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Degree Two Normal Forms of Control Systems and the Generalized Legendre Clebsch Condition

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Abstract The Brunovsky (or Controller) form is a useful normal form of linear systems under the group of linear state coordinate changes and linear state feedbacks. We discuss normal forms of quadratic systems under quadratic change of coordinates and quadratic feedback modulo cubic and higher terms. We discuss the relationship of these normal forms to the Generalized Legendre–Clebsch of Singular Optimal Control.

1. Linear and Quadratic Normal Forms We begin by reviewing the well–known Brunovsky or controller form of a linear system [Br]. For simplicity of exposition we restrict our attention to scalar input systems. Consider a linear system of the form

$$\dot{\xi} = F\,\xi + G\,\mu \tag{1.1}$$

where ξ is n dimensional, μ is one dimensional, and F, G are sized accordingly. Assume that (F,G) is a controllable pair, i.e., the smallest F invariant subspace containing the vector G is all of $\mathbb{R}^{n\times 1}$. We consider transformations of (1.1) under linear change of coordinates.

$$x = \phi^{[1]}(\xi) \tag{1.2}$$

where $\phi^{[1]}(\xi)$ is an invertible linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ and also under linear state feedback

$$\mu = \alpha^{[1]}(\xi) + \beta^{[0]}(\xi)\mu \tag{1.3}$$

where $\alpha^{[1]}(\xi)$ is a linear functional and $\beta^{[0]}(\xi)$ a nonzero constant. (We use superscript $[k]$ to denote a matrix valued function each entry of which is a homogeneous polynomial of degree k in its arguments).

The totality of transformations of the form (1.2) and (1.3) is called the linear feedback group. Brunovsky [Br] showed that a controllable linear system (1.1) could always be transformed into the form

$$\dot{x} = A\,x + B\,u \tag{1.4}$$

when

$$A = \begin{pmatrix} 0 & 1 & & \\ & & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{1.5}$$

by some element (1.2,3) of the linear feedback group. The normal form (1.4,5) is unique but the particular transformation that achieves it is not.

If (1.2,3) transforms (1.1) into Brunovsky form then so does

$$x = c\,\phi^{[1]}(\xi) \tag{1.6}$$

$$\mu = c\,\alpha^{[1]}(\xi) + c\,\beta^{[0]}(\xi)\,\mu \tag{1.7}$$

where $c \neq 0$. Conversely all such transformations taking (1.1) into

Brunovsky form differ by (1.6,7) for some $c \neq 0$.

We consider the generalization of this to nonlinear, in particular, quadratic systems of the form

$$\dot{\xi} = f(\xi) + g(\xi)\mu = F\,\xi + G\,\mu + f^{[2]}(\xi) + g^{[1]}(\xi)\mu + O(\xi,\mu)^3 \qquad (1.8)$$

where $O(\xi,\mu)^3$ indicates a quantity that is cubic and higher in (ξ,μ). Again $\xi \in \mathbb{R}^{n\times 1}$, $\mu \in \mathbb{R}^1$ and we assume (F,G) is controllable.

The quadratic feedback group consists of the linear feedback group (1.2,3) plus

quadratic changes of state coordinates.

$$z = x - \phi^{[2]}(x) \qquad (1.9)$$

and quadratic state feedback

$$v = \alpha^{[2]}(x) + \beta^{[1]}(x)\,u. \qquad (1.10)$$

The action of these transformations on quadratic systems (1.8) is considered modulo cubic and higher terms. In [K–K] it is shown that a system such as (1.8) can always be changed by linear (1.2,3) and quadratic (1.9,10) transformations into the form

$$\dot{z} = A\,z + B\,v + \phi^{[2]}(z) + \phi(z,v)^3 \qquad (1.11)$$

where

$$\phi_i^{[2]}(z) = \frac{1}{2}\sum_{j=i+2}^{n} \theta_{ij}\, z_j^2 . \qquad i = 1,\dots, n-1 \qquad (1.12)$$

The transformation taking (1.8) into (1.11) is not unique, there is as before (1.6,7) a real parameter $c \neq 0$ associated with the linear part and also one real parameter associated with the quadratic part. We refer the reader to [K–K] for the full details.

The $\begin{bmatrix} n-1 \\ 2 \end{bmatrix}$ parameters θ_{ij}; $i = j,\ldots, n-1$; $j = i+2,\ldots, n$ are a complete set of invariants of controllable nonlinear systems (1.8) under the quadratic feedback group (1.2,3,9,10).

