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The Construction of Optimal Linear and Nonlinear Regulators

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1 Introduction

The regulator servomechanism problem is to design a feedforward and feedback control law to make the output of a given system called the plant, track a signal from a given class. There are various ways to make this precise, we shall follows Francis [5] and Isidori and Byrnes [11]. The class of signals to be tracked are described as the output of a second system called the signal generator (or exosystem). The control law consists of feedforward terms involving the state of signal generator, feedback terms involving the state of the plant and mixed terms involving both.

Francis [5] posed and solved the linear regulator problem and Isidori and Byrnes [11] generalized this to the nonlinear case. The former showed that the linear regulator problem is solvable only if a certain system of linear equation is solvable. The latter showed that the nonlinar regulator problem is solvable only if a certain system of first order partial differential equations is solvable. Huang and Rugh [9], [10] did a formal (term by term) analysis of these PDEs and gave sufficient conditions for its solvability. The degree one terms of the Isidori and Byrnes PDE yield the linear equations of Francis, so we refer to the system as the Francis-Byrnes -Isidori (FBI) equations.

In this paper, we shall show necessary and sufficient conditions for the term by term solvability of the FBI equations when either the signal generator has a semisimple pole structure or the plant has a semisimple zero structure. We present a proof of Huang-Rugh sufficient conditions.

We also give optimal methods for constructing the nonlinear regulator based on term by term analysis of the Hamilton-Jacobi Bellman HJB equations in the spirit of Al'brecht. This optimal approach to constructing a regulator may be novel even in the linear case.

2 Linear Regulation

Following Francis [5] and Isidori and Byrnes [11], we review the solution of the linear regulator problem. We are given a plant

(1)
$$\dot{x} = Fx + Gu$$
$$y = Hx + Ju$$

where $x \in \mathbb{R}^{n\times 1}$, $u \in \mathbb{R}^{m\times 1}$ and $y \in \mathbb{R}^{p\times 1}$. Throughout we shall assume (F, G) is stabilizable, (H, F) is detectable, $\begin{bmatrix} G \\ J \end{bmatrix}$ of full column rank and $[H \quad J]$ of full row rank. We are also given a signal generator

(2)
$$\dot{w} = Aw$$

(3)
$$\bar{y} = Cw$$

where $w \in \mathbb{R}^{d\times 1}$ and $\bar{y} \in \mathbb{R}^{p\times 1}$. Without loss of generality, we assume (C, A) is observable. The initial state of the signal generator is nonzero and the state and output evolve in a smooth fashion. The goal is to find a feedforward and feedback law from (x, w) to u which both stabilizes the plant (1) and drives the output error

(4)
$$\tilde{y} = y - \bar{y} = Hx + Ju - Cw$$

to zero as $t \to \infty$. Alternatively, one can occasionally reinitialize the state of the signal generator and obtain piecewise smooth signals. The goal then is to achieve stabilization and tracking on a significantly faster time scale then the times between reinitialization of the signal generator. A third alternative is to include an input channel in the dynamics of the signal generator

(5)
$$\dot{w} = Aw + Bv$$

where v is a piecewise constant input. For example, if

$$A = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \qquad C = [1 \; 0 \; \cdots \; 0]$$

then the output $\bar{y}$ of (5, 3) is a polynomial spline of degree d. Again the goal is to achieve stabilization and tracking on a significantly faster time scale than the times between steps of the input. Throughout, we shall consider signal generators of the form (2, 3), the extension to (5, 3) is straightforward.

There are several other interesting extensions of the regulator problem. One is to assume that the plant dynamics is directly disturbed by the signal generator

(6)
$$\dot{x} = Fx + Gu - Bw$$

and then the goal of the feedback can be viewed as stabilization, regulation and disturbance rejection. Another is to assume that the state of signal generator is not directly measurable, only the output error $\tilde{y}$ is measurable. Dynamic feedback can be used to solve this problem. We shall not discuss these extensions, they are treated in [11] and the methods that we shall present are generalizable to them. The dynamic feedback problem requires the construction of nonlinear observers.

Francis [5] showed that the original problem is solvable iff (F, G) is stabilizable and there exists mappings

$$(7)\qquad \begin{aligned} x &= Pw \\ u &= Kw \end{aligned}$$

which solve the Francis equations,

$$(8)\qquad \begin{aligned} FP + GK &= PA \\ HP + JK &= C \end{aligned}$$

Hautus [8] has shown that solvability of these equations is connected with the transmission polynomials of the original plant (1) and the transmission polynomials of the combined plant, signal generator and output error (1, 2, 3, 4). This is also related to the Internal Model Principle of Francis and Wonham [6].

We now offer a necessary and sufficient condition for the solvability of the Francis equations that is slightly different from these and is valid regardless of the relative sizes of m and p.

Following standard terminology, a pole λ of the signal generator is an eigenvalue of A. Introducing new terminology, an output pole triple $(\lambda, w, y) \in \mathbb{C} \times \mathbb{C}^{d\times 1} \times \mathbb{C}^{p\times 1}$ satisfies $w \neq 0$ and

$$(9)\qquad \begin{aligned} Aw &= \lambda w \\ Cw &= y \end{aligned}$$

Of course w is a right eigenvector of A corresponding to λ. If A is not semisimple, i.e., A has nontrivial Jordan blocks, then there are sequences of generalized output pole triples, (λ, w^j, y^j) for $j = 1, \cdots, r$ satisfying $w^1, \ldots, w^r$ linearly independent and

$$(10)\qquad \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} [w^1 w^2 \cdots w^r] = \begin{bmatrix} 0 & w^1 & \cdots & w^{r-1} \\ y^1 & y^2 & \cdots & y^r \end{bmatrix}$$

The w^j's are generalized right eigenvectors of A and it is always possible to choose a basis for $\mathbb{C}^{d\times 1}$ consisting of right eigenvectors and generalized right eigenvectors of A.

The terminology for zeros of a multivariable system (1) varies among authors, we adopt the following. An $s \in \mathbb{C}$ is an output zero if

$$(11)\qquad rank \begin{bmatrix} F - sI & G \\ H & J \end{bmatrix} < n + p.$$

This matrix is called the system matrix of (1). An output zero triple $(s,\zeta,\psi) \in \mathbb{C} \times \mathbb{C}^{1\times m} \times \mathbb{C}^{1\times p}$ satisfies $\zeta \neq 0$ and

$$(\zeta \quad \psi)\begin{bmatrix} F - sI & G \\ H & J \end{bmatrix} = (0 \quad 0)\,. \tag{12}$$

Notice that if $m < p$ then every $s \in \mathbb{C}$ is an output zero.

