Controller and Observer design for cubic systems

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Abstract

In this article we study the design of controllers and observers for nonlinear systems. The approach is that adopted in Karahan[8]. The truncated Taylor series of the system are used as an approximation and the design is made for the truncated series. The flight control system studied in Garrard-Jordan[1] is used as an example.

1 Introduction.

Most real systems are nonlinear. Nonetheless, it is quite common to design controllers and observers for the linear approximation of a system around a reference point. The higher order terms are thus ignored following the assumption that they are negligable when the system is in a state close to the reference point chosen for the linearization. The purpose of this article is to go two steps further into the approximation of a nonlinear system by taking into account not only its linear approximation but also the quadratic and cubic terms which appear in its Taylor series. Higher order terms (quartic...) could also be considered but, since they bring little improvement to the cubic approximation while adding a lot to the computationnal burden, they will be left aside in this study. Section 2 describes the class of systems to which the design procedures presented in this article can be applied as well as the necessary tools and related concepts used. Section 3 presents the design methodology. It shows how the design of a quadratic-cubic controller or observer can be achieved in two successive steps, one for the quadratic and one for the cubic terms, or in one unique step. Differences between the two designs are examined. Section 4 applies the above procedures to an example provided by Garrard-Jordan[1]. It shows how a nonlinear controller can greatly improve the ability of an aircraft to recover from a stall.

2 Preliminaries.

The nonlinear systems which will be dealt with in this article are of the general form :

$$\left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x).u \\ y & = & h(x) \end{array} \right. \qquad (1)$$

where the dimensions of x, u and y are n, m and p respectively. Moreover, $f(x)$, $g(x)$ and $h(x)$ are nonlinear functions of x. Without lost of generality, we can assume that $f(0) = 0$, $g(0) = 0$ and $h(0) = 0$. Note that the input enters the system linearly and that the system is 'strictly proper' in the sense that the input does not appear in the output equation. We then consider the Taylor series of system (1) around the reference point 0. System (1) is then approximated up to the third order as :

$$\left\{ \begin{array}{rcl} \dot{x} & = & A.x + B.u + f^{[2]}(x) + g^{[1]}(x).u + f^{[3]}(x) + g^{[2]}(x).u \\ y & = & C.x + h^{[2]}(x) + h^{[3]}(x) \end{array} \right. \qquad (2)$$

where $f^{[2]}(x)$ and $f^{[3]}(x)$ are n-dimensionnal polynomial vector fields of order two and three in the components of x, $h^{[2]}(x)$ and $h^{[3]}(x)$ are p-dimensionnal polynomial vector fields of order two and three in the components of x and $g^{[1]}(x)$ and $g^{[2]}(x)$ are $n \times m$-dimensionnal polynomial matrix fields of order one and two in the components of x.

Let us first introduce briefly the normal forms for nonlinear systems as defined in Krener[4]. The controller normal form is :

$$\left\{ \begin{array}{rcl} \dot{x} & = & A.x + B.u + B.\alpha(x) + B.\beta(x).u \\ y & = & \gamma(x) \end{array} \right. \qquad (3)$$

where $\alpha(x)$ is an m-dimensionnal vector field, $\beta(x)$ an $m \times m$-dimensionnal matrix field and $\gamma(x)$ a p-dimensionnal vector field. $\alpha(x)$, $\beta(x)$ and $\gamma(x)$ are nonlinear The observer normal form is :

$$\left\{ \begin{array}{rcl} \dot{x} & = & A.x + B.u + \alpha(y) + \beta(y).u \\ y & = & C.x + \gamma(y) \end{array} \right. \qquad (4)$$

where $\alpha(y)$ is an n-dimensionnal vector field, $\beta(y)$ an $n \times m$-dimensionnal matrix field and $\gamma(x)$ a p-dimensionnal vector field.

It can easily be seen that, if a system is in controller normal form (3), its dynamics can easily be linearized by choosing the appropriate feedback law, namely the feedback u should satisfy :

$$u + \alpha(x) + \beta(x).u = v \qquad (5)$$

where v is the reference input.

Similarly, if a system is in observer normal form (4), an observer with linear error dynamics can be found :

$$\dot{\hat{x}} = A.\hat{x} + B.u + \alpha(y) + \beta(y).u + K.[y - C.\hat{x} - \gamma(y)] \tag{6}$$

Just as linear systems can be transformed into controller or observer form through a linear change of coordinates, the possibility of using a nonlinear change of coordinates to transform a nonlinear system into controller or observer normal form has been thoroughly investigated (see for example Krener[2][5], Krener and Respondek[6], Karahan[8]). Note also that transforming a system into controller form through coordinate change is equivalent to linearizing the system through coordinate change and feedback. In the same way, transforming a system into observer form through coordinate change is equivalent to linearizing the equation error through coordinate change and output injection into the observer equation.

Unfortunately, it is not possible to transform any given nonlinear system into controller or observer form but this idea lead to a methodology for the design of controllers and observers for such systems. Namely, we can look for a set of coordinates and a nonlinear feedback (resp. output injection) which will linearize the system (resp. equation error) 'as much as possible' in a certain sense and apply linear design to the resulting 'almost linear' system.

