Lecture Notes in Control and Information Sciences

Edited by M. Thoma and A. Wyner

165

G. Jacob,
F. Lamnabhi-Lagarrigue (Eds.)

Algebraic Computing in Control

Proceedings of the First European Conference
Paris, March 13–15, 1991

Springer-Verlag
Berlin Heidelberg New York
London Paris Tokyo
Hong Kong Barcelona Budapest

Poincaré's Linearization Method

Applied to the Design of Nonlinear Compensators

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1. Introduction Over the past three years with funding from AFOSR, we have developed methodology for the design of compensators for highly nonlinear plants. This project has led to the development of POINCARE, a MATLAB based package for the design of nonlinear feedback control laws and nonlinear state observers. POINCARE has generally proven to be a substantial improvement over standard linear designs when applied to systems with significant nonlinearities. This paper describes the software package, POINCARE, as well as related research on several important theoretical issues. The lon term goal of our efforts is the development of methodologies for the design of compensat for plants exhibiting highly nonlinear behavior. These include high performance and V/STOL aircraft, high performance robots, advanced jet engines and nonlinear chemical processes. The existing methods of linear compensator design, the classical Bode plot, phase gain margins, etc., the semiclassical LQG approaches and the more modern H^∞ LQG/LTR approaches are not totally adequate to design controllers for plants which

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$$\dot{x} = f(x) \tag{2.1}$$

around a critical point x

$$z = \phi(x). \tag{2.2}$$

multiaxis, highly coupled nonlinearities, significant parameter variations and unmodeled dynamics, when these plants must perform well over a wide range of operating regimes. More sophisticated control methodologies are needed to obtain the full range of performance that such devices and processes are capable of achieving.

That is not to say that these linear methodologies are of no use when it comes to nonlinear design. On the contrary, they have proven highly successful in various linear applications and we believe that any broad and successful approach to nonlinear design must precede from this solid foundation and reduce to it when the plant is reasonably close to linear.

It is for this reason that our approach is based on linearizing the system model to the greatest extent possible by various coordinate changes, state feedback, input–output injection, etc. as available and then proceeding with tried and trustworthy linear methodologies on the linearized model. The resulting compensator is then transformed back to the original coordinates for implementation. The technique of H. Poincaré is a conceptually and numerically simple method for the term by term linearization of nonlinear dynamics. Its generalization to control problems forms the basis of our approach.

2. The Linearization Technique of Poincaré and Approximate Normal Forms of Control Systems.

Poincaré [Ar, G–H] considered the problem of linearizing an n–dimensional vector field i.e. an ordinary differential equation.

$$\dot{x} = f(x) \tag{2.1}$$

around a critical point x^0, $f(x^0) = 0$ by a change of state coordinates

$$z = \phi(x). \tag{2.2}$$

Rather than attack the problem in its entirety, he sought a term by term Taylor series solution. Suppose the critical point is $x^0 = 0$ and (2.1) can be expanded as

$$\dot{x} = Ax + f^{[2]}(x) + O(x)^3. \tag{2.3}$$

when $f^{[2]}(x)$ is an n–dimensional vector field each component of which is a homogeneous polynomial of degree 2 in the coordinates $x_1,\dots x_n$ and $O(x)^3$ denotes cubic and higher terms.

We seek a change of coordinates

$$z = x - \phi^{[2]}(x) \tag{2.4}$$

where again $\phi^{[2]}$ is n–dimensional vector field of homogeneous polynomials of degree 2 which carries (2.3) into

$$\dot{z} = Az \tag{2.5}$$

A straightforward calculation shows that (2.4) transforms (2.3) into

$$\dot{z} = Az + (f^{[2]}(x) - [Ax,\phi^{[2]}(x)]) + O(x)^3 \tag{2.6}$$

Hence (2.3) can be linearized to degree 2 iff the so–called Homological Equations [Ar]

$$f^{[2]}(x) = [Ax,\phi^{[2]}(x)] \tag{2.7}$$

are satisfied. Notice that Lie bracketing by Ax is a linear mapping from quadratic ve

fields $\phi^{[2]}(x)$ into quadratic vect
linear equations relating the $n^2(n$
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A are λ_i; $1 \le i \le n$, then the eige
vector fields by Ax are $\lambda_i + \lambda_j -$
solvable for any $f^{[2]}(x)$ iff the so–
$\lambda_i + \lambda_j - \lambda_k \neq 0$, for $1 \le i \le j \le n$,

If a system is linearized to

(2.8) $\dot{x} = Ax + f^{[\rho]}(x) +$

then we can seek a degree ρ chan

(2.9) $z = x - \phi^{[\rho]}(x)$

carrying (2.8) into (2.5).

As before this is possible fo

(2.10) $f^{[\rho]}(x) = [Ax, \phi^{[\rho]}(x$

are solvable. This is a larger linea
for $1 \le i_1 \le i_2 \le \dots \le i_\rho \le n$ and 1
conditions.

Now consider a smooth non

(2.11a) $\dot{x} = f(x) + g(x)\,u$

(2.11b) $y = h(x)$

fields $\phi^{[2]}(x)$ into quadratic vector fields $f^{[2]}(x)$. Therefore (2.7) is a square system of linear equations relating the $n^2(n+1)/2$ coefficients of the quadratic monomials making up the components of $\phi^{[2]}(x)$ to those of $f^{[2]}(x)$. Poincaré noted that if the eigenvalues of A are λ_i; $1 \le i \le n$, then the eigenvalues of the linear operation of bracketing quadratic vector fields by Ax are $\lambda_i + \lambda_j - \lambda_k$ where $1 \le i \le j \le n$; $1 \le k \le n$. Hence (2.7) is solvable for any $f^{[2]}(x)$ iff the so–called nonresonance condition holds, i.e. $\lambda_i + \lambda_j - \lambda_k \neq 0$, for $1 \le i \le j \le n$, $1 \le k \le n$.

If a system is linearized to degree $\rho - 1$, i.e.

$$\dot{x} = Ax + f^{[\rho]}(x) + O(x)^{\rho+1} \tag{2.8}$$

then we can seek a degree ρ change of coordinates

$$z = x - \phi^{[\rho]}(x) \tag{2.9}$$

carrying (2.8) into (2.5).

As before this is possible for all $f^{[\rho]}$ iff the homological equations

$$f^{[\rho]}(x) = [Ax, \phi^{[\rho]}(x)] \tag{2.10}$$

are solvable. This is a larger linear system with eigenvalues $\lambda_{i_1} + \ldots + \lambda_{i_\rho} - \lambda_k$ for $1 \le i_1 \le i_2 \le \ldots \le i_\rho \le n$ and $1 \le k \le n$, hence leads to additional nonresonance conditions.

Now consider a smooth nonlinear system of the form

$$\dot{x} = f(x) + g(x)\,u \tag{2.11a}$$

$$y = h(x) \tag{2.11b}$$

where u, x and y are m, n and p dimensional. For convenience we assume that control enters the dynamics linearly and does not appear explicitly in the output. These restrictions can easily by relaxed. Hunt–Su [H–S], Jakubczyck–Respondek [J–R] and others have considered the question of when such a system can be transformed by a change of state coordinates (2.2) into a system of the form

$$z = Az + B(\alpha(x) + \beta(x)\,u). \tag{2.12}$$

Such a system can be linearized by state feedback

$$\alpha(x) + \beta(x)\,u = v \tag{2.13a}$$

or, equivalently,

$$\alpha(\phi^{-1}(x)) + \beta(\phi^{-1}(z)\,u = v. \tag{2.13b}$$

Such systems are said to be feedback linearizable. The above authors have derived the integrability conditions for the system of partial differential equations that must be satisfied by ϕ, α and β.