The output zero structure need not be semisimple. There may exist sequences of generalized output zero triples $(s,\ \zeta_i,\ \psi_i)$ for $i = 1,\ \cdots, r$ satisfying $\zeta_1, \cdots, \zeta_r$ linearly independent and

$$\begin{bmatrix} \zeta_1 & \psi_1 \\ \zeta_2 & \psi_2 \\ \vdots & \vdots \\ \zeta_r & \psi_r \end{bmatrix} \begin{bmatrix} F - sI & G \\ H & J \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \zeta_1 & 0 \\ \vdots & \vdots \\ \zeta_{r-1} & 0 \end{bmatrix}. \tag{13}$$

Intuitively the output pole triples describe the modal directions of the state and output of the signal generator. The output zero triples describe the components of the plant's state and output which cannot be excited at a given frequency. There are dual definitions of input pole triples and input zero triples which we omit as we have no need of them. The intuitive content of the next theorem is that the Francis equations are not solvable iff an output pole triple of the signal generator excites an output zero triple of the plant.

Theorem 2.1 *The Francis equations (8) are not solvable iff*

(i) there exists an output pole triple (λ, w, y) *of the signal generator (2, 3) and an output zero triple* (s,ζ,ψ) *of the plant (1) such that*

$$\lambda = s \tag{14}$$

and

$$\psi y \neq 0 \tag{15}$$

or

(ii) there exists a sequence of generalized output pole triples $(\lambda,\ w^j, y^j)$, $j = 1,\cdots,r$, *of the signal generator (2, 3) and a sequence of generalized output zero triples* $(s,\ \zeta_i,\ \psi_i)$, $i = 1,\cdots,\rho$, *of the plant (1) such that (14) and for some* $j \leq \min(r,\rho)$

$$\psi_1 y^j + \psi_2 y^{j-1} + \cdots + \psi_j y^1 \neq 0 \tag{16}$$

Proof. We choose a basis for $\mathbb{C}^{d\times 1}$ consisting of eigenvectors and generalized eigenvectors of A. Multiplying the Francis equations on the right by an eigenvector w corresponding to the eigenvalue λ and $y = Cw$, we obtain

$$\begin{bmatrix} F - \lambda I & G \\ H & J \end{bmatrix} \begin{bmatrix} Pw \\ Kw \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}. \tag{17}$$

This is an equation for Pw and Kw. It is solvable iff every $(\zeta \quad \psi) \in \mathbb{C}^{1\times n} \times \mathbb{C}^{1\times p}$ annihilating the system matrix on the left annihilates the vector on the right. There is such a $(\zeta \quad \psi)$ annihilating the system matrix iff $\lambda = s$, where s is an output zero. If $\lambda = s$ and $(s,\ \zeta,\ \psi)$ is an output zero triple then (17) is solvable iff $\psi y = 0$. If A is semisimple, its eigenvectors span $\mathbb{C}^{d\times 1}$ so we are done.

Suppose A is not semisimple, say there exists a sequence of generalized output pole triples (λ, w^j, y^j), $j = 1, \cdots, r$. Multiplying the Francis equations on the right by w^j yields

$$\begin{bmatrix} F-\lambda I & G \\ H & J \end{bmatrix} \begin{bmatrix} Pw^j \\ Kw^j \end{bmatrix} = \begin{bmatrix} Pw^{j-1} \\ y^j \end{bmatrix} \tag{18}$$

and we have a coupled set of equations for Pw^j, Kw^j as $j = 1, \cdots, r$. This can be rewritten as

$$\begin{bmatrix} D(\lambda) & 0 & \dots & 0 & 0 \\ E & D(\lambda) & \dots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \dots & D(\lambda) & 0 \\ 0 & 0 & \dots & E & D(\lambda) \end{bmatrix} \begin{bmatrix} \theta^1 \\ \vdots \\ \theta^r \end{bmatrix} = \begin{bmatrix} \gamma^1 \\ \vdots \\ \gamma^r \end{bmatrix} \tag{19}$$

where

$$D(s) = \begin{bmatrix} F-sI & G \\ H & J \end{bmatrix} \qquad E = \begin{bmatrix} -I & 0 \\ 0 & 0 \end{bmatrix}$$

$$\theta^i = \begin{bmatrix} Pw^i \\ Kw^i \end{bmatrix} \qquad \gamma^i = \begin{bmatrix} 0 \\ y^i \end{bmatrix}$$

If there is a sequence $(s,\ \zeta_i,\ \psi_i)$, $i = 1,\ \cdots,\ \rho$, of generalized output zero triples of (1) where $s = \lambda$ then the matrix

$$\begin{bmatrix} \xi_1 & 0 & \dots & 0 \\ \xi_2 & \xi_1 & \dots & 0 \\ & & \ddots & \\ \xi_\mu & \xi_{\mu-1} & \dots & \xi_1 \end{bmatrix} \tag{20}$$

where $\mu = \min(r,\ \rho)$ and

$$\xi_i = [\zeta_i \ \psi_i]$$

annihilates the large matrix on the left side of (19). Moreover, it is not hard to see that any vector annihilating the left side of (19) must define a sequence of generalized output zero triples and hence a matrix of the form (20). Therefore (19) is solvable iff

$$\psi_1 y^1 = 0$$
$$\psi_2 y^1 + \psi_1 y^2 = 0$$
$$\psi_\mu y^1 + \cdots + \psi_1 y^\mu = 0$$

QED

If the Francis equations are solvable then the desired feedback law to enable the output of the plant (1) to track the output of the signal generator is

$$u = Kw + L(x - Pw) \tag{21}$$

where L is any feedback matrix such that $F+GL$ is asymptotically stable. Under this feedback the combined system (1, 2) is driven to the subspace $x = Pw$, which is an invariant subspace for the closed loop dynamics [5], [11]. On this subspace the closed loop dynamics is the same as that of the signal generator (2) and this is referred to as the Internal Model Principle [6].

We would like to suggest an optimal way to choose L and also to choose K when the Francis equations admit many solutions, e.g. $m > p$. Suppose we wish to minimize the infinite time quadratic cost

$$\frac{1}{2}\int_0^\infty x^* Qx + 2x^* Su + u^* Ru \, dt \tag{22}$$

where

$$R > 0 \tag{23}$$

and

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \geq 0. \tag{24}$$

There has been some discussion in the literature about how to suitably interpret this because the cost (22) could very well be infinite, see for example [2]. We believe it should be interpreted in two stages. If the initial condition of the combined system (1, 2) lies in the subspace defined by $x = Pw$ and we use the feedback $u = Kw + v$ where P and K solve the Francis equations and v will be chosen later, then the closed loop dynamics with $v = 0$ will evolve on this subspace. If there are many solutions to the Francis equations then to minimize the integrand of the cost (22) whenever we are on the subspace, we should choose one that minimizes

$$\text{trace} \begin{bmatrix} P \\ K \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} P \\ K \end{bmatrix}. \tag{25}$$

If (24) is positive definite then the minimizing solution is unique.

