we can see that if $\omega_1 = 0$, then ω_2, ω_3 can not be estimated by $y = \omega_1$.

Theorem 4.2 *Under the same hypotheses as Theorem 4.1, then*

$$\omega_2^2 = \frac{\beta}{\alpha}\omega_1^2 - \beta \cdot \alpha \cdot \max\{\omega_1^2\} \tag{4.2}$$

$$\omega_3^2 = \frac{\gamma}{\alpha}\omega_1^2 - \gamma \cdot \alpha \cdot \max\{\omega_1^2\}. \tag{4.3}$$

Proof. From (4.1), we have

$$\frac{1}{\alpha}\frac{d\omega_1^2}{dt} = \frac{1}{\beta}\frac{d\omega_2^2}{dt} = \frac{1}{\gamma}\frac{d\omega_3^2}{dt}.$$

Therefore,

$$\frac{\omega_2^2}{\beta} = \frac{\omega_1^2}{\alpha} + c_1$$
$$\frac{\omega_3^2}{\gamma} = \frac{\omega_1^2}{\alpha} + c_2.$$

Because $I_1 > I_2 > I_3$ or $I_1 < I_2 < I_3$, the constants α and β have different signs, the constants α and γ have the same signs

$$\frac{\omega_2^2}{\beta} - \frac{\omega_1^2}{\alpha} = c_1$$

is an ellipse. So $(\max\{\omega_1\}, 0)$ is on the ellipse. So,

$$c_1 = -\alpha \cdot \max\{\omega_1^2\}.$$

Since

$$\frac{\omega_3^2}{\gamma} - \frac{\omega_1^2}{\alpha} = c_2$$

is a hyperbola, $\omega_1 \neq 0$ means that ω_1^2 takes its minimum value if $\omega_3 = 0$. So

$$c_2 = -\alpha \cdot \min\{\omega_1^2\}.$$

Therefore, the formulas in Theorem 4.2 are proved.

In the following, we are going to find a kind of change of coordinates so that (4.1) can be transformed to observer normal form, i.e., we want to change (4.1) and make it look like

$$\begin{aligned} \dot{x} &= Ax + f(y, u) \\ \dot{y} &= Cx \end{aligned}$$

where (C, A) is an observable pair.

In [1], this method is discussed in detail. In Example 7.3 of [1], the author proved a necessary and sufficient condition for a system like (4.1) to be transformed to observer form. Unfortunately, it can be proved that system (4.1) does not satisfy this condition. Therefore, we have to think about this problem from another point of view.

In Theorem 4.3, we will find a family of changes of coordinates $x = x(\omega, c)$ such that for each output trajectory, there is a constant c_0 so that $x = x(\omega, c_0)$ transforms (4.1) to an observer normal form, $x(\omega, c)$ and the observer normal forms are continuous with respect to c.

Theorem 4.3 *Under the same hypotheses as Theorem 4.1, we define*

$$\begin{aligned} x_1 &= \omega_1 \\ x_2 &= \alpha\omega_2\omega_3 \\ x_3 &= \alpha\beta\omega_1\omega_3^2 + \alpha\gamma\omega_1\omega_2^2 \\ &\quad - \frac{2}{3}(\beta\gamma + \beta^2 + \gamma^2)y^3 - \alpha\beta\gamma c y \end{aligned} \tag{4.4}$$

$$c = -\alpha\{\max(y^2) + \min(y^2)\}. \tag{4.5}$$

Then $x(\omega(t), c)$ *satisfies*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 + \frac{2}{3}(\beta\gamma + \beta^2 + \gamma^2)y^3 + \alpha\beta\gamma c y \\ \dot{x}_3 &= 0 \\ y &= x_1. \end{aligned} \tag{4.6}$$

Proof. Substitute (4.4) into (4.5), and use (4.2), (4.3).

In this note, we assume the spacecraft moves freely. An obvious question is, how to observe the attitude when the system has nonzero input? This is an important open question.

Another interesting problem is, for what range of the output, system (4.1) can be estimated by Theorem 4.3 without changing the parameter c.

The results in this note are applicable to the rigid body problem, but most recent spacecraft research is directed towards large flexible space structures and the models are much more complicated. However, the rigid dynamics are still interesting and important.

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