Practical applications of this approach predate the general theoretical development. It is at the core of many nonlinear control schemes such as the Total Automatic Flight Control System (TAFCOS) developed by G. Meyer and associates [M–C] at NASA Ames Research Center and the Resolved Acceleration approach to robot control of Luh, Walker and Paul [L–W–P]. Fortunately, for many mechanical systems the PDE's for ϕ, α and are integrable and a solution is obvious. The solution is not unique. However, such systems are exceptional, for a generic system with 2m < n, the PDE's are not integrable. Moreover, even if they are, a solution is not always easy to find.

By utilizing Poincaré's

obtaining only approximate so

Suppose that system dynamics

(2.14a) $\dot{x} = Ax + Bu +$

and the output map is similar

(2.14b) $y = Cx + h^{[2]}($

We consider the effect

space (2.4) and output space

(2.15) $w = y - \gamma^{[2]}(y)$

The resulting system is

(2.16a) $z = Az + Bu$

$+ f^{[2]}(x) + g^{[1}$

(2.16b) $w = Cz$

$+ h^{[2]}(x) + C\,\phi$

By adding to and subt

(2.17) $u + \alpha^{[2]}(x) + $

we obtain the system

(2.17a) $\dot{z} = Az + Bv +$

(2.17b) $w = Cz + R_2^{[2]}$

By utilizing Poincaré's method we can bypass these difficulties at the cost of obtaining only approximate solutions to the PDE's irrespective of their integrability. Suppose that system dynamics (2.11) is expanded as follows

$$\text{(2.14a)} \qquad \dot{x} = Ax + Bu + f^{[2]}(x) + g^{[1]}(x)\, u + O(x,u)^3$$

and the output map is similarly expanded

$$\text{(2.14b)} \qquad y = Cx + h^{[2]}(x) + O(x)^3.$$

We consider the effect on (2.14) of quadratic changes of coordinates in the state space (2.4) and output space

$$\text{(2.15)} \qquad w = y - \gamma^{[2]}(y).$$

The resulting system is

$$\text{(2.16a)} \qquad z = Az + Bu$$
$$+\ f^{[2]}(x) + g^{[1]}(x)u - [Ax + Bu, \phi^{[2]}(x)] + O(x,u)^3$$
$$\text{(2.16b)} \qquad w = Cz$$
$$+\ h^{[2]}(x) + C\,\phi^{[2]}(x) - \gamma^{[2]}(y) + O(x)^3$$

By adding to and subtracting from (2.16a) the quadratic feedback expression

$$\text{(2.17)} \qquad u + \alpha^{[2]}(x) + \beta^{[1]}(x)\, u = v$$

we obtain the system

$$\text{(2.17a)} \qquad \dot{z} = Az + Bv + R_1^{[2]}(x,u) + O(x,u)^3$$
$$\text{(2.17b)} \qquad w = Cz + R_2^{[2]}(x) + O(x)^3$$

where the quadratic error terms are given by

(2.18a) $$R_1^{[2]}(x,u) = f^{[2]}(x) + g^{[1]}(x)u - [Ax + Bu, \phi^{[2]}(x)] - B(\alpha^{[2]}(x) + \beta^{[1]}(x)u)$$

(2.18b) $$R_2^{[2]}(x,y) = h^{[2]}(x) + C\,\phi^{[2]}(x) - \gamma^{[2]}(y) + O(x)^3.$$

We refer to (2.18a) with $R_1^{[2]}(x,u) = 0$ as the controller homological equation of degree 2. It is a system of $n^2(n+1)/2 + m\,n^2$ linear equations in $n^2(n+1)/2 + m\,n(n+1)/2 + m^2 n$ unknown coefficients of $\phi^{[2]}$, $\alpha^{[2]}$ and $\beta^{[1]}$. If there exists a solution the controller homological equation of degree two then we say that the nonlinear system (2.14) <u>is feedback linearizable to degree two</u>.

It is easy to see that if a system is feedback linearizable then it is feedback linearizable to degree two. Given a system that is feedback linearizable to degree two, o can ask whether it is feedback linearizable to degree three, etc. This suggests an obvio question. If a real analytic system (2.11) is feedback linearizable to arbitrary degree, is feedback linearizable? This may or may not be a hard question to resolve. The corresponding question for the Poincaré's problem is quite difficult, see Arnold [Ar]. the point of view of design of nonlinear compensators, the question is somewhat irrelev as it would be extremely impractical to solve the sequence of higher degree controller homological equations. The size of these equations grows exponentially in the degree

Typically the controller homological equations of degree 2 are not solvable and one is forced to seek an approximate solution, e.g. in the least square sense. Moreover many applications such as tracking, it is desirable not only to have $R_1^{[2]}(x,u)$ small also $R_2^{[2]}(x)$. If we set the left sides of (2.18) to zero we obtain a system of $n^2(n +$ $+ m^2 n + p\,n(n+1)/2$ in $n^2(n+1)/2 + m\,n(n+1)/2 + m^2 n + p^2(p+1)/2$ unk

coefficients of $\phi^{[2]}$, $\alpha^{[2]}$, $\beta^{[1]}$ an
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feedback gain F is found. Supp

(2.19a) $u = F\,x + \bar{u}$

is chosen because of the desirabl

(2.19b) $\dot{x} = \bar{A}\,x + B\,\bar{u}$

where

(2.19c) $\bar{A} = A + B\,F$

POINCARE allows one t
quadratic part of the system (2.1

(2.20a) $\dot{\lambda} = \bar{A}x + \bar{f}^{[2]}(x)$

where

(2.20b) $\bar{f}^{[2]}(x) = f^{[2]}(x) +$

If the stability of (2.20a)
interest then one need not go an
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find such a quadratic feedback b
(2.4) and feedback

(2.21) $\bar{u} + \alpha^{[2]}(x) + \beta^{[1]}$

on (2.20a).

(2.22a) $\dot{z} = \bar{A}z + Bv + \bar{R}$

coefficients of $\phi^{[2]}$, $\alpha^{[2]}$, $\beta^{[1]}$ and $\gamma^{[2]}$. Again, typically these equations do not have an exact solution and an approximate solution must be found instead.

The current version of POINCARE takes as data the linear parts A, B, C and the quadratic parts $f^{[2]}$, $g^{[1]}$, and $h^{[2]}$ if the system (2.14). If the desired goal is the design of a stabilizing state feedback control law, then, using the pole placement or LQR algorithm of A. Laub's Control Systems Toolbox for MATLAB, a stabilizing linear feedback gain F is found. Suppose the feedback

(2.19a) $$u = F\,x + \bar{u}$$

is chosen because of the desirable close loop stability of

(2.19b) $$\dot{x} = \bar{A}\,x + B\,\bar{u}$$

where

(2.19c) $$\bar{A} = A + B\,F$$

POINCARE allows one to simulate the linear feedback (2.19a) applied to the quadratic part of the system (2.14)

(2.20a) $$\dot{\lambda} = \bar{A}x + \bar{f}^{[2]}(x) + B\bar{u} + g^{[1]}(x)\,\bar{u}$$

where

(2.20b) $$\bar{f}^{[2]}(x) = f^{[2]}(x) + g^{[1]}(x)\,Fx.$$

If the stability of (2.20a) is satisfactory with $\bar{u} = 0$ over the range of the x of interest then one need not go any further. On the other hand if $\bar{f}^{[2]}$ has degraded the performance then one may seek an additional quadratic feedback to improve things. We find such a quadratic feedback by considering the effect of a quadratic coordinate change (2.4) and feedback

(2.21) $$\bar{u} + \alpha^{[2]}(x) + \beta^{[1]}(x)\,\bar{u} = v$$

on (2.20a).