It should be noted that this criterion is not invariant under change of coordinates in the state space of the signal generator. Therefore it is important that the state coordinates of the signal generator are chosen in such a way that the expected states of the signal generator are uniformly distributed over the ellipsoids.

$$w^*\Gamma w = \text{constant}$$

where $\Gamma > 0$. Then we should choose a solution of the Francis equations which minimizes

$$\text{trace}\left[\begin{bmatrix} P \\ K \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix}\begin{bmatrix} P \\ K \end{bmatrix}\Gamma^{-1}\right] \tag{26}$$

If the initial condition of the combined system is off of the subspace $x = Pw$ then the goal is to drive the solution onto the subspace in an optimal fashion. We use $z = x - Pw$ as coordinates transverse to the subspace $x = Pw$. A simple calculation shows that under the feedback $u = Kw + v$,

$$\dot{z} = Fz + Gv \tag{27}$$

$$\tilde{y} = Hz + Jv \tag{28}$$

If we replace the cost (22) with

$$\frac{1}{2}\int_0^\infty z^*Qz + 2z^*Sv + v^*Rv\ dt \tag{29}$$

then we get a standard LQR problem for the optimal feedback

$$v = Lz = L(x - Pw). \tag{30}$$

Under the standard assumptions, (F, G) stabilizable, $(Q^{1/2}, F)$ detectable and (23) we have the cost (29) is finite and the feedback (30) asymptotically drives the system (27) to zero and hence (20) under the closed loop dynamics remains on the invariant subspace given by $z = x - Pw = 0$.

Of course there is no reason why the cost (29) must use the Q, R, S of (22). One could use instead

$$\frac{1}{2}\int_0^\infty \|\tilde{y}\|^2 + \|v\|^2\ dt \tag{31}$$

or any other cost that satisfies the standard assumptions, as long as the cost is zero whenever $z = 0$ and $v = 0$. This will ensure that the LQR problem will have a finite solution.

3 The FBI Equations

Isidori and Byrnes [11] have posed and solved a nonlinear extension of Francis' linear regulator [5]. We are given a plant

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \\ 0 &= f(0,0) \end{aligned} \tag{32}$$

and a signal generator

$$\begin{aligned} \dot{w} &= a(w) \\ \bar{y} &= c(w) \\ 0 &= a(0). \end{aligned} \tag{33}$$

As before, the goal is to find a feedback law from (x, w) to u which stabilizes the plant (1) and drives the output error

$$\tilde{y} = y - \bar{y} = h(x, u) - c(w) \tag{34}$$

to zero as $t \to \infty$ for all initial conditions. Actually Isidori and Byrnes treated a slightly different problem in which the plant dynamics (32) is affine in both u and y and where the output does not depend explicitly on u. Their methods and results extend immediately to the above. They also treat the problem where (x, w) is not directly measurable.

Isidori and Byrnes assume that the signal generator dynamics (33) is Poisson stable around the origin, $w = 0$ and the linear approximation around the origin of the plant dynamics (32) is stabilizable. Under these assumptions they show that the nonlinear regulator problem is solvable iff there exists

$$\begin{aligned} x &= \pi(w) \\ u &= \kappa(w) \end{aligned} \tag{35}$$

satisfying the first order P.D.E.s

$$\begin{aligned} &f(\pi(w), \kappa(w)) - \frac{\partial \pi}{\partial w}(w) a(w) = 0 \\ &h(\pi(w), \kappa(w)) - c(w) = 0. \end{aligned} \tag{36}$$

We expand (32), (33) and (35) in Taylor series

$$\begin{aligned} \dot{x} &= Fx + Gu + F^{[2]}(x,\ u) + f^{[3]}(x,\ u) + \cdots \\ y &= Hx + Ju + h^{[2]}(x,\ u) + h^{[3]}(x,\ u) + \cdots \end{aligned} \tag{37}$$

$$\begin{aligned} \dot{w} &= Aw + a^{[2]}(s) + a^{[3]}(x,\ u) + \cdots \\ \bar{y} &= Cw + c^{[2]}(w) + c^{[3]}(w) + \cdots \end{aligned} \tag{38}$$

$$\begin{aligned} x &= Pw + \pi^P[2](w) + \pi^{[3]} + \cdots \\ u &= Kw + \kappa^{[2]}(w) + \kappa^{[3]}(w) + \cdots \end{aligned} \tag{39}$$

where superscripts [2], [3], $\cdots$ denote homogeneous polynomial function of degree $2, 3, \ldots$ respectively. If we insert these series expansions into (36) and truncate after degree one terms, we obtain the Francis equations (8). For this reason we refer to (36) as the Francis-Byrnes-Isidori equations of the FBI equations.

Huang and Rugh [9] [10] have stated sufficient conditions for the formal solvability of the FBI equations. Formal solvability means term by term solvability of the FBI equations without regard to convergence.

To simplify the exposition we restrict our criteria to signal generators which are linear, but the methods can easily be extended to nonlinear signal generators. The methods can also be easily extended to the case where dynamics depends explicitly on the state of the signal generator, i.e., the disturbance rejection and regulation problem. We shall give necessary and sufficient conditions for the formal solvability of the FBI equations when either the signal generator's linear pole structure is semisimples or the plant's linear output zero structure is semisimple. We shall also give a proof of the Huang-Rugh sufficient condition.

We start by considering the degree two part of the FBI equations assuming the degree one part admits the solution $P,\ K$,

$$F\pi^{[2]}(w) + G\kappa^{[2]}(w) - \frac{\partial \pi^{[2]}}{\partial w}(w)Aw = -f^{[2]}(Pw, Kw) \qquad (40)$$

$$H\pi^{[2]}(w) + J\kappa^{[2]}(w) = -h^{[2]}(Pw,\ Kw)\,.$$

Suppose $(\omega_1, \ldots, \omega_d)$ is a basis of $\mathbb{C}^{1\times d}$ consisting of left eigenvectors of A,

$$\omega_i A = \lambda_i \omega_i, \qquad (41)$$

or generalized left eigenvectors of A,

$$\omega_i A = \lambda_i \omega_i + \omega_{i+1}. \qquad (42)$$

We can expand $\pi^{[2]}(w), \kappa^{[2]}(w), f^{[2]}(Pw,\ Kw)$ and $h^{[2]}(Pw,\ Kw)$ in terms of the ω_i as

$$\begin{aligned} \pi^{[2]}(w) &= \sum \pi^{ij}\,(\omega_i w)\,(\omega_j w) \\ \kappa^{[2]}(w) &= \sum \kappa^{ij}\,(\omega_i w)\,(\omega_j w) \\ f^{[2]}(Pw,\ Kw) &= \sum f^{ij}\,(\omega_i w)\,(\omega_j w) \\ h^{[2]}(Pw, Kw) &= \sum h^{ij}\,(\omega_i w)\,(\omega_j w) \end{aligned} \qquad (43)$$

where $\pi^{ij} \in \mathbb{C}^{n\times 1}, \kappa^{ij} \in \mathbb{C}^{m\times 1}, f^{ij} \in \mathbb{C}^{n\times 1},\ h^{ij} \in \mathbb{C}^{p\times 1}$ and the sums range over $1 \le i \le j \le d$.