3 Nonlinear Controller and Observer Design

3.1 Controller Design

Let us first examine the design of a quadratic controller for the system (2). Since we are interested only in quadratic design, all terms of order higher than two will be neglected. We will thus consider a quadratic change of coordinates $z = x - \phi^{[2]}(x)$, a feedback law defined by $u + \alpha^{[2]}(x) + \beta^{[1]}(x).u = v$ and a change of coordinates in the output $w = y - \gamma^{[2]}(y)$ (we assume here that a linear change of coordinates and a linear feedback have already been performed to obtain the desired first order dynamics). After the quadratic change of coordinates in the state and output is performed and the feedback added, the system becomes, after neglecting terms of order higher than two :

$$\begin{cases} \dot{z} &= A.z + B.v + f^{[2]}(z) - \bar{f}^{[2]}(z) + \left(g^{[1]}(z) - \bar{g}^{[1]}(z)\right).v \\ w &= C.z + h^{[2]}(z) - \bar{h}^{[2]}(z) \end{cases} \tag{7}$$

where :

$$\begin{cases} \bar{f}^{[2]}(z) = \left[A.z, \phi^{[2]}(z)\right] + B.\alpha^{[2]}(z) \\ \bar{g}^{[1]}(z) = \left.\frac{\partial \phi^{[2]}(z)}{\partial z}\right|_{(z)}.B + B.\beta^{[1]}(z) \\ \bar{h}^{[2]}(z) = -C.\phi^{[2]}(z) + \gamma^{[2]}(C.z) \end{cases} \tag{8}$$

where $[f,g] = \frac{\partial g}{\partial x}.f - \frac{\partial f}{\partial x}.g$ denotes the Lie bracket as defined in Isidori[3] for example. The set of equations :

$$\begin{cases} \bar{f}^{[2]}(z) &= f^{[2]}(z) \\ \bar{g}^{[1]}(z) &= g^{[1]}(z) \\ \bar{h}^{[2]}(z) &= h^{[2]}(z) \end{cases} \tag{9}$$

is called *Second Order Controller Homological Equation.*

Let us call $\Psi^{[2]}$ the operator that maps $(\phi^{[2]}, \alpha^{[2]}, \beta^{[1]}, \gamma^{[2]})$, element of a vector space of dimension $\frac{n^2(n+1)}{2} + \frac{mn(n+1)}{2} + m^2n + \frac{p^2(p+1)}{2}$ to $\left(\bar{f}^{[2]}, \bar{g}^{[1]}, \bar{h}^{[2]}\right)$, element of a vector space of dimension $\frac{n^2(n+1)}{2} + mn^2 + \frac{p^2(p+1)}{2}$. This mapping, which was studied in Karahan[8], cannot be inverted in general. Nonetheless, we would like to find a quadruplet $(\phi^{[2]}, \alpha^{[2]}, \beta^{[1]}, \gamma^{[2]})$ which is as small as possible in L^2-norm and which minimize the $L^{[2]}$ norm of the difference $(f^{[2]}, g^{[1]}, h^{[2]}) - \Psi^{[2]}\left(\phi^{[2]}, \alpha^{[2]}, \beta^{[1]}, \gamma^{[2]}\right)$. This guarantees that, in a coordinate system hopefully not to far from the original one (note that x and z agree at the first order), the system is 'as linear as possible' in the least-square sense. The solution found for $\alpha^{[2]}(x)$ and $\beta^{[1]}(x)$ defines the desired quadratic feedback law.

The design of the cubic controller is similar. After the quadratic feedback given by $\alpha^{[2]}(x)$ and $\beta^{[1]}(x)$ is implemented, the system (2) is updated, yielding a system having the same general form as (2) if terms of order higher than three are ignored. Let us also rename the updated $f^{[2]}$, $g^{[1]}$, $h^{[2]}$, $f^{[3]}$, $g^{[2]}$ and $h^{[3]}$ and call them again $f^{[2]}$, $g^{[1]}$, $h^{[2]}$, $f^{[3]}$, $g^{[2]}$, $h^{[3]}$. Our task now consists in finding a cubic change of coordinates $z = x - \phi^{[3]}(x)$, a feedback law $u + \alpha^{[3]}(x) + \beta^{[2]}(x).u = v$ and a change of coordinates in the output $w = y - \gamma^{[3]}(y)$ to minimize the cubic terms in the new coordinate set (z, w). The system in the new coordinate set and under the feedback law defined above is :

$$\begin{cases} \dot{z} &= A.z + B.v + f^{[2]}(z) + g^{[1]}(z).v + f^{[3]}(z) - \bar{f}^{[3]}(z) + \left(g^{[2]}(z) - \bar{g}^{[2]}(z)\right).v \\ w &= C.z + h^{[2]}(z) + h^{[3]}(z) - \bar{h}^{[3]}(z) \end{cases} \tag{10}$$

where :

$$\begin{cases} \bar{f}^{[3]}(z) = \left[A.z, \phi^{[3]}(z)\right] + B.\alpha^{[3]}(z) \\ \bar{g}^{[2]}(z) = \left.\frac{\partial \phi^{[3]}(z)}{\partial z}\right|_{(z)}.B + B.\beta^{[2]}(z) \\ \bar{h}^{[3]}(z) = -C.\phi^{[3]}(z) + \gamma^{[3]}(C.z) \end{cases} \tag{11}$$

The set of equations :

$$\begin{cases} \bar{f}^{[2]}(z) &= f^{[2]}(z) \\ \bar{g}^{[1]}(z) &= g^{[1]}(z) \\ \bar{h}^{[2]}(z) &= h^{[2]}(z) \end{cases} \tag{12}$$

is naturally called *Third Order Controller Homological Equation*.

Let us call $\Psi^{[3]}$ the operator that maps $(\phi^{[3]},\alpha^{[3]},\beta^{[2]},\gamma^{[3]})$, element of a vector space of dimension $\frac{n^2(n+1)(n+2)}{6}+\frac{mn(n+1)(n+2)}{6}+\frac{m^2n(n+1)}{2}+\frac{p^2(p+1)(p+2)}{6}$ to $\left(\tilde f^{[3]},\tilde g^{[2]},\tilde h^{[3]}\right)$, element of a vector space of dimension $\frac{n^2(n+1)(n+2)}{6}+\frac{mn^2(n+1)}{2}+\frac{p^2(p+1)(p+2)}{6}$. Again this mapping cannot be inverted in general but we can again find a quadruplet $(\phi^{[3]},\alpha^{[3]},\beta^{[2]},\gamma^{[3]})$ which is as small as possible in L^2-norm and which minimize the $L^{[2]}$ norm of the difference $(f^{[3]},g^{[2]},h^{[3]}) - \Psi^{[3]}(\phi^{[3]},\alpha^{[3]},\beta^{[2]},\gamma^{[3]})$. Thus in a coordinate system hopefully not to far from the original one (note that x and z agree at the first and second order), the system is 'as quadratic (since we may not have eliminated all the quadratic terms in the first design step) as possible' in the least-square sense. The solution found for $\alpha^{[3]}(x)$ and $\beta^{[2]}(x)$ then defines the desired cubic feedback law.