(2.22a) $$\dot{z} = \bar{A}z + Bv + \bar{R}^{[2]}(x, \bar{u}) + O(x, \bar{u})^3$$

where

(2.22b) $\bar{R}^{[2]}(x, \bar{u}) = \bar{f}^{[2]}(x) + g^{[1]}(x)\,\bar{u}$

$$- [\bar{A}\,x + B\,\bar{u}\,,\, \phi^{[2]}(x)] - B(\alpha^{[2]}(x) + \beta^{[1]}(x)\,\bar{u}).$$

If we assume that the open loop control $v = 0$ then $\bar{u} = O(x^2)$ and (2.22a,b) become

(2.22c) $\dot{z} = \bar{A}\,z + \bar{R}_1{}^{[2]}(x,0) + O(x)^3$

(2.22d) $\bar{R}_1{}^{[2]}(x, 0) = \bar{f}^{[2]}(x) - [\bar{A}x\,,\, \phi^{[2]}] - B\,\alpha^{[2]}(x).$

If we have chosen the linear feedback given F so that the eigenvalues of $\bar{A}$ are resonant at degree two then there exist $\phi^{[2]}$ and $\alpha^{[2]}$ so that (2.22d) is zero. In fact there are m n(n + 1/2 linearly independent solutions parameterized by α. Which should be chosen? It is a trade off between the size of the quadratic feedback (2.21) as measured by $\alpha^{[2]}$ and the size of the quadratic change of coordinates (2.4) as measured by $\phi^{[2]}$. One approach is to make $\phi^{[2]}$ and $\alpha^{[2]}$ as small as possible in a least squares sense. This implies a choice of a metric on the space of coefficients of $\phi^{[2]}$ and $\alpha^{[2]}$.

An obvious choice is to take the standard inner product on the space of coefficients of $\phi^{[2]}$, $\alpha^{[2]}$. This is not as naive as it sounds because POINCARE has previously asked us for unit lengths of the components of x, u and y and has scaled the equation (2.14) accordingly. One can make more sophisticated choices as we shall describe later.

Given a choice of metric, POINCARE uses MATLAB's singular value decomposition (SVD) algorithm to find $\phi^{[2]}$ and $\alpha^{[2]}$ satisfying (2.22). It then uses the quadratic feedback

(2.23a) $\bar{u} = -\,\alpha^{[2]}(x)$

to satisfy (2.14). The linear (2.19a) and quadratic (2.23a) feedback yield the total feedback.

(2.23b) $u = Fx - \alpha^{[2]}x.$

Notice it does not depend on $\phi^{[2]}$. The closed loop dynamics in the original x coordinates has a quadratic component,

(2.24) $$\dot{x} = \bar{A}\, x + \bar{f}^{[2]}(x) - B\, \alpha^{[2]}(x) + O(x)^3$$

but in the transformed z coordinates (2.4) it does not,

(2.25) $$\dot{z} = \bar{A}\, z + O(z^3).$$

Because $\phi^{[2]}(x)$ and $\alpha^{[1]}(x)$ have been chosen to be small, one expects the performance of (2.24) to be similar to that of (2.25). The result should be an improvement over the system with only linear feedback,

(2.26) $$\dot{x} = \bar{A}\, x + \bar{f}^{[2]}(x).$$

To verify this, POINCARE simulates the response of three systems to a nonzero initial condition. It graphs the norm of the state $\|x(t)\|$ and the components of the output $y_i(t)$, $i = i,...,p$ of (2.14b) for the linear system with linear feedback,

(2.27) $$\dot{x} = \bar{A}x,$$

the quadratic system with linear feedback, (2.26) and the quadratic system with linear and quadratic feedback (2.24). The first is the linear ideal closed loop system, the second is the result of a standard design obtained by ignoring nonlinear terms and the third is the POINCARE approach. Generally in these simulations we have found that the performance of POINCARE system closely approximates the linear ideal, while the standard design is somewhat different and usually worse. By performance we mean things like settling times and basis of attraction. We present some simulations in Section 3.

The above problem is somewhat simplistic. Usually one wants to exercise open loop control on the stabilized system. For example one might desire the output $y(t)$ of the system (2.14) to track a reference signal $r(t)$. As before the first step is to do a linear design. One might try to use static state feedback and reference signal feedforward of the form

(2.28) $$u = F\, x + \bar{u}$$

to achieve the desired tracking for the linear part of (2.14). The input $\bar{u}(t)$ is a linear functional of the current and past values of the reference signal $\{r(s) : s \leq t\}$ typically obtained by passing the reference signal through an approximate right inverse of the linear system

(2.29a) $\dot{x} = \bar{A}\, x + B\, \bar{u}$

(2.29b) $y = C\, x$

We denote this dependency of $\bar{u}$ by $\bar{u}(t) = \bar{u}(t;r)$. We can use this as a basis for a nonlinear design via POINCARE.

We start by applying the feedback (2.28) to the nonlinear system (2.14) and obtain

(2.30a) $\dot{x} = \bar{A}\, x + B\, \bar{u} + \bar{f}^{[2]}(x) + \bar{g}^{[1]}(x)\bar{u} + O(x,\bar{u})^3$

(2.30b) $y = C\, x + h^{[2]}(x) + O(x)^3$

where

(2.30c) $\bar{g}^{[1]}(x) = g^{[1]}(x)\, G.$

We then apply the quadratic changes of coordinates and feedback (2.4), (2.15) and (2.21) to obtain

(2.31a) $\dot{z} = \bar{A}\, z + B\bar{u} + \bar{R}_1^{[2]}(x,\bar{u}) + O(x,\bar{u})^3$

(2.31b) $y = C\, z + \bar{R}_2^{[2]}(x,y) + O(x)^3$

where the quadratic residual $\bar{R}_1^{[2]}$ is as before (2.18a) after substitution of $\bar{A}$, B, $\bar{f}^{[2]}$ for A, B, $f^{[2]}$, $g^{[1]}$ and $\bar{R}_2^{[2]} = R_2^{[2]}$.

We wish to make the quadratic residuals in (2.31) as small as possible while at the same time using as keeping $\phi^{[2]}$, $\alpha^{[2]}$, $\beta^{[1]}$, $\gamma^{[2]}$ as small as possible. To make this we must define metrics on the space of coefficients of $(\bar{R}_1^{[2]}, \bar{R}_2^{[2]})$ and the space of coefficients of $(\phi^{[2]}, \alpha^{[2]}, \beta^{[1]}, \gamma^{[2]})$. An obvious choice is to use the standard metr these spaces and again this is not a bad choice provided one has already scaled the equations (2.14). More sophisticated metrics can be obtained as follows. Choose m on the state, input and output spaces defined by positive definite matrices M_s, M_i

$$\|x\|_s^2 = x^* M_s x$$
$$\|u\|_i^2 = u^* M_i u$$
$$\|y\|_o^2 = y^* M_o y$$

These could be the same metrics as those used in the LQR algorithm. Also choose a probability density $\rho(x,u)$ on the input cross state space. This density describes the expected operating regime; the likelihood that the system will be at state x with input u.

Then we define

(2.32a) $$\left\| \begin{matrix} R_1^{[2]} \\ R_2^{[2]} \end{matrix} \right\|^2 = \int\int(\|R_1(x,u)\|_s^2 + \|R_2(x,C_x)\|_o)\,\rho(x,u)\,dx\,du$$

and

(2.32b) $$\left\| \begin{matrix} \phi^{[2]} \\ \alpha^{[2]} \\ \beta^{[2]} \\ \gamma^{[2]} \end{matrix} \right\|^2 = \int\int\|\phi^{[2]}(x)\|_s^2 + \|\alpha^{[2]}(x) + \beta^{[1]}(x)u\|^2 + \|\gamma^{[2]}(x)\|_o^2)\,\rho(x,u)\,dx\,du$$

At present the only choices of metric available in POINCARE are the standard metrics on the spaces of coefficients and the metrics defined by (2.31a,b) when M_s, M_i and M_o are identity matrices and $\rho(x,u)$ is a Gaussian density with zero mean and unit covariance. These two choices are actually quite similar. We expect to add other options in the future.