Theorem 3.1 *Assume the FBI equations of degree one are solvable and the pole structure of the signal generator (2, 3) is semisimple. The FBI equations of degree two are not solvable iff there exist left eigenpairs (λ_i, ω_i), (λ_j, ω_j) of A and an output zero triple $(s,\ \zeta,\ \psi)$ of the linear part of the plant (1) such that*

$$s = \lambda_i + \lambda_j \qquad (44)$$

and

$$(\zeta \quad \psi)\begin{bmatrix} f^{ij} \\ h^{ij} \end{bmatrix} \neq 0. \qquad (45)$$

Proof. Plugging (43) into the FBI equations of degree two (40), we obtain $(d+1)$ choose 2 linear equations for the unknown π^{ij}, κ^{ij} in terms of the known f^{ij}, h^{ij} and the plant system matrix. Because A is semisimple, these equations are in diagonal from

$$\begin{bmatrix} F-(\lambda_i+\lambda_j) & G \\ H & J \end{bmatrix} \begin{bmatrix} \pi^{ij} \\ \kappa^{ij} \end{bmatrix} = - \begin{bmatrix} f^{ij} \\ h^{ij} \end{bmatrix}. \tag{46}$$

for $1 \le i \le j \le d$.

If $s = \lambda_i + \lambda_j$ is not an output zero of the linear part of the plant then (46) is solvable for any f^{ij}, h^{ij}. If there exist an output zero triple $(s,\ \zeta,\ \psi)$ satisfying (44) then (ζ, ψ) annihilates the system matrix on the left side of (46). Hence (46) is solvable iff

$$(\zeta \quad \psi) \begin{bmatrix} f^{ij} \\ h^{ij} \end{bmatrix} = 0.$$

QED

Suppose the FBI equations are formally solvable up to degree $k-1$ then the degree k equations are

$$\begin{aligned} F\pi^{[k]}(w) + G\kappa^{[k]}(w) - \tfrac{\partial \pi^{[k]}}{\partial w}(w)Aw &= -f^{[k]}(w) \\ H\pi^{[k]}(w) + J\kappa^{[k]}(w) &= -h^{[k]}(w) \end{aligned} \tag{47}$$

where $f^{[k]}(w)$ and $h^{[k]}(w)$ are the degree k parts of the composition of $f(x,u)$ and $h(x,u)$ with the expansion of (39) up to degree $k-1$.

As before we can expand $\pi^{[k]}(w)$, $\kappa^{[k]}(w)$, $f^{[k]}(w)$ and $h^{[k]}(w)$ in terms of a basis of $\mathbb{C}^{1\times d}$ of left eigenvectors and generalized eigenvectors of A,

$$\begin{aligned} \pi^{[k]}(w) &= \sum \pi^{i_1\cdots i_k}(\omega_{i_1}w)\cdots(\omega_{i_k}w) \\ \kappa^{[k]}(w) &= \sum \kappa^{i_1\cdots i_k}(\omega_{i_1}w)\cdots(\omega_{i_k}w) \\ f^{[k]}(w) &= \sum f^{i_1\cdots i_k}(\omega_{i_1}w)\cdots(\omega_{i_k}w) \\ h^{[k]}(w) &= \sum h^{i_1\cdots i_k}(\omega_{i_1}w)\cdots(\omega_{i_k}w) \end{aligned} \tag{48}$$

where the sum ranges over $1 \le i_1 \le i_2 \le \cdots \le i_k \le d$.

Theorem 3.2 *Assume the FBI equations are formally solvable up to degree $k-1$ and the pole structure of the signal generator (2) is semisimple. The FBI equations of degree k are not solvable iff there exist left eigenpairs $(\lambda_{i_1}, \omega_{i_1}), \ldots, (\lambda_{i_k}, \omega_{i_k})$ of A and an output zero triple (s, ζ, ψ) of the linear part of the plant (32) such that*

$$s = \lambda_{i_1} + \cdots + \lambda_{i_k} \tag{49}$$

$$(\zeta \quad \psi) \begin{bmatrix} f^{i_1\cdots i_k} \\ h^{i_1\cdots i_k} \end{bmatrix} \neq 0. \tag{50}$$

Proof. Plugging (48) into the degree k FBI equations (47), we obtain $(d+k-1)$ choose k decoupled sets of linear equations of the form

$$\begin{bmatrix} F-(\lambda_{i_1}+\cdots+\lambda_{i_k})I & G \\ H & J \end{bmatrix} \begin{bmatrix} \pi^{i_1\cdots i_k} \\ \kappa^{i_1\cdots i_k} \end{bmatrix} = -\begin{bmatrix} f^{i_1\cdots i_k} \\ h^{i_1\cdots i_k} \end{bmatrix} \tag{51}$$

If $s=\lambda_{i_1}+\cdots+\lambda_{i_k}$ where (s,ζ,ψ) is an output zero triple then (51) is solvable iff the left side of (50) is zero,

QED

Conditions like (44) and (49) are called resonance conditions in the literature, see [3], [7]. The intuitive meaning is that the signal generator excites the nonlinearities of the plant so as to create harmonics which the linear part of the plant has difficulty tracking and canceling.

The above theorem excludes many interesting signal generators, for example, those that generate polynomial splines as was discussed in Section 2. Such a signal generator is not semisimple. Unfortunately, a general analysis of regulator problems with nonsemisimple signal generators can be quite complicated. We illustrate this by a relatively simple example.

Suppose $p=1$, $d=3$ and the Jordan form of A consist of one block. In other words there exists $\lambda,\omega_1,\omega_2,\omega_3$ such that

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}(A-\lambda I) = \begin{bmatrix} \omega_2 \\ \omega_3 \\ 0 \end{bmatrix}$$

Notice that ω_3 is a left eigenvector of A and ω_1 and ω_2 are generalized left eigenvectors. Assume the degree one FBI equations have a solution P,K. To solve the degree two equations(40) we expand as before (43) and obtain as the coefficients $(\omega_i w)(\omega_j w)$

$$\begin{bmatrix} F-2\lambda I & G \\ H & J \end{bmatrix} \begin{bmatrix} \pi^{ij} \\ \kappa^{ij} \end{bmatrix} - \begin{bmatrix} \pi^{i-1,j}+\pi^{i,j-1} \\ 0 \end{bmatrix} = \begin{bmatrix} f^{ij} \\ h^{ij} \end{bmatrix}$$

where $\pi^{ij}=0$, $\kappa^{ij}=0$ if i or $j=0$. Because the pole structure of the signal generator is not semisimple, the equations are not decoupled. We rewrite them as

$$\begin{bmatrix} D(2\lambda) & 0 & 0 & 0 & 0 & 0 \\ E & D(2\lambda) & 0 & 0 & 0 & 0 \\ 0 & E & D(2\lambda) & 0 & 0 & 0 \\ 0 & 2E & 0 & D(2\lambda) & 0 & 0 \\ 0 & 0 & E & E & D(2\lambda) & 0 \\ 0 & 0 & 0 & 0 & 2E & D(2\lambda) \end{bmatrix} \begin{bmatrix} \theta^{11} \\ \theta^{12} \\ \theta^{13} \\ \theta^{22} \\ \theta^{23} \\ \theta^{33} \end{bmatrix} = -\begin{bmatrix} \gamma^{11} \\ \gamma^{12} \\ \gamma^{13} \\ \gamma^{22} \\ \gamma^{23} \\ \gamma^{33} \end{bmatrix}$$

