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EXTENDED QUADRATIC CONTROLLER NORMAL FORM AND DYNAMIC STATE FEEDBACK LINEARIZATION OF NONLINEAR SYSTEMS*

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Abstract. In this paper, a set of extended quadratic controller normal forms of linearly controllable nonlinear systems is given, which is the generalization of the Brunovsky form of linear systems. A set of invariants under the quadratic changes of coordinates and feedbacks is found. It is then proved that any linearly controllable nonlinear system is linearizable to second degree by a dynamic state feedback.

Key words. nonlinear systems, quadratic normal forms, invariants, dynamic state feedbacks

AMS(MOS) subject classifications. 93C10, 93C15

1. Introduction. It is well known that there are four normal forms of linear systems: controllable, observable, controller, and observer form. The nonlinear generalizations of these four linear normal forms were given and discussed in Krener [12], Hunt and Su [5], Jakubczyk and Respondek [8], Brockett [1], and Sommer [16], among others. For a system in controller normal form, the design of a stabilizing state feedback control law is a straightforward task. Unfortunately, most controllable systems do not admit a controller normal form, and even when one does, the transformation of a system into controller normal form involves solving a system of first-order partial differential equations (PDEs), which numerically can be quite difficult. For these reasons, the approximate versions of nonlinear controller and observer normal forms were introduced in Krener [11], Krener et al. [13], Phelps and Krener [14], and Karahan [10], among others. It was proved that for certain kinds of nonlinear controllable systems, we can find a nonlinear change of coordinates and nonlinear state feedback that transforms the system into the linear approximation of the plant dynamics, which is accurate to second or higher degree. The computation of such a change of coordinates and state feedback is reduced to solving a set of linear equations. However, these linear equations are not always solvable, and most of the nonlinear systems do not admit such a linear approximation.

In this paper, a set of extended quadratic controller normal forms of linearly controllable systems with single input is given (Theorems 2 and 3). We can consider these normal forms as the extension of the Brunovsky form to the nonlinear systems. Then we prove that, given a nonlinear system, there exists a dynamic state feedback so that the extended system has a linear approximation that is accurate to at least second degree (Theorem 4). This means that any linearly controllable nonlinear system is linearizable to second degree by a dynamic state feedback (see the corollaries).

In this paper, we only consider the single-input systems. The generalization to multi-input systems will be given in another paper.

2. Extended quadratic controller form and dynamic state feedback linearization. From Brunovsky [2] (see also Kailath [9]), we know that any controllable linear system can be transformed into a controller form by a linear change of coordinates. If, in addition, we also allow linear change of coordinates in the input space and linear state

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where

$$\tilde{g}^{[1]}(x)=\begin{bmatrix}\tilde{g}_1^{[1]}(x)\\ \tilde{g}_2^{[1]}(x)\\ \vdots\\ \tilde{g}_n^{[1]}(x)\end{bmatrix}, \tag{2.11b}$$

$$\tilde{g}_i^{[1]}(x)=\begin{cases}0 & i=1 \text{ or } n,\\ \sum_{j=n-i+2}^{n} a_{ij}x_j & \text{others}.\end{cases} \tag{2.11c}$$

Remark 1. If both the linear and quadratic changes of coordinates and state feedbacks are used, what is the normal form of a nonlinear control system such as (2.4) under this larger transformation group? In fact, all the linear changes of coordinates and state feedbacks that leave the Brunovsky form invariant are $z=cx$, $\tilde{v}=c^{-1}v$, where c is a constant. When we apply this linear transformation to the normal form (2.10), the resulting quadratic part is $c^{-1}\tilde{f}^{[2]}(x)$. Let P denote the projective space induced by the linear space

$$\{\tilde{f}^{[2]}(x);\ \tilde{f}^{[2]}(x) \text{ is in the normal form (2.10)}\}.$$

The above linear transformation does not change $\tilde{f}^{[2]}(x)$ in the projective space P. Therefore, under both the linear and quadratic transformations, the family of the normal forms of systems such as (2.4) is a projective space plus the origin $\tilde{f}^{[2]}(x)=0$.