POINCARE uses MATLAB's SVD algorithm to find $\phi^{[2]}$, $\alpha^{[2]}$, $\beta^{[1]}$, $\gamma^{[2]}$. This leads to the control law with linear part (2.28a) and quadratic part

(2.33a) $$\bar{u} + \alpha^{[2]}(x) + \beta^{[1]}(x)\bar{u} = \bar{u}(t;r)^{[2]}\,\bar{u}(t;\gamma^{[2]}(r))$$

By neglecting cubic terms, we have the control law

(2.33b) $$\begin{aligned} u &= Fx + Gr \\ &\quad - G(\alpha^{[2]}(x) + \beta^{[1]}(x)r + \bar{u}(t;\gamma^{[2]}(r)) \end{aligned}$$

and the closed loop system

$$\dot{x} = \bar{A}x + Br + \bar{f}^{[2]}(x) + \bar{g}^{[1]}(x)r - G(\alpha^{[2]}(x) + \beta^{[1]}(x)r + u(t;\gamma^{[2]}(r)) \tag{2.34a}$$

$$y = Cx + h^{[2]}(x). \tag{2.34b}$$

This (2.34) should be compared with the ideal linear system (2.29) and the standard approach of applying the linear law (2.28) to the linear and quadratic parts of (2.14), i.e

$$\dot{x} = \bar{A}x + B\bar{u}(t;r) + \bar{f}^{[2]}(x) + \bar{g}^{[1]}(x)\bar{u}(t;r) \tag{2.35a}$$

$$y = Cx + h^{[2]}(x). \tag{2.35b}$$

Notice that the linear parts of all three systems are the same. POINCARE simulates a compares the behavior of the three systems (2.29), (2.35) and (2.34).

Of course in many situations full state observations are not possible and hence must use a filter or observer to estimate the state from the past observations. If one to ignore the nonlinearities of the system (2.14) one could design a linear observer of th form

$$\dot{\hat{x}} = (A + KC)\hat{x} + Bu - Ky \tag{2.36a}$$

with linear error dynamics, $\tilde{x} = x - \hat{x}$, given by

$$\dot{\tilde{x}} = (A + KC)\tilde{x}. \tag{2.36b}$$

If the linear part of (2.14) is observable then one can choose K to set the spectrum $A + KC$ arbitrarily. One can also choose K by using a Kalman filtering formula MATLAB's Control Systems Toolbox has algorithms for either approach.

Of course if we use the linear observer (2.36a) to establish the state of the li quadratic part of (2.14) then the observer error dynamic has quadratic terms

$$\dot{\tilde{x}} = (A + KC)\tilde{x} + f^{[2]}(x) + g^{[1]}(x)u + Kh^{[2]}(x). \tag{2.36c}$$

These extra terms may sufficiently degrade the performance of the observer (2.36a it is unacceptable. POINCARE allows one to ameliorate the effect of the quadratic nonlinearities in the following fashion.

Consider once again the effect of quadratic changes of state (2.4) and output (2.15) coordinates on the quadratic part of (2.14). The result is (2.16). If we add and subtract an input–output injection term

$$\alpha^{[2]}(y) + \beta^{[1]}(y)u$$

to and from (2.16a) we obtain

(2.37a) $$\dot{z} = Az + Bu + \alpha^{[2]}(y) + \beta^{[1]}(y)u + R_3^{[2]}(x,u,y) + O(x,u)^3$$

(2.37b) $$w = Cz + R_4^{[2]}(x,y) + O(x,u)^3$$

where

(2.38a) $$R_3^{[2]}(x,u,y) = f^{[2]}(x) - g^{[1]}(x)u - [Ax + Bu, \phi^{[2]}(x)] - \alpha^{[2]}(y) - \beta^{[1]}(y)u,$$

(2.38b) $$R_4^{[2]}(x,y) = h^{[2]}(x) + C\phi^{[2]}(x) - \gamma^{[2]}(y).$$

We wish to emphasize that the current $\phi^{[2]}, \alpha^{[2]}, \beta^{[1]}, \gamma^{[2]}$ under discussion are different from those employed previously. In particular the dimensions and arguments of $\alpha^{[2]}$ and $\beta^{[1]}$ have changed. Previously $\alpha^{[2]}$ and $\beta^{[1]}$ were $m \times 1$ and $m \times m$ valued functions of y. Later on, when it will be necessary to distinguish between these, we shall denote the previous ones with a subscript $_c$ as in $\phi_c^{[2]}, \alpha_c^{[2]}, \beta_c^{[1]}, \gamma_c^{[2]}$ since they are used for the design of control laws. The current ones shall be denoted with a subscript $_o$, as in $\phi_o^{[2]}$, $\alpha_o^{[2]}, \beta_o^{[1]}, \gamma_o^{[2]}$ since they will be used for the design of observers.

If we set $R_3^{[2]}$ and $R_4^{[2]}$ to zero then the system (22.37) is said to be in observer form of degree 2 and (2.38) are the observer homological equations of degree two.

Krener–Respondek [K–R], Bestle–Zeitz [B–Z] and others have looked at the question of when a nonlinear system can be transformed exactly into observer form by a smooth change of coordinates. For most systems, this requires checking a large set of integrability conditions and finding the change of coordinates by solving a set of PDE's. Phelps [P] has developed a method that greatly simplifies this, but it is still a formidable task.

Instead, we have approached the question term by term using the linearization technique of Poincaré.

We can design an observer for (2.37) with quadratic input–output injection

$$\dot{\hat{z}} = (A + KC)\hat{z} + Bu - Ky + \alpha^{[2]}(y) + \beta^{[1]}(y)u + K\gamma^{[2]}(y) \tag{2.39a}$$

then the error $\tilde{z} = z - \hat{z}$ satisfies

$$\dot{\tilde{z}} = (A + KC)\tilde{z} + R_3^{[2]}(x,u,y) + KR_4^{[2]}(x,y) + O(x,\hat{z},u)^3. \tag{2.39b}$$

POINCARE uses MATLAB's SVD algorithm to choose $\phi^{[2]}$, $\alpha^{[2]}$, $\beta^{[1]}$, $\gamma^{[2]}$ to minimize the quadratic part of (2.37b),

$$\| R_3^{[2]} + KR_4^{[2]} \|^2 \tag{2.40}$$

The metric used is either the standard one on the space of coefficients or

$$\| R_3^2 + KR_4^{[2]} \|^2 = \int\int \|R_3^{[2]}(x,u,Cx) + KR_4^{[2]}(x,Cx)\|_s^2 \, \rho(x,u)\, dx\, du$$

where M_s is the identity and $\rho(x,u)$ is a Gaussian of zero mean and unit covariance. Notice that the linear observer gain K enters the quantity (2.40) to be minimized and hence influences the solution.

Of course, the error dynamics (2.39b) is nearly linear in the transformed coordinates. The implementation of the observer (2.39a) should be done in the original coordinates so we define $\hat{x}$ by

$$\begin{aligned} \dot{\hat{x}} &= (A + KC)\hat{x} + Bu - Ky \\ &\quad + [(A + KC)\hat{x} + Bu, \phi^{[2]}(\hat{x})] \\ &\quad + \alpha^{[2]}(y) + \beta^{[1]}(y) + K\gamma^{[2]}(y) - \frac{\partial \phi^{[2]}}{\partial \hat{x}}(\hat{x})\, Ky\,. \end{aligned} \tag{2.41a}$$

The error $\tilde{x} = x - \hat{x}$ dynamics is given by

$$\begin{aligned} \dot{\tilde{x}} &= (A + KC)\tilde{x} + f^{[2]}(x) + g^{[1]}(x)u + Kh^{[2]}(x) \\ &\quad - [(A + KC)\hat{x} + Bu, \phi^{[2]}(x)] - \alpha^{[2]}(y) - \beta^{[1]}(y)u - K\gamma^{[2]}(y) + O(\end{aligned} \tag{2.41b}$$

Then to $O(x,\hat{x})$, (2.39a) and (2.41a) are related by (2.41c) $\hat{z} = \hat{x} - \phi^{[2]}(\hat{x})$. Notice that the linear part of the observer (2.41a) is the same as the linear observer (2.36a). The extra quadratic terms in (2.41a) correct in part for the quadratic part of the original system (2.14). The linear part of the observer (2.41a) is stable by proper choice of K but the quadratic terms may destabilize it. Of course one might expect this since the quadratic terms of (2.14) are also destabilizing. But as a system with inputs u and y and output $\hat{x}$, (2.41a) it is not BIBO stable in contrast to the linear observer (2.36a).