These parameters can be found in the following fashion. Given (1.8) where (F, G) is a controllable pair, there exists a unique $H \in \mathbb{R}^{1\times n}$ such that

$$H \; F^{k-1} \, G = \begin{cases} 0 & k = 1,\ldots, n-1 \\ 1 & k = n. \end{cases}$$

then

$$\theta_{ij} = H \, F^{i-1} \, [ad^{n-j}(-f)g, \, ad^{n-j+1}(-f)g] \, (0) \tag{1.11}$$

This is easily seen by noting that under the linear change of coordinates H is transformed to the unit vector

$$C = (1 \; 0 \; 0 \ldots 0) \tag{1.12}$$

and

$$\theta_{ij} = C \, A^{i-1} \, [\, ad^{n-j}(-Az - \phi^{[2]}(z)) \, B \, , \; ad^{n-j+1}(-Az - \phi^{[2]}(z))B] \tag{1.13}$$

(0).

2. The Generalized Legendre Clebsch Condition.

Consider the problem of minimizing some function of the final state

$$g_0 \, (x(T)) \tag{2.1a}$$

subject to the dynamics

$$\dot{\xi} = f(\xi) + g(\xi) \, \mu \tag{2.1b}$$

and boundary condition

$$g_i \, (x(0), x(T)) = 0 \qquad \epsilon = 1,\ldots, k. \tag{2.1c}$$

According to the Pontryagin Maximum Principle, if $\xi(t)$ and $\mu(t)$ are optimal then there exists an adjoint variable $\lambda(t) \in \mathbb{R}^{1xn}$ satisfying

(2.2) $$\dot{\lambda} = -\frac{\partial H}{\partial \xi}(\lambda(t), \xi(t), \mu(t))$$

where the Hamiltonian is given by

(2.3) $$H(\lambda, \xi, \mu) = \lambda(f(\xi) + g(\xi)\mu),$$

For all admissible v, $\lambda(t)$, $\xi(t)$ and $\mu(t)$ must satisfy

(2.4) $$H(\lambda(t), \xi(t), \mu(t)) \geq H(\lambda(t), \xi(t), v).$$

In addition, $\lambda(0)$, $\lambda(T)$ must satisfy certain transversality conditions. A triple $\lambda(t)$, $\xi(t)$ and $\mu(t)$ satisfying (2.1b), (2.2) and (2.4) is called an extremal. If $\lambda(t) = \lambda$ and $\xi(t) = \xi$ are given then $\mu(t) = \mu$ is usually found by the conditions

(2.5) $$\frac{\partial H}{\partial \mu}(\lambda, \xi, \mu) = 0$$

(2.6) $$\frac{\partial^2 H}{\partial \mu^2}(\lambda, \xi, \mu) \leq 0$$

which follow from (2.4). The second condition (2.6) is classically known as the Legendre–Clebsch Condition and it helps to distinguish between extremals that are minimizing and those that are maximizing. Unfortunately for systems such as (2.11) which are affine in μ, (2.6) is satisfied with identity. The Hamiltonian H is linear in μ and so

(2.7) $$\frac{\partial H}{\partial \mu}(\lambda, \xi, \mu) = \lambda g(\xi)$$

does not depend on μ. If there is no constraint on the size of the control and (2.7) is not zero then the extremal control is unbounded, i.e., an impulsive control. If there is a control constraint, e.g.

(2.8) $$|\mu| \leq c$$

then the extremal is called bang bang and $\mu(t)$ is given by

$$\mu(t) = c \operatorname{sign} (\lambda(t)\, g(\xi(t))) \tag{2.9}$$

There is the possibility that along an extremal, (2.7) will be zero at an isolated point in time, or at a sequence of isolated times leading to a limiting time or over an interval of time. The isolated times where (2.9) is zero are associated with switching of the sign of the bang bang extremal control according to (2.9).

An extremal over an interval of time where

$$\frac{\partial H}{\partial \mu}(\lambda(t), \xi(t), \mu(t)) = \lambda(t)\, g(\xi(t)) = 0\,, \qquad t \in [t_0, t_1] \tag{2.10}$$

is called a singular extremal. If there is a sequence of times with limiting time where (2.10) holds then we have chattering, i.e. an infinite number of sign switches of the control (according to (2.9)) occur over a finite interval. Chattering frequently occurs at junctions between bang bang and singular extremals.