Since most nonlinear systems (2.4) do not admit a controller form, they cannot be completely linearized by a change of coordinates and a state feedback. We wish to use (2.9) to transform (2.4) into a linear system plus an error of second or higher degree, below:

$$\dot{x}=Fx+Gv+O(x,v)^3. \tag{2.12}$$

A system with this property is said to be quadratically linearizable by (2.9). From the result of Theorem 2, we know that system (2.8) is quadratically linearizable by (2.9) if and only if the corresponding extended quadratic controller form (2.10) satisfies

$$\tilde{f}^{[2]}(x)=0. \tag{2.13}$$

Therefore most nonlinear systems are not quadratically linearizable by state feedback. In the following, we introduce a method of linearizing a nonlinear system to the second degree by a dynamic state feedback. The concept of dynamic state feedback was introduced and studied in Singh [15] and Charlet, Lévine, and Marino [3], [4].

DEFINITION 3. A dynamic state feedback is a system

$$\begin{aligned}\dot{\omega}&=a(\xi,\omega)+b(\xi,\omega)v, && \omega(t)\in R^q,\\ \mu&=c(\xi,\omega)+d(\xi,\omega)v, && v(t)\in R,\end{aligned} \tag{2.14}$$

where q is called the dimension of the dynamic state feedback; $a(\xi,\omega)$, $b(\xi,\omega)$ are q-dimensional vector fields; and $c(\xi,\omega)$, $d(\xi,\omega)$ are scalar functions. In general, they are nonlinear.

Consider system (2.4) with a dynamic state feedback (2.14). The extended system is as follows:

$$\begin{aligned}\begin{bmatrix}\dot{\xi}\\ \dot{\omega}\end{bmatrix}&=\begin{bmatrix}f(\xi)+g(\xi)c(\xi,\omega)\\ a(\xi,\omega)\end{bmatrix}+\begin{bmatrix}g(\xi)d(\xi,\omega)\\ b(\xi,\omega)\end{bmatrix}v\\ &=f_e(\xi,\omega)+g_e(\xi,\omega)v.\end{aligned} \tag{2.15}$$

Let F_e be the Jacobian matrix of $f_e(\xi, \omega)$ at $(0, 0)$; let G_e be $g_e(0, 0)$.

DEFINITION 4. If we can find a dynamic state feedback such that the extended system (2.15) is linearly controllable and it can be transformed into

$$\dot{z} = F_e z + G_e v + O(z, v)^3 \tag{2.16}$$

by a change of coordinates (in the extended state space)

$$\begin{bmatrix} \xi \\ \omega \end{bmatrix} = z + \psi^{[2]}(z), \tag{2.17}$$

then system (2.4) is called quadratically linearizable by a dynamic state feedback.

THEOREM 4. *Any linearly controllable system* (2.8) *is quadratically linearizable by a dynamic state feedback.*

COROLLARY 1. *Any linearly controllable system* (2.4) *is quadratically linearizable by a dynamic state feedback.*

In Corollary 2, below, we show that finding a suitable dynamic state feedback and a change of coordinates in the extended space is equivalent to solving a set of linear equations. Suppose that the Taylor series of the vector fields $f(\xi)$ and $g(\xi)$ in system (2.4) are

$$f(\xi) = F\xi + f^{[2]}(\xi) + O(\xi)^3, \qquad g(\xi) = G + g^{[1]}(\xi) + O(\xi)^2. \tag{2.18}$$

COROLLARY 2. *Suppose that the dimension of the state space of system* (2.4) *is* n. *To quadratically linearize this system by a dynamic state feedback, we can use the following* $(n-1)$*-dimensional dynamic state feedback*:

$$\dot{\omega} = A\omega + Bv, \qquad \mu = \omega_1 + \gamma^{[1]}(\xi, \omega) + \gamma^{[2]}(\xi, \omega), \tag{2.19}$$

where (A, B) *is in Brunovsky form* (2.6a) *of dimension* $n-1$. *The change of coordinates* (2.17) *in the extended state space is*

$$\begin{bmatrix} \xi \\ \omega \end{bmatrix} = \begin{bmatrix} z \\ \omega \end{bmatrix} + \begin{bmatrix} \phi^{[2]}(z, \omega_1, \cdots, \omega_{n-2}) \\ 0 \end{bmatrix}. \tag{2.20}$$