(52)

where

$$D(s) = \begin{bmatrix} F - sI & G \\ H & J \end{bmatrix} \qquad E = \begin{bmatrix} -I & 0 \\ 0 & 0 \end{bmatrix}$$

$$\theta^{ij} = \begin{bmatrix} \pi^{ij} \\ \kappa^{ij} \end{bmatrix} \qquad \gamma^{ij} = \begin{bmatrix} f^{ij} \\ h^{ij} \end{bmatrix}$$

Clearly if $s = 2\lambda$ is not an output zero of the linear part of the plant then (52) is always solvable. Suppose $s = 2\lambda$ is an output zero frequency and there exist a sequence of generalized output zero triples satisfying (13) for $r \geq 5$. It is not hard to see that that $6 \times 6(n + p)$ matrix

$$\begin{bmatrix} \xi_1 & 0 & 0 & 0 & 0 & 0 \\ \xi_2 & \xi_1 & 0 & 0 & 0 & 0 \\ \xi_3 & \xi_2 & \xi_1 & 0 & 0 & 0 \\ 2\xi_3 & 2\xi_2 & 0 & \xi_1 & 0 & 0 \\ 3\xi_4 & 3\xi_3 & \xi_2 & \xi_2 & \xi_1 & 0 \\ 6\xi_5 & 6\xi_4 & 2\xi_3 & 2\xi_3 & 2\xi_2 & \xi_1 \end{bmatrix} \tag{53}$$

where $\xi_i = (\zeta_i \quad \psi_i)$ annihilates the $6(n+p) \times 6(n+m)$ matrix on the left side of (52). Moreover, any $1 \times 6(n + p)$ vector annihilating this matrix must be in the form of a row of (53) for some sequence of generalized output zero triples where $s = 2\lambda$. Hence (52) is solvable iff every matrix of the form (53) constructed from sequences of five generalized output zero triples at $s = 2\lambda$ annihilates the right side of (52). If there are several linearly independent sequences of output zero triples at $s = 2\lambda$, then these yield additional conditions for the solvability of (52).

If the sequence of generalized output zero triples is shorter than five triples, then there are fewer conditions and they are obtained by deleting rows of (53) containing the nonexistent triples. In particular, if the output zero structure of the plant is semisimple, i.e., there are no generalized output zero triples, then the degree two FBI equations are solvable iff whenever $s = 2\lambda$ then

$$(\zeta \quad \psi) \begin{bmatrix} f^{11} \\ h^{11} \end{bmatrix} = 0. \tag{54}$$

This last remark can be generalized to the following theorem.

Theorem 3.3 *Assume the FBI equations are formally solvable up to degree $k-1$ and the output zero structure of the linear part of the plant (32) is semisimple. The degree k FBI equations are not solvable iff there exists left eigenpairs $(\lambda_{i_j}, \omega_{i_j})$ where ω_{i_j} is a left eigenvector corresponding to λ_{i_j} or the first generalized left eigenvector of a sequence corresponding to λ_{i_j} for $j = 1, \ldots, n$ and an output zero triple (s, ζ, ψ) of the linear part of the plant such that*

$$s = \lambda_{i_1} + \cdots + \lambda_{i_k} \tag{55}$$

(56)
$$(\zeta \quad \psi)\begin{bmatrix} f^{i_1\cdots i_k} \\ h^{i_1\cdots i_k}\end{bmatrix} \neq 0$$

Proof. Plugging (48) into the degree k FBI equations (47), we obtain $(d+k-1)$ choose k set of linear equations. A set of equations corresponding to all simple eigenpairs $(\lambda_{i_1},\omega_{i_1})\cdots(\lambda_{i_k},\omega_{i_k})$ takes the form (51) and is decoupled from all the rest. Hence it is not solvable iff (55, 56) hold.

If one of the $(\lambda_{i_j},\omega_{i_j})$ is part of a sequence of generalized eigenpairs, then this set of equations is coupled to several other sets. All these coupled sets can be written in a large system like (52). If the sets of equations are written in lexographic order with $i_1 \leq i_2 \leq \cdots \leq i_k$ and the rightmost indices moving fastest then this large system is lower triangular and the diagonal consists of $D(\lambda_{i_1}+\cdots+\lambda_{i_k})$ as in (52). If $\lambda_{i_1}+\cdots+\lambda_{i_k}$ is not an output zero of the plant then this matrix is left invertible and the large system is solvable. If (55) holds, where $(s,\ \zeta,\psi)$ is an output triple of the plant, then the vector

$$(\zeta\ \psi\ 0\ 0\quad 0\ 0\ \cdots\ 0\ 0)$$

annihilates the matrix on the left side of the large equations. Hence the system is not solvable iff (56). QED

The Huang-Rugh sufficient condition is proved in a similar fashion.

Theorem 3.4 [9], [10] *Assume the FBI equations are formally solvable up to degree $k-1$. The degree k FBI equations are solvable if for all output zeros s of linear part of the plant (3.1) and all poles $\lambda_{i_1},\ldots,\lambda_{i_k}$ of the signal generator,*

(57)
$$s \neq \lambda_{i_1}+\cdots+\lambda_{i_k}.$$

Proof. Plugging (48) into the degree k FBI equations (47), we obtain $(d+k-1)$ choose k sets of linear equations. A set of equations corresponding to all simple eigenpairs $(\lambda_{i_1},\omega_{i_1})\cdots(\lambda_{i_k},\omega_{i_k})$ takes the form (51) and is decoupled from the rest. If (57) holds then it is solvable.

If one of the $(\lambda_{i_j},\omega_{i_j})$ is part of a sequence of generalized eigenpairs then this set of equations is coupled to several other sets. All these coupled sets can be written in lexographic order as a large lower triangular system with $D(\lambda_{i_1}+\cdots+\lambda_{i_k})$ on the diagonal. If (57) holds then this large matrix is left invertible, hence the system is solvable.

QED

4 Nonlinear Regulation

Now suppose the FBI equations are solvable to degree 2. There are two approaches to constructing the degree two regulator. The first might be called the pole placement approach. For the linear part of the regulator we proceed as in Section 2 and choose a P, K satisfying the Francis equations and a L such that $F + GL$ is asymptotically stable. For the quadratic part of the regulator we choose any

$$\begin{aligned} &\bar{\pi}^{[2]}(z, w) \\ &\bar{\kappa}^{[2]}(z,\ w,\ v) \end{aligned} \tag{58}$$

such that the restrictions

$$\begin{aligned} \pi^{[2]}(w) &= \bar{\pi}^{[2]}(0, w) \\ \kappa^{[2]}(w) &= \bar{\kappa}^{[2]}(0, w, 0) \end{aligned} \tag{59}$$

satisfy the FBI equations of degree 2. We consider the effect of the change of state coordinates

$$x = z + Pw + \bar{\pi}^{[2]}(z, w) \tag{60}$$

and feedback

$$u = v + Kw + Lz + \bar{\kappa}^{[2]}(z, w, v) \tag{61}$$

on the combined system (32, 33, 34) neglecting cubic and higher terms. The result is

$$\dot{z} = (F + GL)z + Gv + \bar{f}^{[2]}(z, w, v) + O(z, w, v)^3 \tag{62}$$

$$\dot{w} = Aw \tag{63}$$

$$\tilde{y} = (H + JL)z + Jv + \bar{h}^{[2]}(z, w, v) + O(z, w, v)^3 \tag{64}$$

where

$$\begin{aligned} \bar{f}^{[2]}(z, w, v) &= f^{[2]}(z + Pw,\ v + Kw + Lz) + F\bar{\pi}^{[2]}(z, w) \\ &\quad - \frac{\partial \bar{\pi}^{[2]}}{\partial (z, w)}(z, w) \begin{bmatrix} (F + GL)z + Gv \\ Aw \end{bmatrix} \\ \bar{h}^{[2]}(z, w, v) &= h^{[2]}(z + Pw,\ v + Kw + Lz) + H\bar{\pi}^{[2]}(z, w). \end{aligned} \tag{65}$$