POINCARE simulates three pairs of systems and observers to compare their performance. The first pair is a system consisting of the linear part of (2.14) and the linear observer (2.36a). This is the linear ideal and provides a benchmark for the other two. The second pair is the system consisting of the linear and quadratic parts of (2.14) and the linear observer (2.36a). This is a standard approach. The third is the system consisting of the linear and quadratic parts of (2.14) with the linear and quadratic observer (2.41a). This is the POINCARE approach. The three pairs of systems and observers are excited by initial condition errors and various kinds of inputs, u(t). The norms of the errors in x coordinates are graphed with respect to time. Generally one finds that the errors of the POINCARE approach are smaller and closer to those of the linear ideal than those of the standard approach, although this is not always the case. Simulations are given in the next section.

It is also possible to combine the two halves of POINCARE to obtain an input–output driven nonlinear compensator. The stable estimate of the nonlinear observer (2.41a) us fed into a linear and quadratic feedback control law. Once again the POINCARE approach can be simulated and compared with an ideal linear system with linear observer and linear state estimate feedback and a standard approach of a linear and quadratic system with linear observer and linear stable estimate feedback.

These three pairs of systems are as follows. The linear ideal is

$$\dot{x} = Ax + Bu \tag{2.42a}$$

$$\dot{\hat{x}} = (A + KC)\hat{x} + Bu - Ky \tag{2.42b}$$

$$y = Cx \tag{2.42c}$$

$$u = F\hat{x} + v \tag{2.42d}$$

The standard approach yields

$$\dot{x} = Ax + Bu + f^{[2]}(x) + g^{[1]}(x)u \tag{2.43a}$$

$$\dot{\hat{x}} = (A + KC)\hat{x} + Bu - Ky \tag{2.43b}$$

$$y = Cx + h^{[2]}(x) \tag{2.43c}$$

$$u = F\hat{x} + v \tag{2.43d}$$

The POINCARE approach yields

$$\dot{x} = Ax + Bu + f^{[2]}(x) + g^{[1]}(x)u \tag{2.44a}$$

$$\begin{aligned}\dot{\hat{x}} = {} & (A + KC)\hat{x} + Bu - Ky \\ & + [(A + KC)\hat{x} + Bu, \phi_o^{[2]}(\hat{x})] \\ & + \alpha_o^{[2]}(y) + \beta_o^{[1]}(y) + K\gamma_o^{[2]}(y) - \frac{\partial \phi_o^{[2]}(\hat{x})}{\partial \hat{x}} Ky\end{aligned} \tag{2.44b}$$

$$y = Cx + h^{[2]}(x) \tag{2.44c}$$

$$u = F\hat{x} + v - \alpha_c^{[2]}(\hat{x}) - \beta_c^{[1]}(\hat{x})v \tag{2.44d}$$

Equation (2.44d) should be plugged into (2.44b) before computing the bracket.

Simulations can be found in the next section.

3. Simulations
SESSION 1 Controller/Observer Example

```
        _ QUADRATIC APPROXIMATION _
        _  Controller - Observer  _

----- What is your Macintosh running under? -----

      1) Finder
      2) Multifinder


 Select a menu number:  1
----- Enter mode -----

      1) Novice
      2) Expert


 Select a menu number:  1
----- M A I N   M E N U -----

      1) Help
      2) Enter nonlinear system (plant and output equations)
      3) Select the type of problem to be solved
      4) Simulate and plot results
      5) Quit


 Select a menu number: 1
- QC is a menu driven script file for 2nd order approximate
observers
- for nonlinear control systems.  Your control system should
be in
- the following format:
-
-  dx               [2]      [1]              3
-  -- = Ax + Bu + f  (x) + g  (x)u + O(x, u)
-  dt
-              [2]
-  y = Cx + h  (x)
```

_
_ where A is the nxn plant matrix, B is the nxm input vector,
f[2] is
_ the second degree part of the vector field, and g[1] is the
first
_ degree part of the input vector. All of these terms should
be
_ obtained by the user from the Taylor expansion of the
control
_ system at the nominal operating point 0.
_ We compute the quadratic observer-controller parameters
and plot
_ the system against linear and quadratic observer-controller
pairs.
_ The corresponding function modules are documented on-line
under
_ HELP QCRUN. For further details on the individual
subroutines
_ and function programs called by the routines, use the help
utility
_ in MATLAB.

__ Glossary of Q.C. Functions __

QCHELP introduces the quadratic approximation program.
QCSETUP inputs in the system functions F, G, f2, g1.
QCCOMP solves the homological equations for _, _ and _.
_ QCSCALE rescales the system and control matrices.
_ QCBIGMX sets up the linear equations in the
coefficients.
_ QCSVD solves these equations using the SVD algorithm.
QCFDBK gets the user-specified feedback.
QCGLOB sets up global symbolic variables for controller.
QCSIM controls the simulation subroutines.
QCINIT sets the initial conditions for the ODE's.
QCCNTRL creates the control M-file with several options.
QCSOLVE solves the ODE's for the system and observers.
QCPLOT plots the solutions with the following options:
_ QCPHASE provides the phase plots;
_ QCEXT extends the solutions;
_ QCTIME provides the time plots.

```
 QCRUN is this help display.

----- M A I N   M E N U -----

     1) Help
     2) Enter nonlinear system (plant and output equations)
     3) Select the type of problem to be solved
     4) Simulate and plot results
     5) Quit

 Select a menu number: 2

----- Input one of the following: -----

     1) Enter data manually
     2) Enter filename for loading data


 Select a menu number: 2

  Enter filename: example_1

Dimensions of the system:

[n, m] =
     3     1

Homogeneous order of system:
nchoose2 =
     6

 Linear plant matrix, A:
______________________________

     0     1     0
     0     0     1
    -3    -2    -1

 Eigenvalues of the open-loop plant:
__________________________________________

 -1.2757
  0.1378 + 1.5273i
```

0.1378 - 1.5273i

Second order part of the plant, f2:

1	0	-1	0	0	0
0	-1	0	1	0	1
0	0	0	0	1	0

Constant part of the input vector, B:

0
0
1

First order part of the input vector, g1:

1	0	0
0	0	0
0	1	0

Constant part of the output vector, C:

1	0	0

Second order part of the output vector, h2:

0	0	0	0	0	0

```
Scale factors of the states, x1 through x3:
xscale =
     1     1     1

Scale factor of the input, u1:
uscale =
     1

Scale factor of the output y1:
yscale =
```

1

```
----- M A I N   M E N U -----

      1) Help
      2) Enter nonlinear system (plant and output equations)
      3) Select the type of problem to be solved
      4) Simulate and plot results
      5) Quit


 Select a menu number:  3
----- Select the type of problem to be solved -----

      1) Quadratic Controller design (full observability)
      2) Quadratic Observer design
      3) Quadratic Controller-Quadratic Observer design
      4) Get more information on the above choices


 Select a menu number: 3
CLOSED LOOP FEEDBACK DESIGN FOR LINEAR PART
                OF THE PLANT

----- Input one of the following: -----

      1) Specify closed loop eigenvalues
      2) Design linear quadratic regulator with state
weighing
      3) Design linear quadratic regulator with output
weighing


 Select a menu number: 2

Quadratic regulator will minimize:


    _
    _ integral(x'Qx + u'Ru)dt


 _
 _ Enter 3x3 <sym. pos. semi-def.> matrix, Q:
 _
```

```
___> _ [1 0 0;0 1 0;0 0 1]
_ Enter 1x1 <sym. pos. def.> matrix, R:
_
___> 1
```

Closed loop eigenvalues:

```
-1.3294
-0.4975 + 1.4599i
-0.4975 - 1.4599i
```

Gain matrix F that places the poles of A + BF at above

```
-0.1623   -1.7015   -1.3244
```

Calculating scaled variables according to the scale factors...