The homogeneous polynomials $\gamma^{[1]}(\xi, \omega)$, $\gamma^{[2]}(\xi, \omega)$ *and the vector fields* $\phi^{[2]}(z, \omega_1, \cdots, \omega_{n-2})$ *are chosen such that the extended system is linearly controllable and that*

$$\begin{aligned} &[Fz + G(\omega_1 + \gamma^{[1]}(z, \omega)), \phi^{[2]}(z, \omega_1, \cdots, \omega_{n-2})] + \frac{\partial \phi^{[2]}}{\partial \omega} A\omega \\ &\quad = G\gamma^{[2]}(z, \omega) + f^{[2]}(z) + g^{[1]}(z)(\omega_1 + \gamma^{[1]}(z, \omega)). \end{aligned} \tag{2.21}$$

Furthermore, by (2.19) *and* (2.20), *system* (2.4) *will be transformed into*

$$\begin{bmatrix} \dot{z} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} Fz + G(\omega_1 + \gamma^{[1]}(z, \omega)) \\ A\omega \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} v + O(z, \omega, v)^3. \tag{2.22}$$

Remark 2. In Charlet, Lévine, and Marino [4], it was proved that if a single-input system is not exactly linearizable by state feedback, then this system is not linearizable by a dynamic state feedback. The result of Corollary 1 means that in the problem of finding the quadratic linearization, the opposite result is true; i.e., any single-input linearly controllable system is quadratically linearizable by a dynamic state feedback.

The theorems and the corollaries in this section will be proved in § 5.

3. Quadratic equivalence. In this section, we will define the family of all the systems, such as (2.8), of certain dimension to be a linear space. An equivalence relation on this linear space will be introduced. Then several theorems on this equivalence relation and the associated classification will be given. All these results will be used in the proofs of the theorems in § 2, given in § 5. The definition of an equivalent relation can be found in [7].

DEFINITION 5. Consider two systems

$$\dot{\xi} = A\xi + B\mu + f_1^{[2]}(\xi) + g_1^{[1]}(\xi)\mu + O(\xi, \mu)^3, \tag{3.1a}$$

$$\dot{x} = Ax + Bv + f_2^{[2]}(x) + g_2^{[1]}(x)v + O(x, v)^3. \tag{3.1b}$$

System (3.1a) is said to be quadratically state feedback equivalent to system (3.1b) if and only if there exists a change of coordinates and state feedback (2.9) such that system (3.1a) is transformed into

$$\dot{x} = Ax + Bv + f_2^{[2]}(x) + g_2^{[1]}(x)v + O(x, v)^3; \tag{3.2}$$

i.e., system (3.1a) is transformed into a system that agrees with (3.1b) up to an error of third degree.

The first kth terms in the Taylor expansion of a vector field is called a k-jet. Therefore the linear and quadratic parts of system (2.8) is the second jet of this system. Similarly, transformation (2.9) is the second jet of the analytic transformation

$$\begin{aligned} \xi &= \xi(x) = x + \phi^{[2]}(x) + O(x)^3, \\ v &= \alpha(x) + \beta(x)\mu = \mu + \alpha^{[2]}(x) + \beta^{[1]}(x)\mu + O(x, \mu)^3. \end{aligned} \tag{3.3}$$

The family of all the transformations of the form (3.3) is a group. The quotient of this group over the normal subgroup of the transformations with vanishing second jets is also a group, and there is a natural one-to-one correspondence between this quotient group and the family of all the second jet transformations (2.9). Therefore the family of the second jet transformations is also a group. It is denoted by **G**. Let T_1 and T_2 be two elements in **G**, as follows:

$$T_1: \begin{cases} \xi = \xi_1 + \phi_1^{[2]}(\xi_1), \\ \mu_1 = \mu + \alpha_1^{[2]}(\xi_1) + \beta_1^{[1]}(\xi_1)\mu \end{cases} \tag{3.4a}$$

and

$$T_2: \begin{cases} \xi_1 = \xi_2 + \phi_2^{[2]}(\xi_2), \\ \mu_2 = \mu_1 + \alpha_2^{[2]}(\xi_2) + \beta_2^{[1]}(\xi_2)\mu_1. \end{cases} \tag{3.4b}$$

Then $T_2 \circ T_1$ are the linear and quadratic parts of the composition of the following two transformations:

$$T_2 \circ T_1: \begin{cases} \xi = \xi_2 + \phi_1^{[2]}(\xi_2) + \phi_2^{[2]}(\xi_2), \\ \mu_2 = \mu + \alpha_1^{[2]}(\xi_2) + \beta_1^{[1]}(\xi_2)\mu + \alpha_2^{[2]}(\xi_2) + \beta_2^{[1]}(\xi_2)\mu. \end{cases} \tag{3.5}$$