We focus our attention on singular extremals. From (2.10) one cannot directly determine the extremal $\mu(t)$ as a function of $\lambda(t)$ and $\xi(t)$. If we assume (2.10) holds an $[t_0, t_1]$ and we differentiate with respect to time we obtain

$$\frac{d}{dt}\frac{\partial H}{\partial \mu}(\lambda(t), \xi(t), \mu(t)) = \lambda(t)[f, g](\xi(t)) = 0 \tag{2.11}$$

and

$$\frac{d^2}{dt^2}\frac{\partial H}{\partial \mu}(\lambda(t), \xi(t), \mu(t)) = \lambda(t)[f[f, g]](\xi(t)) + \lambda(t)[g[f, g]](\xi(t))\, \mu(t) = 0 \tag{2.12}$$

or $[t_0, t_1]$

The first of these quations (2.11) is no help in determining $\mu(t)$ but the second does determine $\mu(t)$ when

$$\frac{\partial}{\partial\mu}\frac{d^2}{dt^2}\frac{\partial H}{\partial\mu}(\lambda(t), \xi(t), \mu) = \lambda(t)\,[g[f, g]](\xi(t)) \neq 0 \tag{2.13}$$

But even when it holds, (2.13) does not distinguish between minimizing and maximizing extremals. To remedy this situation, Kelly [K] introduced a generalization of the Legendre–Clebsch Condition, namely, that when the Legendre–Clebsch Condition (2.6) holds with identity over an interval [to, t1] then the minimizing arcs must have (2.13) nonnegative. Of course (2.13) could be identically zero on $[t_0, t_1]$ and then one must keep on differentializing (2.10) until μ explicitly appears.

The generalized Legendre–Clebsch Conditions of Kelley–Kopp–Moyer [K–K–M] asserts that the first time μ explicitly appears, it will be in an even time derivative of H_μ and its coefficient must have the right sign for the extremal to be minimizing. In other words, if along a minimizing extremal

$$\frac{\partial}{\partial\mu}\left[\frac{d}{dt}\right]^j \frac{\partial H}{\partial\mu}(\lambda(t), \xi(t), \mu) = 0 \tag{2.14}$$

for $t \in [t_0, t_1]$ and $j = 0, 2, 4..., 2k-2$ then (2.14) also holds for $j = 1,...,2k-1$ and to be minimizing,

$$(-1)^k \frac{\partial}{\partial\mu}\left[\frac{d}{dt}\right]^{2k} \frac{\partial H}{\partial\mu}(\lambda(t), \xi(t), \mu) \leq 0. \tag{2.15}$$

The multiple input version of this result can be found in Goh [G]. These results can be proved rigorously using the High Order Maximum Principle of Krener [Kr]

It is straightforward calculation from (2.16) and (2.2) to show that (2.14, 15) are equivalent to

$$\lambda(t)\,[g\,,\,ad^{j}(f)\,g]\,(\xi(t)) = 0 \qquad (2.16)$$

for $t \in [t_0, t_1]$ and $j = 1, 3, \ldots, 2k-3$ and

$$(-1)^{k}\,\lambda(t)\,[g,\,ad^{2k-1}(f)\,g](\xi(t)) \leq 0. \qquad (2.17)$$

Moreover, using the skew–symmetry and Jacobi identities of the Lie bracket, it follows that (2.16, 17) are equivalent to (2.16,18) when

$$\lambda(t)\,[\,ad^{k-1}(f)g,\,ad^{k}(f)g] \geq 0. \qquad (2.18)$$

is exactly the expression arising in the parameters of the quadratic normal form (1.13) when $\lambda(t) = CA^{i-1}$. A similar result holds in the multiple input case.

3. Conclusion We have shown that there is a close connection between the invariants of quadratic systems under quadratic change of coordinates and quadratic state feedback and the generalized Legendre–Clebsch Condition, a quadratic necessary condition of optimal control. We have not satisfactorily explained why this is the case and direct the interested reader to the paper of B. Bonnard [Bo] describing his work with I. Kupka, which sheds considerable light on this question.

[Bo] Bo th co

[Br] Br Ky

[K-K] Ka an UC

[Ke] Ke AL

[K-K-M] Kel Top 63-

[Go] Gol mul

[Kr] Kre app 256

REFERENCES

[Bo] Bonnard, B., Feedback equivalence for nonlinear systems and the time optimal control problem. To appear in SIAM J. on control and Opt.

[Br] Brunovsky, P., A classification of linear controllable systems, Kybernetika, 6, 173-183 (1970).

[K-K] Kang, W. and A.J. Krener, Extended quadratic controller from any dynamic feedback linearization of nonlinear systems, preprint, UC Davis, 1990.

[Ke] Kelley, H.J., A second variation test for singular extremals, AIAA J., 1380-1382 (1964).

[K-K-M] Kelley, H.J., R.E. Kopp and H.G. Moyer, Singular extremals in Topics in Optimization, G. Leitmann, ed., Academic Press, NY, 63-101, 1967.

[Go] Goh, B.S., Necessary conditions for singular extremals involving multiple control variables, SIAM J. Con. 4, 716-731, (1966).

[Kr] Krener, A.J., The high order maximal principle and its applications to singular extremals, SIAM J. Con. and Opt., 15, 256-293, 1977.

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