_ We construct a large linear system L X = B in the $O(x,u)^2$

_ coefficients of the homological equations. This system has

_ ROWS: $n^2 (n+1)/2 + mn^2$

_ COLUMNS: $n^2 (n+1)/2 + mn(n+1)/2 + m^2 n$.

_ In general, the column rank is deficient and the solution is

_ overdetermined.

_ We use the SVD algorithm to get the "nearest" possible solut

----- Please choose the method for minimization: -----

1) Identity norm
2) Norm weighed by a normal distribution

Select a menu number: 2

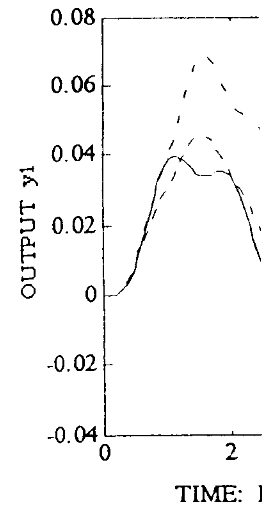
Solving for the coordinate change and feedback...

```
Q u a d r a t i c   C o n t r o l l e r   P r o b l e m :

----- Please choose a menu item -----

1) Display coordinate change and feedback
2) Show remainder between the actual and closest linearizable system
3) Exit to main menu


 Select a menu number: 1
    [2]                                                [2]
  phi in second degree coordinate change z = x - phi_con(x):
  ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~

phi_con[2](1) = -0.0076646*x(1)*x(1) - 0.15515*x(1)*x(2) +
                0.67221*x(1)*x(3) - 0.023882*x(2)*x(2) +
                0.1054*x(2)*x(3) + 0.046875*x(3)*x(3)
phi_con[2](2) = -2.9808*x(1)*x(1) - 1.1353*x(1)*x(2)
                + 0.15515*x(1)*x(3) - 0.5626*x(2)*x(2) +
                0.016217*x(2)*x(3) - 0.043276*x(3)*x(3)
phi_con[2](3) = -0.49063*x(1)*x(1)-5.5871*x(1)*x(2) -
                1.2223*x(1)*x(3) - 2.1954*x(2)*x(2) -
                0.68737*x(2)*x(3) -0.7826*x(3)*x(3)

                                          [2]
  Second degree part of feedback:  alpha_con
  ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~

alpha_con[2](1) = +2.8083*x(1)*x(1)+4.7978*x(1)*x(2)-
1.4355*x(1)*x(3)+7.6045*x(2)*x(2)+3.1339*x(2)*x(3)-
0.1493*x(3)*x(3)
                                        [1]
  First degree part of feedback:  beta_con
  ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~

beta_con[1](1, 1) = +1.2223*x(1)+1.6874*x(2)+1.5652*x(3)

 Q u a d r a t i c   C o n t r o l l e r   P r o b l e m :

----- Please choose a menu item -----

     1) Display coordinate change and feedback
     2) Show remainder between the actual and closest
linearizable system
     3) Exit to main menu
```

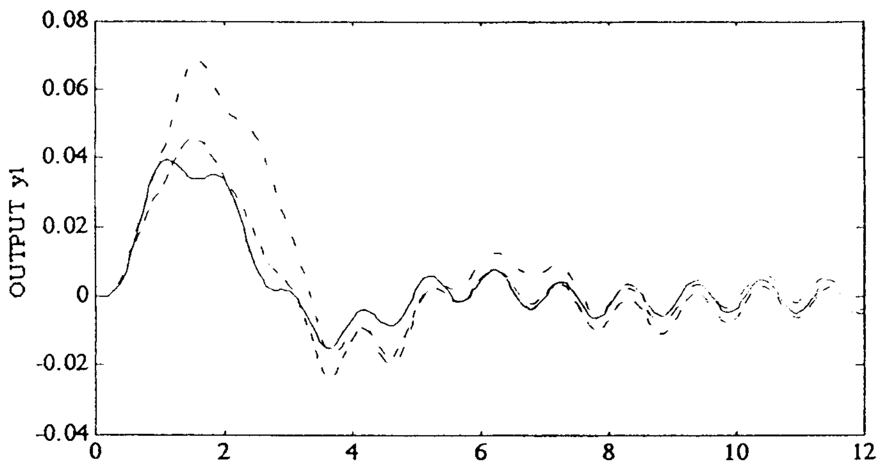
```
Select a menu number: 2

                                               [2]
Second degree part in the remainder:  R(x)
~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~

R[2](1) = -0.01731*x(1)*x(1)-0.15516*x(1)*x(3)-
0.01731*x(2)*x(2)-0.016222*x(2)*x(3)+0.069239*x(3)*x(3)
R[2](2) = 0
R[2](3) = 0

                                             [1]
First degree part in the remainder: R (i'th col. of R[1]
multiplies u(i))
~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
~~~~~~~~~~~~

R[1](1, 1) = +0.32779*x(1)-0.1054*x(2)-0.09375*x(3)
R[1](2, 1) = -0.15515*x(1)-0.016217*x(2)+0.086552*x(3)
R[1](3, 1) = 0

                                                         [2]
Second degree part in the remainder of the output:  H(x)
~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~

H[2](1) = -0.0076646*x(1)*x(1)


CLOSED LOOP FEEDBACK DESIGN FOR LINEAR PART OF THE
            O-B-S-E-R-V-E-R

----- Input one of the following: -----

      1) Specify observer eigenvalues
      2) Design linear Kalman filter


 Select a menu number: 2

Kalman filter will minimize the error covariance.

_
_ Enter 3x3 <sym. pos. semi-def.> driving noise covariance
matrix, Q:
_
```

```
___> _ [1 0 0;0 1 0;0 0 1]
_ Enter 1x1 <sym. pos. def.> observation noise covariance
matrix, R:
_
___> 1
 Observer eigenvalues:
_________________________

 -1.5385
 -0.4085 + 1.5331i
 -0.4085 - 1.5331i

 Gain matrix K that places the poles of A + KC at above
____________________________________________________________

  -1.3556
  -0.4188
   2.2571

S o l v i n g   f o r   t h e   Q u a d r a t i c   O b s e r v e r

----- Please choose the method for minimization: -----

      1) Identity norm
      2) Norm weighed by a normal distribution


 Select a menu number: 2

----- M A I N   M E N U -----

      1) Help
      2) Enter nonlinear system (plant and output equations)
      3) Select the type of problem to be solved
      4) Simulate and plot results
      5) Quit


 Select a menu number: 4
--Press any key--
Simulation with external disturbances:

Please define input u1
```

```
----- Disturbance input type -----

      1) Zero disturbance on the above input
      2) Impulse (not implemented yet)
      3) Step
      4) Sinusoid
      5) Random noise (not implemented yet)
      6) External data file (not implemented yet)


 Select a menu number: 4
 Enter sinusoid amplitude: 1
 Enter sinusoid frequency (*scaled* time): 6

_
_ Enter 3x1 vector of initial conditions, x10c: [0;0;0]

_
 ___>
Simulation will start from initial time t0c = 0.