Because the restrictions (59) of $\bar{\pi}^{[2]}$ and $\bar{\kappa}^{[2]}$ satisfy the FBI equations of degree two, it follows that

$$\begin{aligned} \bar{f}^{[2]}(0,\ w,\ 0) &= 0 \\ \bar{h}^{[2]}(0,\ w,\ 0) &= 0. \end{aligned} \tag{66}$$

If we ignore the $O(z, w, v)^3$ terms in (62, 63, 64), the submanifold $z = 0$ is an invariant manifold of the dynamics (62, 63) when $v = 0$ and on this submanifold the tracking error (64) is zero. The linear part of the dynamics is asymptotically stable so in a neighborhood of $z = 0,\ w = 0$, the transverse state coordinates

$z(t)$ and tracking error $\tilde{y}(t)$ will get smaller as $t \to \infty$ and $v(t) = 0$. Because of the presence of quadratic and higher degree terms in (62, 63, 64) and because $w(t) \neq 0$, these will not necessarily converge to zero. To make them smaller it would be desirable to choose (58) satisfying (59) so that

$$\begin{aligned} \bar{f}^{[2]}(z,\ w,\ v) &= 0 \\ \bar{h}^{[2]}(z,\ w,\ v) &= 0. \end{aligned} \tag{67}$$

In general this is not possible and one may have to settle for a least squares approximate solution to (67). The extra freedom in (58) as opposed to (59) allows a better approximate solution to (67).

If the FBI equations are solvable to degree k, then this approach can be used to build regulator of degree k, see [10] for details.

Now we discuss an optimal approach to constructing a degree two regulator. Assume that the state coordinates of the signal generator (33) have been suitably chosen so that the distribution of the state is isotropic and assume that we have a quadratic cost criterion (22). We proceed as in Section 2 and choose a solution $P,\ K$ of the Francis equations which minimizes (25). Next we solve the FBI equation of degree two in a similar fashion. We choose a basis $(\omega_1, \ldots, \omega_d)$ for $\mathbb{C}^{1\times d}$ of left eigenvectors (41) and generalized left eigenvectors (42) such that the ω_i's are unit vectors and orthogonal whenever possible. It is always possible to choose this basis so that all the eigenvectors and generalized eigenvectors corresponding to the same eigenvalue are orthogonal. We seek a solution (43) of the FBI equations of degree two (40) which minimizes

$$\frac{1}{2} \sum_{1 \leq i \leq j \leq d} \pi^{ij^*} Q \pi^{ij} + 2\pi^{ij^*} S \kappa^{ij} + \kappa^{ij^*} R \kappa^{ij} \tag{68}$$

The reason for doing this is that we wish to make cost of tracking small. Of course (68) is not the only criterion one can employ and there may be better choices which take into account the cost due to the interaction between the linear and quadratic solutions of the FBI equations.

Now we apply the change of coordinates

$$x = z + Pw + \pi^{[2]}(w) \tag{69}$$

and feedback

$$u = v + Kw + \kappa^{[2]}(w) \tag{70}$$

to the combined system (32, 33, 34) neglecting cubic and higher terms. The result is

$$\dot{z} = Fz + Gv + \bar{f}^{[2]}(z,w,v) + O(z,w,v)^3 \tag{71}$$

$$\dot{w} = Aw \tag{72}$$

$$\tilde{y} = Hz + Jv + \bar{h}^{[2]}(z,w,v) + O(z,w,v)^3 \tag{73}$$

where

$$(74)\quad \bar{f}^{[2]}(z,w,v) = f^{[2]}(z+Pw,\ v+Kw) + F\pi^{[2]}(w) - \frac{\partial \pi^{[2]}}{\partial w}(w)Aw$$

$$(75)\quad \bar{h}^{[2]}(z,w,v) = h^{[2]}(z+Pw,\ v+Kw) + H\pi^{[2]}(w).$$

Since (69, 70) is constructed from solutions of the FBI equation of degree one and two, it follows that

$$(76)\quad \begin{aligned} \bar{f}^{[2]}(0,w,0) &= 0 \\ \bar{h}^{[2]}(0,w,0) &= 0. \end{aligned}$$

Next we choose a cost criterion, say (29) and consider the nonlinear quadratic regulator problem (NQR) of minimizing the cost subject to (71, 72, 73). Let $\theta(z,\ w)$ denote the minimial value of (29) starting (71, 72, 73) at (z,w) at $t=0$. If $\theta(z,w)$ is finite and a smooth function, then it satisfies the Hamilton-Jacobi-Bellman (HJB) system of partial differential equations,

$$(77)\quad \frac{\partial\theta(z,w)}{\partial(z,w)}\begin{bmatrix} f(z,\ w,\ v) \\ Aw \end{bmatrix} + \ell(z,\ w) = 0$$

$$(78)\quad \frac{\partial\theta(z,w)}{\partial z}\frac{\partial f}{\partial v}(z,w,v) + \frac{\partial \ell}{\partial v}(z,w) = 0$$

where $f(z,\ w,\ v)$ is the right side of (71) and $\ell(z,\ w)$ is the integrand of (29).

Al'brecht [1] has analyzed the formal solvability of the HJB equations. We expect the optimal cost $\theta(z,\ w)$ to begin with a quadratic term in z

$$(79)\quad \theta(z,\ w) = \frac{1}{2}z^*Tz + \theta^{[3]}(z,\ w) + \cdots$$

and the optimal feedback to begin with a linear term in z,

$$(80)\quad v = Lz + \bar{\kappa}^{[2]}(z,w) + \cdots.$$

The lowest degree terms in (77) are the familiar equations of a LQR,

$$(81)\quad \frac{1}{2}z^*\left(TF + F^*T + Q - (TG+S)R^{-1}(TG+S)^*\right)z = 0$$

$$(82)\quad z^*(TG+S) + v^*R = 0.$$

We solve the Riccati equation (81) for T and from (82) obtain the linear feedback gain,

$$(83)\quad L = -R^{-1}(TG+S)^*.$$

Next we look at the cubic part of (77) and the quadratic part of (78)

$$\text{(84)} \qquad \frac{\partial \theta^{[3]}}{\partial(z,w)}(z,\ w)\begin{bmatrix} (F+GL)z \\ Aw \end{bmatrix} + z^*T\bar{f}^{[2]}(z,\ w,\ Lz) = 0$$

$$\text{(85)} \qquad \frac{\partial \theta^{[3]}}{\partial z}(z,\ w)G + z^*T\frac{\partial \bar{f}^{[2]}}{\partial v}(z,\ w,\ Lz) + \bar{\kappa}^{[2]^*}(z,\ w)R = 0$$

where $\bar{f}^{[2]}(z,\ w,\ v)$ is given by (74).