The inverse of T_1 is

$$T_1^{-1}: \begin{cases} \xi_1 = \xi - \phi^{[2]}(\xi), \\ \mu = \mu_1 - \alpha^{[2]}(\xi) - \beta^{[1]}(\xi)\mu_1. \end{cases} \tag{3.6}$$

That systems (3.1a) and (3.1b) are quadratically state feedback equivalent means that there is an element in the group of second jet transformations **G** such that it transforms the second jet of (3.1a) to that of (3.1b). So it is easy to show that quadratic equivalence is an equivalence relation (see [6]). We can define a classification on the family of all the systems of the form (2.8) by this equivalence relation. Each class of this classification contains all systems that are quadratically state feedback equivalent to each other. In § 5 we prove Theorem 2 by showing that the extended quadratic controller forms are the representatives of all the equivalent classes.

THEOREM 5. *Consider two nonlinear systems*

$$\dot{\xi}_1 = A\xi_1 + B\mu_1 + f_1^{[2]}(\xi_1) + g_1^{[1]}(\xi_1)\mu_1 + O(\xi_1, \mu_1)^3, \tag{3.7a}$$

$$\dot{\xi}_2 = A\xi_2 + B\mu_2 + f_2^{[2]}(\xi_2) + g_2^{[1]}(\xi_2)\mu_2 + O(\xi_2, \mu_2)^3. \tag{3.7b}$$

They are quadratically state feedback equivalent to each other if and only if there exist functions $\alpha^{[2]}(\xi_2)$, $\beta^{[1]}(\xi_2)$, *and a vector field* $\phi^{[2]}(\xi_2)$ *such that*

$$[A\xi_2, \phi^{[2]}(\xi_2)] + B\alpha^{[2]}(\xi_2) = f_1^{[2]}(\xi_2) - f_2^{[2]}(\xi_2), \tag{3.8a}$$

$$[B, \phi^{[2]}(\xi_2)] + B\beta^{[1]}(\xi_2) = g_1^{[1]}(\xi_2) - g_2^{[1]}(\xi_2). \tag{3.8b}$$

Proof. These two systems are equivalent if and only if there exists a change of coordinates and state feedback, as follows:

$$\xi_1 = \xi_2 + \phi^{[2]}(\xi_2), \qquad \mu_1 = \mu_2 + \alpha^{[2]}(\xi_2) + \beta^{[1]}(\xi_2)\mu_2 \tag{3.9}$$

such that (3.7a) is transformed into (3.7b) by (3.9). Substituting (3.9) into (3.7a), we have that

$$\begin{aligned} \dot{\xi}_2 = {} & A\xi_2 + B\mu_2 + f_2^{[2]}(\xi_2) + g_2^{[1]}(\xi_2)\mu_2 + B(\alpha^{[2]}(\xi_2) + \beta^{[1]}(\xi_2)\mu_2) \\ & + f_1^{[2]}(\xi_2) - f_2^{[2]}(\xi_2) - [A\xi_2, \phi^{[2]}(\xi_2)] + g_1^{[1]}(\xi_2)\mu_2 - g_2^{[1]}(\xi_2)\mu_2 \\ & - [B, \phi^{[2]}(\xi_2)]\mu_2 + O(\xi_2, \mu_2)^3. \end{aligned} \tag{3.10}$$

The detailed proof of (3.10) can be found in Krener et al. [13]. It is clear that (3.10) agrees with (3.7b) up to an error of third and higher degree if and only if equations (3.8) hold. □

Since the set of all the homogeneous polynomials of $(x_1, x_2, \cdots, x_n)$ is a linear space of finite dimension, we can consider $(\phi^{[2]}(x), \alpha^{[2]}(x), \beta^{[1]}(x))$ of (2.9) as an element of a linear space W and $(f^{[2]}(\xi), g^{[1]}(\xi))$ of (2.8) as an element of a linear space V. In this way, we can consider the family of transformation (2.9) and the family of nonlinear system (2.8) as linear spaces W and V. Since the linear part of (2.8) is always in Brunovsky form, we sometimes use $(f^{[2]}, g^{[1]})$ to represent system (2.8). Define a linear map $\mathfrak{A}$ from W to V by the following Lie bracket:

$$\mathfrak{A}(\phi^{[2]}(\xi), \alpha^{[2]}(\xi), \beta^{[1]}(\xi)) = ([A\xi, \phi^{[2]}] + B\alpha^{[2]}, [B, \phi^{[2]}] + B\beta^{[1]}). \tag{3.11}$$

Denote $V_0 = \mathfrak{A}(W)$ = the image of W under $\mathfrak{A}$. By using these notations, we can rewrite Theorem 5 as follows.