 Enter final time (*scaled*), tfc: 12

_
_ Enter 3x1 Vector of observer initial conditions, x10o:
[0;0;0]
_
```




```
USING THE SAME S

      1) Quadrati
      2) Quadrati
      3) Quadrati
      4) Get more

 Select a menu nu
CLOSED LOOP FEEDE

----- Input one c

     1) Specify cl
     2) Design lir
     3) Design lir

 Select a menu nu

Quadratic regulat

      _
      _ integral(x'
_
- Enter 3x3 <sym.
_
```



USING THE SAME SYSTEM AS ABOVE:

1) Quadratic Controller design (full observability)
2) Quadratic Observer design
3) Quadratic Controller-Quadratic Observer design
4) Get more information on the above choices

Select a menu number: 1
CLOSED LOOP FEEDBACK DESIGN FOR LINEAR PART OF THE PLANT

----- Input one of the following: -----

1) Specify closed loop eigenvalues
2) Design linear quadratic regulator with state weighing
3) Design linear quadratic regulator with output weighing

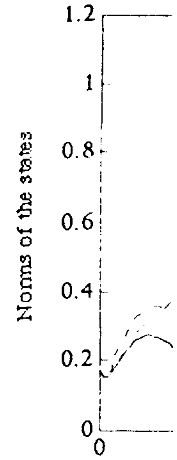
Select a menu number: 2

Quadratic regulator will minimize:

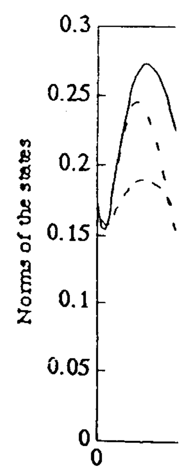
```
    -
    - integral(x'Qx + u'Ru)dt

-
- Enter 3x3 <sym. pos. semi-def.> matrix, Q:
-
```

```
___> [1 0 0;0 1 0;0 0 1]

Enter 1x1 <sym. pos. def.> matrix, R:

___> 1

Closed loop eigenvalues:
________________________

-1.3294
-0.4975 + 1.4599i
-0.4975 - 1.4599i

Gain matrix F that places the poles of A + BF at above
______________________________________________________

-0.1623   -1.7015   -1.3244
```

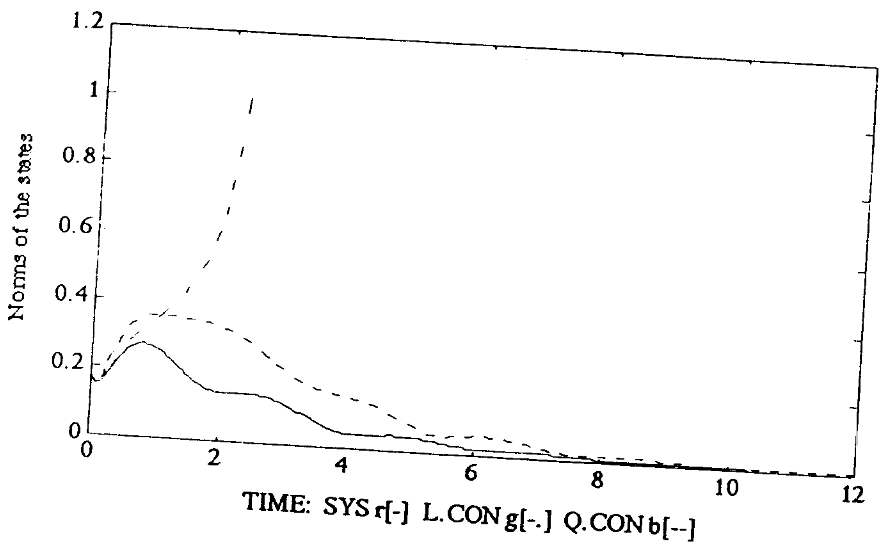
INITIAL CON



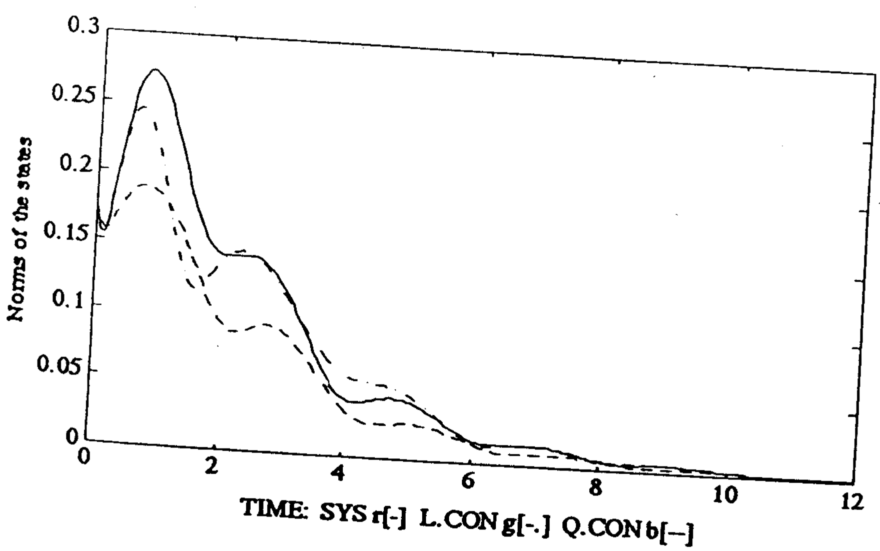
INITIAL CONI
above)



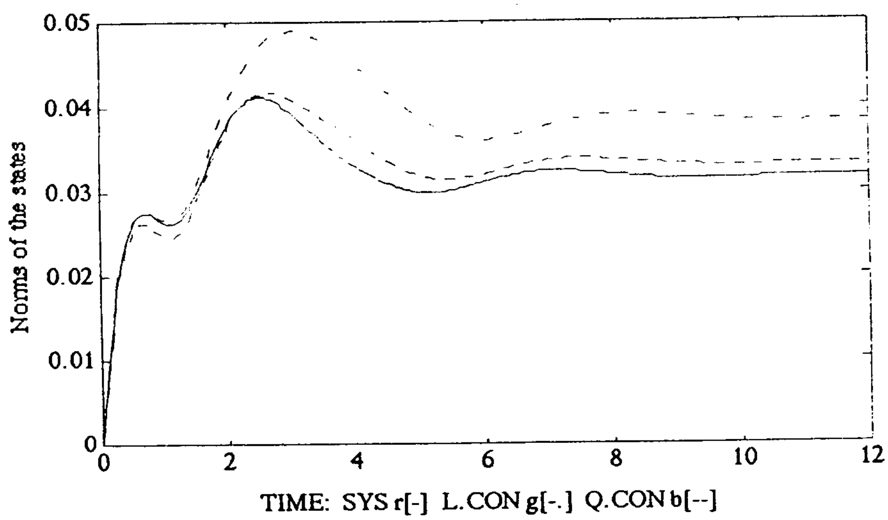
INITIAL CONDITION RESPONSE: X=[0.1 0.1 0.1]



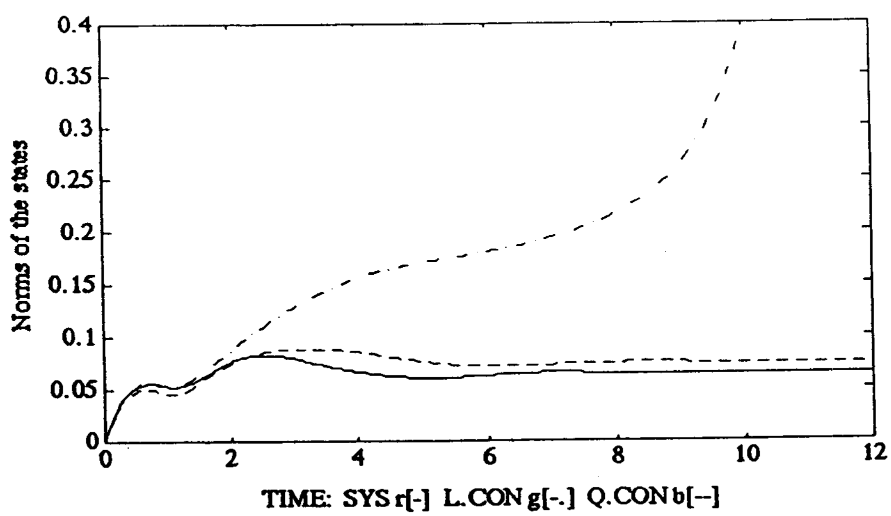
INITIAL CONDITION RESPONSE: X=[-0.1 -0.1 -0.1] (Negative of the above)



STEP RESPONSE: Step input magnitude is 0.1.



STEP RESPONSE: Step input magnitude is 0.2.



SESSION 2 OBSERV

Dimensions of the

[n, m] = 3 1

Homogeneous order

Linear plant mat

0	1	C
0	0	1
0	-1	0

Eigenvalues of t

0
0 + 1.0000
0 - 1.0000

Second order par

0	0	0
0	0	0
0	-5	0

Constant part of

0
0
1

First order part

0	0	0
0	0	0
1	0	0

SESSION 2 OBSERVER EXAMPLE

Dimensions of the system:

[n, m] = 3 1

Homogeneous order of system: nchoose2 = 6

Linear plant matrix, A:

0	1	0
0	0	1
0	-1	0

Eigenvalues of the open-loop plant:

0
0 + 1.0000i
0 - 1.0000i

Second order part of the plant, f2:

0	0	0	0	0	0
0	0	0	0	0	0
0	-5	0	0	3	0

Constant part of the input vector, B:

0
0
1

First order part of the input vector, g1:

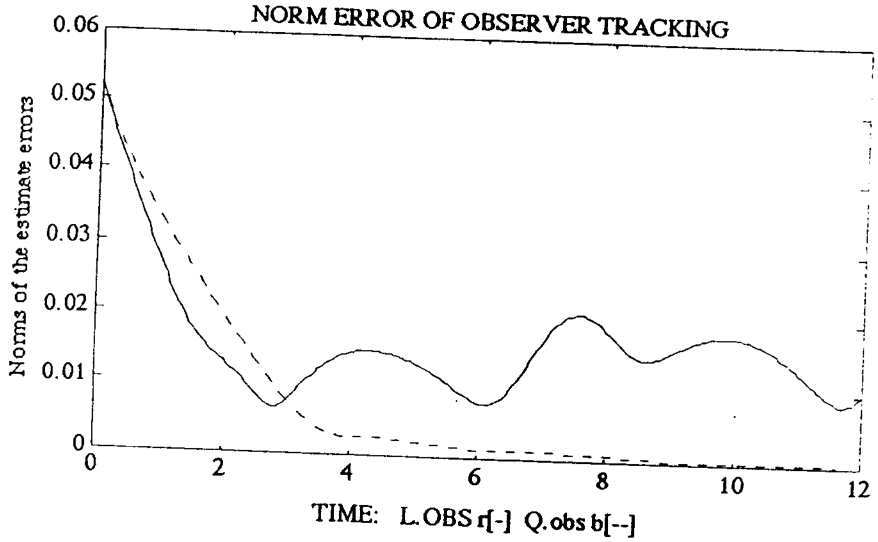
0	0	0
0	0	0
1	0	0