This (84,85) is a system of linear equations for the unknowns, $\theta^{[2]}$ and $\bar{\kappa}^{[2]}$. If we can solve (84) for $\theta^{[3]}(z,\ w)$ then the quadratic feedback $\bar{\kappa}^{[2]}(z,w)$ is given by

$$\text{(86)} \qquad \bar{\kappa}^{[2]}(z,\ w) = -R^{-1}\left(\frac{\partial \theta^{[3]}}{\partial z}(z,w)G + z^*T\frac{\partial \bar{f}^{[2]}}{\partial v}(z,\ w,\ Lz)\right)^*$$

Let $(\omega_1, \ldots, \omega_d)$ be a basis of $\mathbb{C}^{1\times d}$ of left eigenvectors and generalized eigenvectors of A as in (41, 42). Let $(\zeta_1,\ \ldots, \zeta_n)$ be a basis for $\mathbb{C}^{n\times 1}$ of left eigenvectors or generalized left eigenvectors of $(F+GL)$

$$\text{(87)} \qquad \zeta_i(F+GL) = \mu_i\zeta_i$$

or

$$\text{(88)} \qquad \zeta_i(F+GL) = \mu_i\zeta_i + \zeta_{i+1}$$

where $\mu_1, \ldots, \mu_n$ are the eigenvalues of $(F+GL)$. Then $\theta^{[3]}(z,w)$ can be expanded

$$\text{(89)} \qquad \begin{aligned} \theta^{[3]}(z,\ w) \;=\; & \sum_{1\le i\le j\le k\le n} \theta^{ijk}_{zzz}(\zeta_i z)(\zeta_j z)(\zeta_k z) \\ &+ \sum_{\substack{1\le i\le j\le n \\ 1\le k\le d}} \theta^{ijk}_{zzw}(\zeta_i z)(\zeta_j z)(\omega_k w) \\ &+ \sum_{\substack{1\le i\le n \\ 1\le j\le k\le d}} \theta^{ijk}_{zww}(\zeta_i z)(\omega_j w)(\omega_k w) \\ &+ \sum_{1\le i\le j\le k\le d} \theta^{ijk}_{www}(\omega_i w)(\omega_j w)(\omega_k w)\,. \end{aligned}$$

Because of (76), $z^*T\bar{f}^{[2]}(z,\ w,\ Lz)$ is at least quadratic in z so

$$\text{(90)} \qquad \begin{aligned} z^*T\bar{f}^{[2]}(z,\ w,\ Lz) \;=\; & \sum_{1\le i\le j\le k\le n} f^{ijk}_{zzz}(\zeta_i z)(\zeta_j z)(\zeta_k z) \\ &+ \sum_{\substack{1\le i\le j\le n \\ 1\le k\le d}} f^{ijk}_{zzw}(\zeta_i z)(\zeta_j z)(\omega_k w)\,. \end{aligned}$$

If $F+GL$ and A are both semisimple then when we substitute (89, 90) into (84) we obtain a diagonal system of equations

$$\text{(91)} \qquad (\mu_i+\mu_j+\mu_k)\,\theta^{ijk}_{zzz} = -f^{ijk}_{zzz}$$

for $1 \le i \le j \le k \le n$ and

$$\text{(92)} \qquad (\mu_i+\mu_j+\lambda_k)\,\theta^{ijk}_{zzw} = -f^{ijk}_{zzw}$$

for $1 \le i \le j \le n$ and $1 \le k \le d$.

If $(F+GL)$ or A is not semisimple, then when the terms of the expansions (89, 90) are written in lexographic order we obtain a lower triangular system with (91, 92) on the diagonal. Hence if there are no reasonances, i.e.,

$$
\begin{aligned}
&\mu_i + \mu_j + \mu_k \neq 0 \qquad 1 \le i \le j \le k \le n \\
&\mu_i + \mu_j + \mu_k \neq o \qquad 1 \le i \le j \le k \le n, 1 \le k \le d
\end{aligned}
\tag{93}
$$

then (84, 85) is solvable.

Under standard assumptions, the spectrum of $(F+GL)$ will always lie in the open left half of the complex plane. The spectrum of a typical signal generator is in the closed left half of the complex plane. Therefore, under standard assumptions the nonresonance conditions (93) will be satisfied. Notice that the total feedback up to degree two is

$$u = Kw + Lz + \kappa^{[2]}(w) + \bar{\kappa}^{[2]}(z,\ w). \tag{94}$$

Note also that

$$
\begin{aligned}
&\theta^{[3]}(0, w) = 0 \\
&\frac{\partial \theta^{[3]}}{\partial z}(0,\ w) = 0 \\
&\bar{\kappa}^{[2]}(0,\ w) = 0.
\end{aligned}
\tag{95}
$$

Hence when $z = 0$ then the optimal cost up to degree three is zero and the optimal feedback up to degree two is

$$u = Kw + \kappa^{[2]}(w) \tag{96}$$

which nearly keeps the system (37 , 38) on the manifold

$$z = x - Pw - \pi^{[2]}(w) = 0 \tag{97}$$

The extra terms in feedback (94) when $z \neq 0$ drive the system toward (97) in an optimal way up to degree two. This generalizes to higher degrees as follows.

Theorem 4.1 *Assume the following:*

(i) The linear part of the plant (37) is stabilizable.

(ii) The poles of the linear signal generator (2) lie in the closed half of the complex plane.

(iii) A solution (39) exists of the FBI equations up to degree k.

(iv) The quadratic cost (29) satisfies (23, 24) and $(Q^{1/2},\ F)$ is detectable.