This same procedure can be iterated to compute feedback of higher and higher order but, except in marginal cases, it is doubtful that considering higher orders would bring much novelty. Besides, the computationnal cost increases steeply each time higher orders are taken into account.

This procedure of calculating second and third order feedback laws is very easy to derive and implement. Nonetheless, it may not yield the best possible quadratic-cubic feedback law. Indeed, the least-square solution to the equation $\Psi^{[2]}(\phi^{[2]},\alpha^{[2]},\beta^{[1]},\gamma^{[2]}) = \left(\tilde f^{[2]},\tilde g^{[1]},\tilde h^{[2]}\right)$ is generally not unique since any element in the kernel of $\Psi^{[2]}$ may be added to any particular solution $\left(\phi_p^{[2]},\alpha_p^{[2]},\beta_p^{[1]},\gamma_p^{[2]}\right)$ to yield another acceptable solution. It is legitimate to search for the least-square solution which has the smallest $L^{[2]}$ norm since we would like the coordinates (z,w) to be as close as possible to (x,y) and the feedback as small as possible. Such a solution is unique since it is the orthogonal projection of the vector $(0,0,0,0)$ onto the space $\left(\phi_p^{[2]},\alpha_p^{[2]},\beta_p^{[1]},\gamma_p^{[2]}\right) + \mathcal{K}er(\Psi^{[2]})$. Nonetheless, this choice may yield undesirable consequences to the third order equation, preventing some third-order terms to be removed. To ensure that such an unfortunate choice is not made, one might consider solving for the quadratic and cubic feedback laws *simultaneously*. We will thus consider a quadratic-cubic change of coordinates $z = x - \phi^{[2]}(x) - \phi^{[3]}(x)$, a feedback law $u + \alpha^{[2]}(x) + \beta^{[1]}(x).u + \alpha^{[3]}(x) + \beta^{[2]}(x).u = v$ and a change of coordinates in the output $w = y - \gamma^{[2]}(y) - \gamma^{[3]}(y)$ and compute the system in the new coordinates with the feedback. After cancelling all terms of order higher than three, we obtain :

$$\left\{\begin{array}{rcl} \dot z &=& A.z + B.v + f^{[2]}(z) - \tilde f^{[2]}(z) + \left(g^{[1]}(z) - \tilde g^{[1]}(z)\right).v \\ && + f^{[3]}(z) - \tilde f^{[3]}(z) + \left(g^{[2]}(z) - \tilde g^{[2]}(z)\right).v \\ w &=& C.z + h^{[2]}(z) - \tilde h^{[2]}(z) + h^{[3]}(z) - \tilde h^{[3]}(z) \end{array}\right. \tag{13}$$

where :

$$\left\{\begin{array}{rcl} \tilde f^{[2]}(z) &=& [A.z,\phi^{[2]}(z)] + B.\alpha^{[2]}(z) \\ \tilde g^{[1]}(z) &=& \left.\frac{\partial\phi^{[2]}}{\partial z}\right|_{(z)}.B + B.\beta^{[1]}(z) \\ \tilde h^{[2]}(z) &=& -C.\phi^{[2]}(z) + \gamma^{[2]}(C.z) \\ \tilde f^{[3]}(z) &=& [A.z,\phi^{[3]}(z)] + B.\alpha^{[3]}(z) + \left.\frac{\partial\phi^{[2]}}{\partial z}\right|_{(z)}.f^{[2]}(z) - B.\beta^{[1]}(z).\alpha^{[2]}(z) \\ && + \; g^{[1]}(z).\alpha^{[2]}(z) - \left.\frac{\partial\phi^{[2]}}{\partial z}\right|_{(z)}.B.\alpha^{[2]}(z) + \left.\frac{\partial f^{[2]}}{\partial z}\right|_{(z)}.\phi^{[2]}(z) \\ \tilde g^{[2]}(z) &=& \left.\frac{\partial\phi^{[3]}(z)}{\partial z}\right|_{(z)}.B + B.\beta^{[2]}(z) + \left.\frac{\partial\phi^{[2]}}{\partial z}\right|_{z}.g^{[1]}(z) - B.\beta^{[1]}(z).\beta^{[1]}(z) \\ && + \; g^{[1]}(z).\beta^{[1]}(z) - \left.\frac{\partial\phi^{[2]}}{\partial z}\right|_{z}.B.\beta^{[1]}(z) + \left.\frac{\partial g^{[1]}}{\partial z}\right|_{(z)}.\phi^{[2]}(z) \\ \tilde h^{[3]}(z) &=& -C.\phi^{[3]}(z) + \gamma^{[3]}(C.z) + \left.\frac{\partial\gamma^{[2]}}{\partial y}\right|_{(C.z)}.[C.\phi[2](z) + h^{[2]}(z)] \end{array}\right. \tag{14}$$