```
 Constant part of the output vector, C:
 ____________________________________________

     1     0     0

 Second order part of the output vector, h2:
 ________________________________________________

     0     0     0     0     0     0

Scale factors of the states, x1 through x3:
xscale = 1     1     1

Scale factor of the input, u1:
uscale = 1

Scale factor of the output y1:
yscale = 1

----- M A I N   M E N U -----

      1) Help
      2) Enter nonlinear system (plant and output equations)
      3) Select the type of problem to be solved
      4) Simulate and plot results
      5) Quit

 Select a menu number:  3
----- Select the type of problem to be solved -----

      1) Quadratic Controller design (full observability)
      2) Quadratic Observer design
      3) Quadratic Controller-Quadratic Observer design
      4) Get more information on the above choices

 Select a menu number: 2
CLOSED LOOP FEEDBACK DESIGN FOR LINEAR PART OF THE
            O-B-S-E-R-V-E-R
----- Input one of the following: -----

      1) Specify observer eigenvalues
      2) Design linear Kalman filter
```

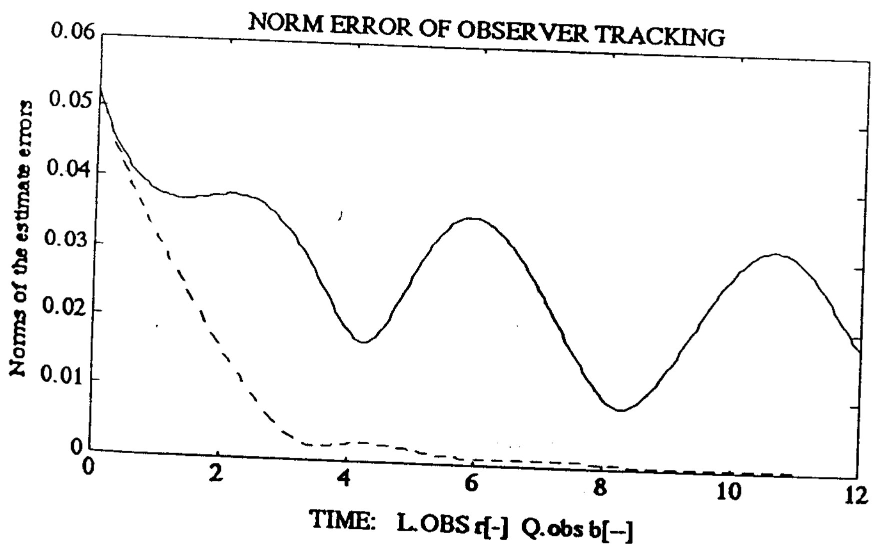
```
Select a menu number: 2

Kalman filter will minimize the error covariance.:
_
_ Enter 3x3 <sym. pos. semi-def.> driving noise covariance
matrix, Q:
_
 ___> _ [1 0 0;0 1 0;0 0 1]
_ Enter 1x1 <sym. pos. def.> observation noise covariance
matrix, R:
_
 ___> 1
 Observer eigenvalues:
 ____________________

 -1.0000
 -0.4551 + 1.0987i
 -0.4551 - 1.0987i

 Gain matrix K that places the poles of A + KC at above
 ______________________________________________________

 -1.9102
 -1.3244
  0.4960

S o l v i n g   f o r   t h e   Q u a d r a t i c   O b s e r v e r

----- Please choose the method for minimization: -----

     1) Identity norm
     2) Norm weighed by a normal distribution

Select a menu number: 2

----- M A I N   M E N U -----

     1) Help
     2) Enter nonlinear system (plant and output equations)
     3) Select the type of problem to be solved
     4) Simulate and plot results
     5) Quit
```

```
 Select a menu number: 3

Initial conditions for the plant to be followed by the observer:

_
_ Enter 3x1 vector of initial conditions, x10c:
_
 ___> [.03 .03 .03]

Simulation will start from initial time t0c = 0.

 Enter final time (*scaled*), tfc: 12

Simulation with external disturbances:

Please define input u1

----- Disturbance input type -----

      1) Zero disturbance on the above input
      2) Impulse (not implemented yet)
      3) Step
      4) Sinusoid
      5) Random noise (not implemented yet)
      6) External data file (not implemented yet)

 Select a menu number: 1
_
_ Enter 3x1 Vector of observer initial conditions, x10o:
_
 ___> [0 0 0]


To integrate the systems:
1- Please quit MATLAB.
2- Double click on the application "intg_68020" in the folder
3- Restart MATLAB
4- After opening the QC folder, type "qc_continue"
   at which point the program will resume plotting.
```

Initial Conditions for the plant : [0.03 0.03 0.03]



Initial Conditions for the plant : [-0.03 -0.03 -0.03]



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