Then the HJB equations for the NQR of minimizing (29) where

$$z = x - Pw - \pi^{[2]}(w) - \cdots - \pi^{k}(w) \tag{98}$$

$$v = u - Kw - \kappa^{[2]}(w) - \cdots - \kappa^{[k]}(w) \tag{99}$$

are formally solvable up to degree $k+1$ *in the optimal cost* $\theta(z,w)$ *and up to degree* k *in the optimal feedback* $\bar{\kappa}(z,\ w)$. *The unique solution*

$$\begin{aligned} \theta(z,w) &= \frac{1}{2}z^*Tz + \theta^{[3]}(z,w) + \cdots + \theta^{[k+1]}(z,w) && (100)\\ \bar{\kappa}(z,w) &= Lz + \bar{\kappa}^{[2]}(z,w) + \cdots + \bar{\kappa}^{[k]}(z,w) && (101) \end{aligned}$$

satisfies

$$\begin{aligned} \theta(0,w) &= 0\\ \frac{\partial\theta}{\partial z}(0,w) &= 0 \qquad (102)\\ \bar{\kappa}(0,w) &= 0. \end{aligned}$$

Proof (By induction on k). The above analysis showed that theorem is true for $k = 1$ and 2. Now suppose the HJB equations are solvable up to degree k in the cost and degree $k-1$ in the feedback. The degrees $k+1$, k parts of the H-J-B equations (84, 85) are

$$\begin{aligned} &\frac{\partial\theta^{[k+1]}}{\partial(z,w)}(z,w)\begin{bmatrix}(F+GL)z\\ Aw\end{bmatrix} + \frac{\partial\theta^{[k]}}{\partial z}(z,w)\bar{f}^{[2]}(z,w,0) \qquad (103)\\ &\qquad + \cdots + z^*T\left(\bar{f}^{[k]}(z,w,0) + G\bar{\kappa}^{[k]}(z,w)\right)\\ &\qquad + z^*(S + L^*R)\,\bar{\kappa}^{[k]}(z,w) + \sum_{i=2}^{k-1}\kappa^{[i]^*}(z,w)R\kappa^{[k+1-i]}(z,w) = 0 \end{aligned}$$

$$\begin{aligned} &\frac{\partial\theta^{[k+1]}}{\partial z}(z,w)G + \frac{\partial\theta^{[k]}}{\partial z}(z,w)\frac{\partial\bar{f}^{[2]}}{\partial v}(z,w,0) \qquad (104)\\ &\qquad + \cdots + z^*T\frac{\partial\bar{f}^{[k]}}{\partial v}(z,w,0)\\ &\qquad + \bar{\kappa}^{[k]^*}(z,w)R = 0 \end{aligned}$$

where $\bar{f}^{[j]}(z,w,v)$ is the degree j part of composition of $f(x,u)$ appearing in (32) with

$$\begin{aligned} x &= z + Pw + \pi^{[2]}(w) + \cdots + \pi^{[k]}(w) \qquad (105)\\ u &= v + Kw + Lz + \kappa^{[2]}(w) + \bar{\kappa}^{[2]}(z,w) + \cdots + \bar{\kappa}^{[k-1]}(z,\ w) \end{aligned}$$

Because of (83), one of the unknowns $\bar{\kappa}^{[k]}(z,w)$ drops out of (103), so we try to solve (103) for the other unknown, $\theta^{[k+1]}(z,w)$. It is possible, then (104) is solvable for $\bar{\kappa}^{[k]}(z,w)$,

$$(106) \qquad \bar{\kappa}^{[k]}(z,w) = -R^{-1}\left[\frac{\partial \theta}{\partial z}^{[k+1]}(z,w)G + \cdots + z^{*}T\frac{\partial \bar{f}}{\partial v}^{[k]}(z,w,0)\right]^{*}$$

Since (98, 99) solve the FBI equations up to degree k and (102) holds by induction it follows that

$$(107) \qquad \bar{f}^{[j]}(0,w,0) = 0$$

for $j = 2,\ldots,k$. Therefore all the terms but the first in (103) are at least quadratic in z. When we expand $\theta^{[k+1]}(z,w)$ and the rest of the terms in (103) in monomials consisting $k+1$ fold products of $(\zeta_i z)$ and $(\omega_j w)$, we obtain a diagonal or at least lower triangular system, which is solvable if the nonresonance condition hold

$$(108) \qquad \begin{aligned} &\mu_{i_1} + \cdots + \mu_{i_{k+1}} \neq 0 \\ &\mu_{i_1} + \cdots + \mu_{i_k} + \lambda_{i_{k+1}} \neq 0 \\ &\cdots \\ &\mu_{i_1} + \mu_{i_2} + \lambda_{i_3} + \cdots + \lambda_{i_{k+1}} \neq 0. \end{aligned}$$

These hold since $Re(\mu_i) < 0$ and $Re(\lambda_i) \leq 0$. Hence there eixts a unique $\theta^{[k+1]}(z,\ w)$ satisfying (103) and it is least quadratic in z. It follows that $\bar{\kappa}^{[k]}(z,\ w)$ exists (106) and is at least linear in z.

QED

We note that if there are several solution (37) to the FBI equations of degree k, we can choose that which minimizes the generalization of (68). Of course there are other reasonable choices.

The term by term approach to the HJB equations can also be used with cost criteria like (31) instead of (29). The key points are that the closed loop linear part of the plant $(F+GL)$ be asymptotically stable and that cost vanish up to order k when $z=0$ and $v=0$ so that all but the first term of (103) is at least quadratic in z.

Suppose the hypothesis of Theorem 4.1 hold so that we can solve the FBI and HJB equations up to degree k. Then we have the total feedback up to degree k,

$$(109) \qquad u(z,\ w) = \kappa(w) + \bar{\kappa}\left(z\left(x,w\right),w\right)$$

when $z(x,w)$ is given by (98), $\kappa(w)$ is (39) to degree k and $\bar{\kappa}(z,w)$ is (101). The optimal cost is

$$\theta(x,w) = \theta\left(z\left(x,w\right),w\right) \tag{110}$$

where $\theta(z,w)$ is (100) and the running cost $\ell(x,w)$ is

$$\begin{aligned}\ell(x,w) &= \tfrac{1}{2}\left(z^*(x,w)Qz(x,w)_2 z^*(x,w)Sv(x,w)\right) \\ &\quad + v(x,w)Rv(x,w)\end{aligned} \tag{111}$$

where $z(x,w)$ is given by (98), $v(x,w)$ is given by (99) with $u = u(x,w)$ as in (109). From the HJB equations (77) we have

$$\frac{d}{dt}\theta\left(x(t),w(t)\right) = -\ell\left(x(t),w(t)\right) + O\left(x(t),w(t)\right)^{k+2}. \tag{112}$$

This suggests that $\theta(x,w)$ is a candidate Lyapunov function for the tracking problem as formalized in the following.

Theorem 4.2 *In addition to the hypothesis of Theorem 4.1, assume there exists a region* Ω *in* $\mathbb{R}^n \times \mathbb{R}^d$ *containing* $(0,0)$ *such that*

(i) Ω *is positively invariant under the closed loop dynamics (32) (2) and (109).*

(ii) $\theta(x,w) \geq 0$ *on* Ω *and* $\theta(x,w) = 0$ *only on* $\{z(x,w) = 0\} \cap \Omega$.

(iii) $\frac{d}{dt}\theta(x,w) \leq 0$ *on* Ω *and the maximal positively invariant subset in* $\left\{\frac{d}{dt}\theta(x,w) = 0\right\} \cap \Omega$ *is contained in* $\{z(x,w) = 0\}$.

Then for any initial condition $(x(0), w(0) \in \Omega)$ *the solution* $(x(t), w(t))$ *of the closed loop dynamics converge to* $\{z(x,w) = 0\} \cap \Omega$ *where the tracking error is* $O(w)^{k+1}$.

The proof of this theorem follows immediately from LaSalle's extension of Lyapunov's direct method.

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