Just as before, the mapping $\Psi^{[2,3]}$ from $(\phi^{[2]},\alpha^{[2]},\beta^{[1]},\gamma^{[2]},\phi^{[3]},\alpha^{[3]},\beta^{[2]},\gamma^{[3]})$ to $\left(\tilde f^{[2]},\tilde g^{[1]},\tilde h^{[2]},\tilde f^{[3]},\tilde g^{[2]},\tilde h^{[3]}\right)$ can usually not be inverted but a least-square solution to the equation : $\Psi^{[2,3]}(\phi^{[2]},\alpha^{[2]},\beta^{[1]},\gamma^{[2]},\phi^{[3]},\alpha^{[3]},\beta^{[2]},\gamma^{[3]}) = \left(\tilde f^{[2]},\tilde g^{[1]},\tilde h^{[2]},\tilde f^{[3]},\tilde g^{[2]},\tilde h^{[3]}\right)$ can be found. To this particular solution, an element of $\mathcal{K}er(\Psi^{[2,3]})$ can be added so as to minimize the $L^{[2]}$ norm of $(\phi^{[2]},\alpha^{[2]},\beta^{[1]},\gamma^{[2]},\phi^{[3]},\alpha^{[3]},\beta^{[2]},\gamma^{[3]})$. Generally, the quadratic-cubic feedback law found with this one-step method will be different from the one found with the two-step method described above. One might wonder then whether there would be a limit to the quadratic-cubic feedback found using equations involving higher and higher order terms simultaneously.

3.2 Observer Design

Observer design is very similar to controller design as described above except that we would like to put the system into *Observer Normal Form* through a change of coordinates in the state and in the output. As before, we can deal with the quadratic and cubic terms separately, thus solving for the *Second and Third Order Observer Homological Equations* successively or deal with these terms altogether in one equation.

Let us first examine the two-step observer design. For the quadratic part, let us apply a quadratic change of state coordinates $z = x - \phi^{[2]}(x)$ and output $w = y - \gamma^{[2]}(y)$. In the new set of coordinates, the quadratic approximation of the system is :

$$\left\{\begin{array}{rcl} \dot z &=& A.z + B.u + \alpha^{[2]}(w) + \beta^{[1]}(w).u + R_1^{[2]}(z,w) + R_2^{[2]}(z,w).u \\ w &=& C.z + \gamma^{[2]}(z) + R_3^{[2]}(z,w) \end{array}\right. \tag{15}$$

where :

$$\left\{\begin{array}{l} R_1^{[2]}(z,w,u) = f^{[2]}(z) - [A.z, \phi^{[2]}(z)] - \alpha^{[2]}(w) \\ R_2^{[2]}(z,w,u) = g^{[1]}(z) - \frac{\partial \phi^{[2]}}{\partial z}\Big|_{(z)} - \beta^{[1]}(w) \\ R_3^{[2]}(z,w,u) = h^{[2]}(z) + C.\phi^{[2]}(z) - \gamma^{[2]}(w) \end{array}\right. \quad (16)$$

We then build an observer as :

$$\dot{\hat{z}} = A.\hat{z} + B.u + \alpha^{[2]}(w) + \beta^{[1]}(w).u - K.\left(w - C.\hat{z} - \gamma^{[2]}(w)\right) \quad (17)$$

so that the error dynamics are :

$$\left\{\begin{array}{lll} \dot{\epsilon} & = & \dot{z} - \dot{\hat{z}} \\ & = & (A + K.C).\epsilon + \left\{R_1^{[2]}(z,w) + K.R_3^{[2]}(z,w)\right\} + R_2^{[2]}(z,w).u \end{array}\right. \quad (18)$$

As a design procedure, we would like to find a quadraplet $(\phi^{[2]}, \alpha^{[2]}, \beta^{[1]}, \gamma^{[2]})$ which minimizes the nonlinear term in (18) so as to linearize as much as possible the equation error (18). The mapping from $(\phi^{[2]}, \alpha^{[2]}, \beta^{[1]}, \gamma^{[2]})$ of dimension $\frac{n^2(n+1)}{2} + \frac{np(p+1)}{2} + nmp + \frac{p^2(p+1)}{2}$ to $\left(R_1^{[2]} + K.R_3^{[2]}, R_2^{[2]}(z,w)\right)$ of dimension $\frac{n^2(n+1)}{2} + \frac{np(p+1)}{2} + n^2m + nmp$ is usually not invertible and a least-square solution has to be found. Moreover, it is necessary, among all the possible solutions, to search for the one which minimizes the $L^{[2]}$ norm of $\phi^{[2]}$ and $\gamma^{[2]}$. Indeed, the error in the estimation of x is the sum of the error ϵ and the term $\phi^{[2]}(z) - \phi^{[2]}(\hat{z})$. If ϵ converges to zero, so will the error on the state x since $\phi^{[2]}$, as a polynomial, is Lipshitz in any bounded domain around zero. Nonetheless, the smaller the norm of $\phi^{[2]}$, the closer the error on z and x. The third-order observer design proceeds from there. The quadratic change of coordinates in the states and output being performed, one obtains a new system and we would like to put its third order terms into observer form 'as much as possible'. To accomplish this we search for a cubic change of coordinates in the state and in the output, namely $z' = z - \phi^{[3]}(z)$ and $w' = w - \gamma^{[3]}(w)$. This yields to the *Third Order Observer Homological Equation* which again can usually only be solved in the least-square sense. As for the controller case, the observer obtained with this two-step procedure may not be yield the smallest possible cubic terms in the equation error for $z' - \hat{z}'$. A one-step approach is then possible at the cost of a higher computationnal burden.

4 Design of a Nonlinear Flight Control System

The procedure described in the above section has been implemented as a package for $MATLAB^{TM}$[1]. The system which we will use as an example to test it is provided by Garrard and Jordan[1]. In their paper, Garrard and Jordan derive the equation of flight for an F-8 Crusader aircraft and look at the linear, quadratic and cubic terms of the resulting three-dimensionnal system. They design a quadratic-cubic controller using an approach different from the one described above and test the ability of the closed-loop system to recover from a stall condition. The system is modelled as in (2) with :

$$A = \begin{pmatrix} -0.877 & 0 & 1 \\ 0 & 0 & 1 \\ -4.208 & 0 & -0.396 \end{pmatrix} \quad B = \begin{pmatrix} -0.215 \\ 0 \\ -20.967 \end{pmatrix}$$

$$f^{[2]}(z) = \begin{pmatrix} 0.47x_1^2 - 0.088x_1x_3 - 0.019x_2^2 \\ 0 \\ -0.47x_1^2 \end{pmatrix} \quad f^{[3]}(z) = \begin{pmatrix} 3.846x_1^3 - x_1^2x_3 \\ 0 \\ -3.564x_1^3 \end{pmatrix}$$

$$g^{[1]}(z) = 0$$
$$g^{[2]}(z) = 0$$

The state x_1 represents the angle of attack, x_2 the angle between the wing plane and the horizon, x_3 the time-derivative of x_2 and the command is the tail deflection angle. All states are assumed to be measured accurately. The linear feedback is the quadratic regulator with matrices (see Kailath[7]) :

$$Q = \begin{pmatrix} 0.25 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 0.25 \end{pmatrix} \quad R = 1 \quad (19)$$

and its gain is F = [-0.0526 0.5000 0.5210] as in Garrard and Jordan.

The quadratic-cubic feedback law obtained with the two-step process is defined by :

$$\begin{array}{lll} \alpha^{[2]}(x) & = & -0.1328x_1^2 - 0.2543x_1x_2 - 0.0592x_1x_3 - 0.1258x_2^2 + 0.0022x_2x_3 + 0.0485x_3^2 \\ \beta^{[1]}(x) & = & -0.5222x_1 - 0.4898x_2 + 0.0045x_3 \\ \gamma^{[2]}(y) & = & 0 \\ \alpha^{[3]}(x) & = & -2.5415x_1^3 - 5.4462x_1^2x_2 - 1.8393x_1^2x_3 - 1.7854x_1x_2^2 - 0.3386x_1x_2x_3 \\ & + & 1.1664x_1x_3^2 + 0.7169x_2^3 + 0.0186x_2^2x_3 - 0.0194x_2x_3^2 - 0.0114x_3^3 \\ \beta^{[2]}(x) & = & -9.9424x_1^2 - 3.4396x_1x_2 + 0.1854x_1x_3 + 2.2053x_2^2 + 0.0361x_2x_3 - 0.0010x_3^2 \\ \gamma^{[3]}(y) & = & 0 \end{array} \quad (20)$$

[1] MATLAB is a TradeMark of The MathWorks

The quadratic-cubic feedback law obtained with the one-step simultaneous design is defined by :

$$
\begin{aligned}
\alpha^{[2]}(x) &= -0.0911x_1^2 - 0.1459x_1x_2 - 0.0413x_1x_3 + 0.0367x_2^2 + 0.0054x_2x_3 + 0.0114x_3^2 \\
\beta^{[1]}(x) &= -0.1991x_1 - 0.0307x_2 + 0.0021x_3 \\
\gamma^{[2]}(y) &= 0 \\
\alpha^{[3]}(x) &= -2.7364x_1^3 - 5.9165x_1^2x_2 - 1.9424x_1^2x_3 - 1.9879x_1x_2^2 - 0.3701x_1x_2x_3 \\
&+ 1.2538x_1x_3^2 + 0.7819x_2^3 + 0.0210x_2^2x_3 - 0.0045x_2x_3^2 - 0.0129x_3^3 \\
\beta^{[2]}(x) &= -10.8158x_1^2 - 4.0173x_1x_2 + 0.2059x_1x_3 + 2.0770x_2^2 + 0.0411x_2x_3 - 0.0027x_3^2 \\
\gamma^{[3]}(y) &= 0
\end{aligned}
\tag{21}
$$

The feedback law used by Garrard and Jordan is :

$$u = -0.053x_1 + 0.5x_2 + 0.521x_3 + 0.04x_1^2 - 0.048x_1x_2 + 0.374x_1^3 - 0.312x_1^2x_2 \tag{22}$$

Let us now examine a few simulations based upon the examples presented in Garrard and Jordan[1]. They show the ability of an F-8 airgraft to recover from a stall. Under the flight conditions taken in these simulations, the airplane stalls when the angle of attack is above 23.5 degrees. The controller is designed so as to limit the time during which the aircraft is in a stall condition and thus have an altitude loss as minimal as possible. The figures show the evolution of the angle of attack and control effort under various initial conditions. The equation which is integrated is the third-order approximation of the system given by (2) and the data at the beginning of this section with a feedback defined as in (20), (21) and (22). The initial conditions are of the form $(x_1(0), 0, 0)$ where $x_1(0)$ is the initial angle of attack. Simulations with $x_1(0) = 22.9$, 25.0, 29.0 and 30.1 degrees are presented. The solid curve represents the two-step controller, the dashed line the one-step controller (it follows the two-step controller so closely you might not even see it) and the dotted line the controller proposed by Garrard and Jordan. Note that the latter is unstable and therefore not represented in the last set of figures ($x_1(0) = 30.1$ degrees). The angle of attack and the control effort are in degrees.

The linear controller becomes unstable when the initial angle of attack reaches 28.3 degrees, the quadratic one when $x_1(0)$ reaches 28.8 degrees and the cubic one when $x_1(0)$ reaches 30.2 degrees. The domain of stability is thus increased when quadratic and cubic terms are taken into account into the feedback design. Also, even when the linear feedback law is stable, it does not perform as well as the quadratic or even the quadratic-cubic controller as shown in the last two figures for $x_1(0) = 28.5$ degrees (solid line : linear controller, dashed line : two-step design quadratic controller, two-step design quadratic-cubic controller).

As far as the feedback law derived by Garrard and Jordan is concerned, we could not reproduce their results exactly because of a lack of information from their article. It seems to be a little less stable and performant than our design

but on the other hand, it requires less control effort. Nonetheless, both designs are fundamentally different in their application. In fact, Garrard and Jordan's design does not include any reference input, thus making it inapropriate for any kind of tracking problem. Our design on the other hand takes into account reference input terms, thus making it a more flexible tool.

References

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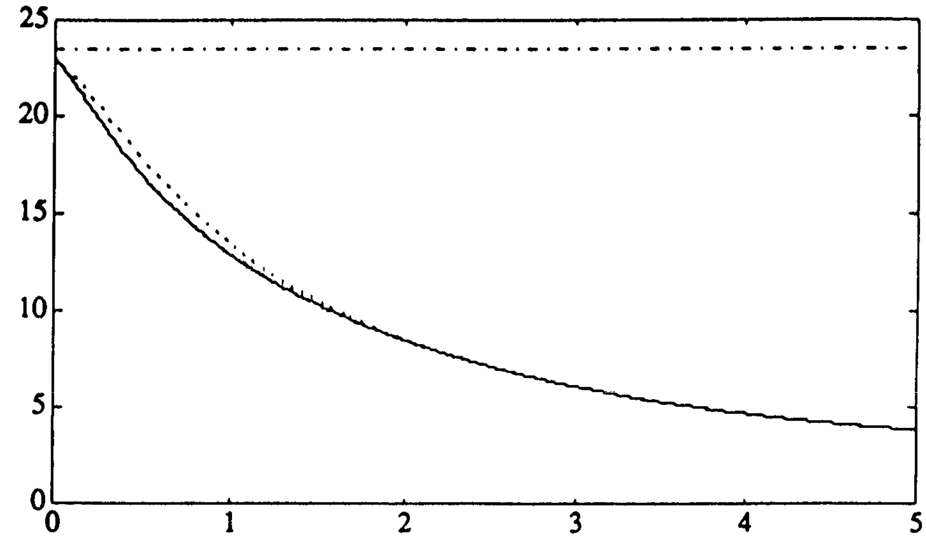
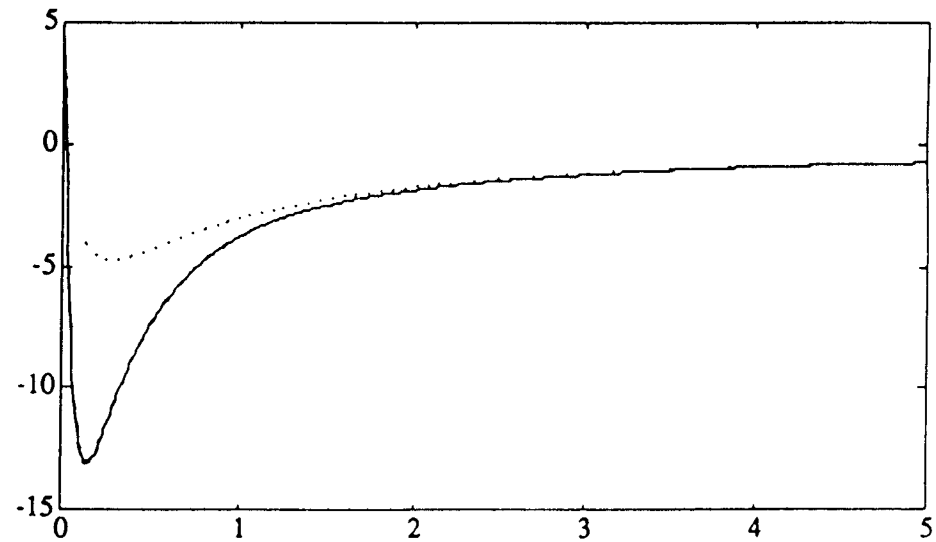
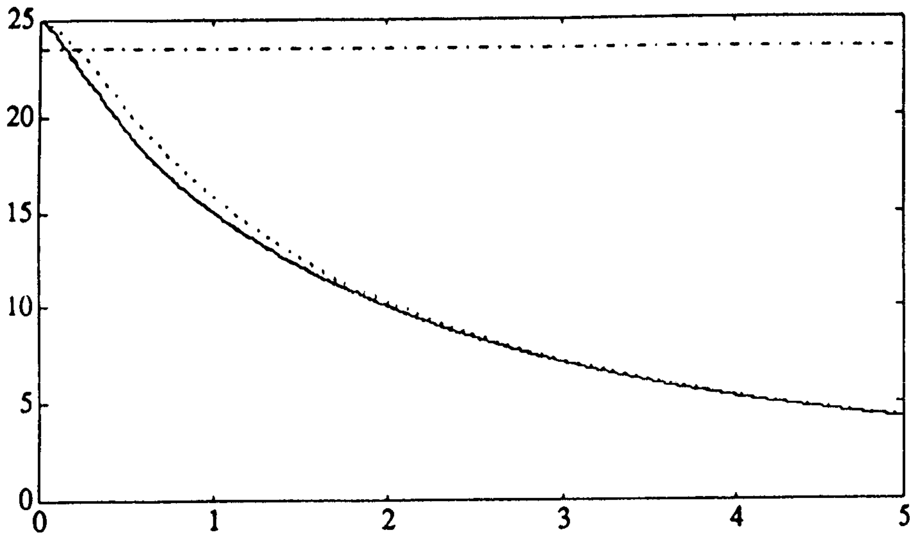


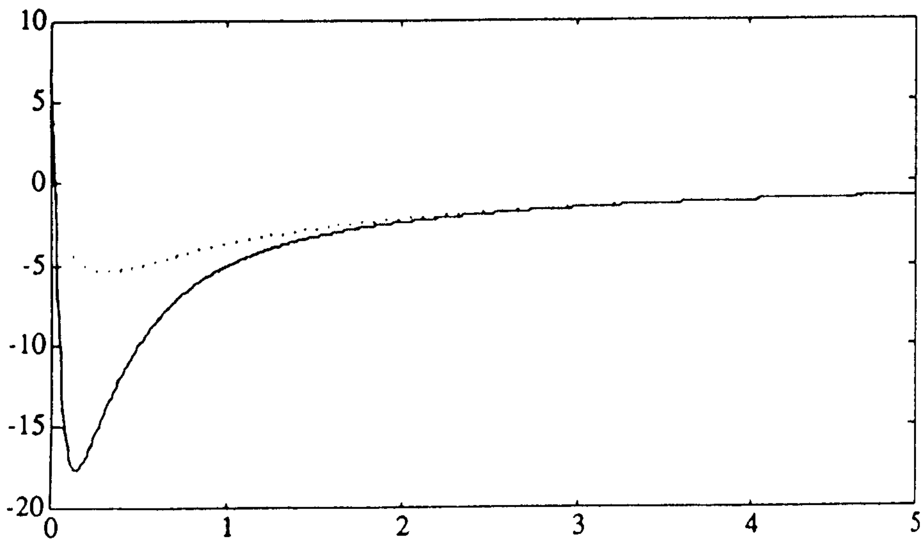
Angle of attack in degrees. Initial condition : x1= 22.9°



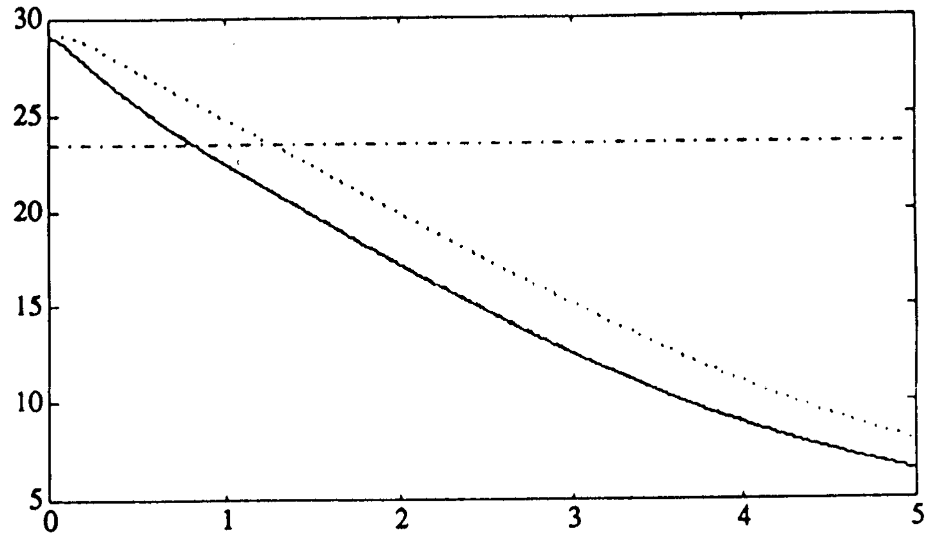
Control effort in degrees. Initial condition : x1= 22.9°



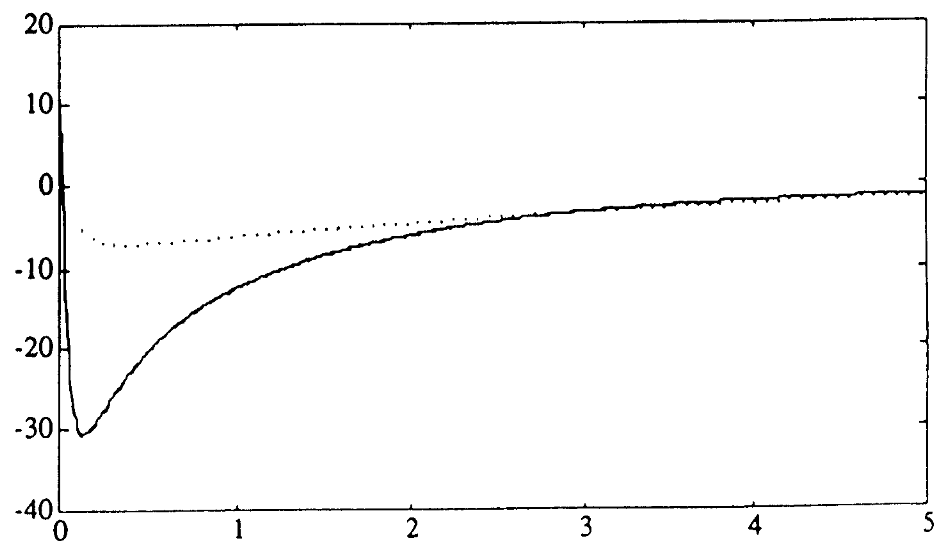
Angle of attack in degrees. Initial condition : x1= 25.0°



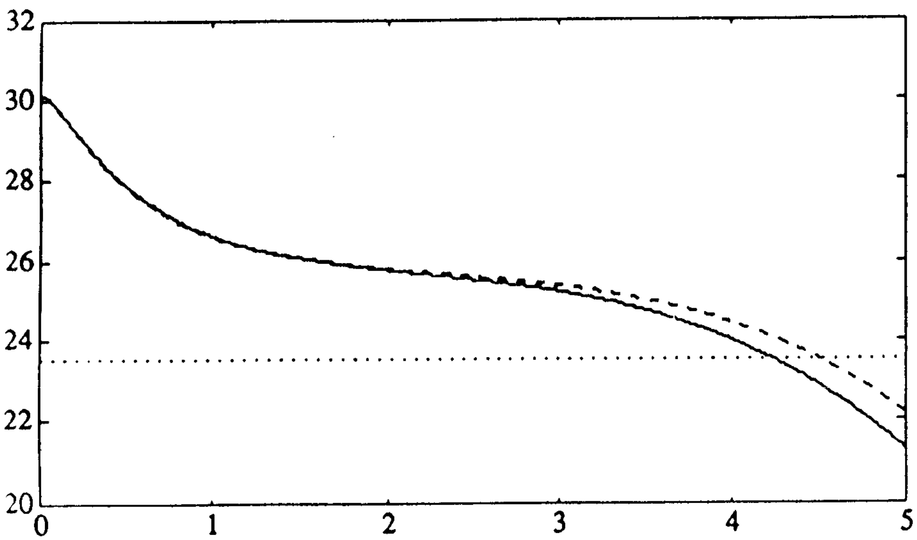
Control effort in degrees. Initial condition : x1= 25.0°



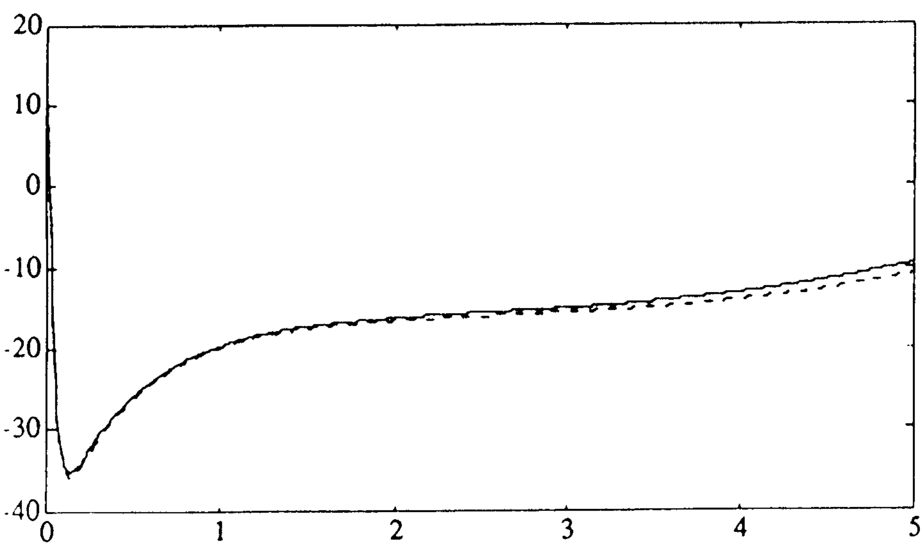
Angle of attack in degrees. Initial condition : x1= 29.0°



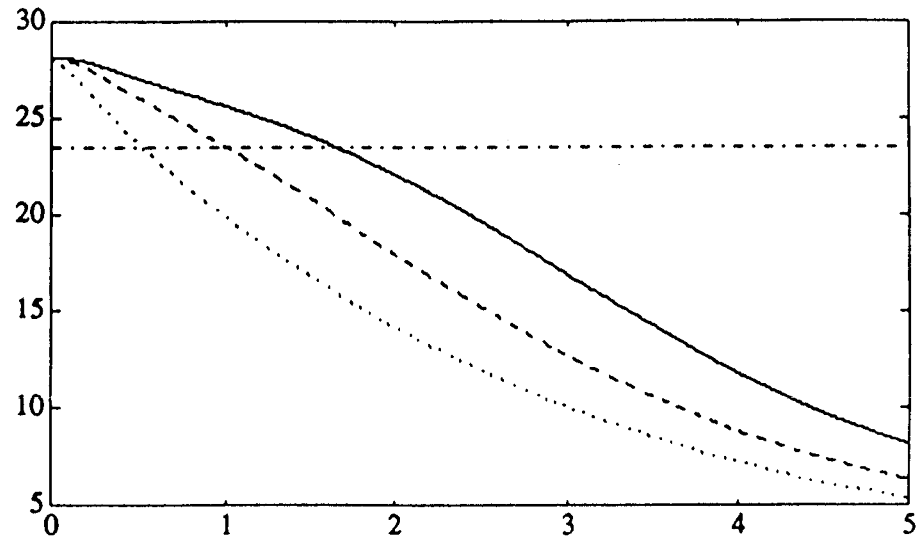
Control effort in degrees. Initial condition : x1= 29.0°



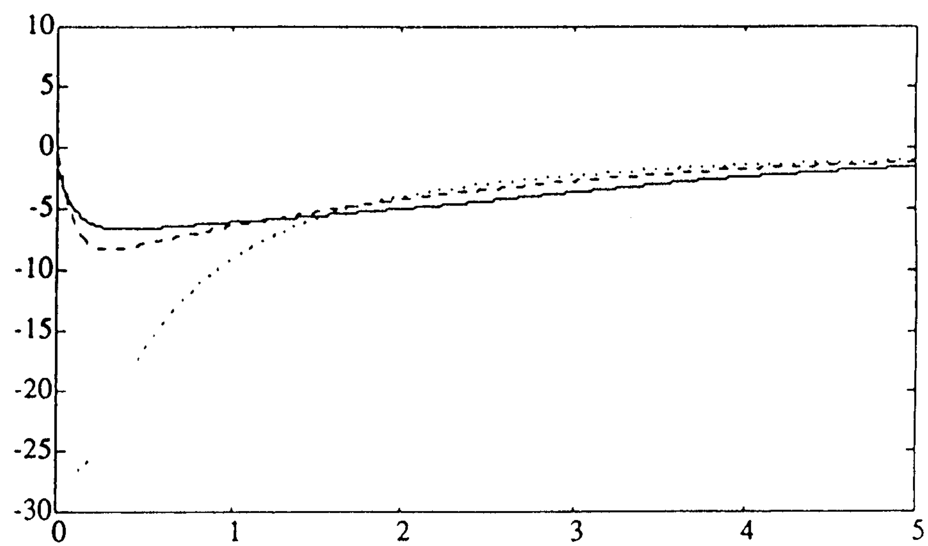
Angle of attack in degrees. Initial condition : x1= 30.1°



Control effort in degrees. Initial condition : x1= 30.1°



Angle of attack in degrees. Initial condition : x1= 28.0°



Control effort in degrees. Initial condition : x1 = 28.0°