THEOREM 5′. *System* (3.7a) *is quadratically state feedback equivalent to system* (3.7b) *if and only if*

$$(f_1^{[2]}, g_1^{[1]}) \in (f_2^{[2]}, g_2^{[1]}) + V_0; \tag{3.12}$$

i.e., $(f_1^{[2]}, g_1^{[1]})$ *and* $(f_2^{[2]}, g_2^{[1]})$ *represent the same element in the quotient space* V/V_0.

Remark 3. Theorem 5′ means that there is a one-to-one correspondence between V/V_0 and the family of all equivalent classes.

Remark 4. A special case of Theorem 5′ is that system (2.8) is quadratically state feedback equivalent to a linear system if and only if $(f^{[2]}, g^{[1]}) \in V_0$. Therefore the elements of V_0 represent all the systems of the form (2.8) that are quadratically linearizable by (2.9).

The following theorem gives us a geometric necessary and sufficient condition for a system to be quadratically linearizable by the change of coordinates and state feedback (2.9).

THEOREM 6. *Consider system* (2.8) *and let*

$$X_r = ad^r_{A\xi+f^{[2]}(\xi)}(B+g^{[1]}(\xi)), \qquad 0 \leqq r \leqq n, \tag{3.13a}$$

$$D^k = C^\infty \operatorname{Span}\{X_r, 0 \leqq r < k\}. \tag{3.13b}$$

System (2.8) *is quadratically state feedback equivalent to the linear system*

$$\dot{\xi} = A\xi + B\mu \tag{3.14}$$

if and only if D^k *is first-degree involutive for* $k = 1, 2, \cdots, n-1$; *i.e., for any* X *and* Y *in* D^k, *we have that*

$$[X, Y] = \sum_{r=0}^{k-1} c_r X_r + O(\xi)^1. \tag{3.15}$$

Proof. This theorem is a particular case of the theorem in Krener [11].

4. Characteristic numbers. In § 3 we defined an equivalence relation by the change of coordinates and state feedback (2.9). In this section, we answer the question of how to determine whether two systems are quadratically state feedback equivalent without trying to solve the system of equations (3.8). We find a set of numbers associated to system (2.8), called characteristic numbers, so that these numbers are invariant under transformation (2.9). Two systems are quadratically state feedback equivalent if and only if they have the same characteristic numbers.

Let C and H be row vectors such that

$$C = [1, 0, 0, \cdots, 0], \tag{4.1a}$$

$$HF^{t-1}G = \begin{cases} 0 & 1 \leqq t \leqq n-1, \\ 1 & t = n. \end{cases} \tag{4.1b}$$

DEFINITION 6. The characteristic numbers of system (2.4) are

$$a^{tr} = HF^{t-1}[ad^{r-1}_{f(\xi)}(g(\xi)), ad^{r-2}_{f(\xi)}(g(\xi))]|_{\xi=0}, \tag{4.2a}$$

where

$$2 \leqq r \leqq n-1, \qquad 1 \leqq t \leqq n-r. \tag{4.2b}$$

Particularly, the characteristic numbers of system (2.8) are

$$\begin{aligned} a^{tr} &= CA^{t-1}[ad^{r-1}_{A\xi+f^{[2]}(\xi)}(B+g^{[1]}(\xi)), ad^{r-2}_{A\xi+f^{[2]}(\xi)}(B+g^{[1]}(\xi))]|_{\xi=0} \\ &= CA^{t-1}[X_{r-1}, X_{r-2}]|_{\xi=0}. \end{aligned} \tag{4.2c}$$

In this section, all the results hold for linearly controllable systems, although they are proved only for the systems whose linear parts are in Brunovsky form.

LEMMA 1. (i) *Let* $X(\xi)$ *and* $Y(\xi)$ *be vector fields*; *then*

$$CA^{t-1}[X(\xi), Y(\xi)] = L_X(CA^{t-1}Y) - L_Y(CA^{t-1}X). \tag{4.3}$$

(ii) *For any integer* $r \geqq 2$, *we have that*

$$\begin{aligned} ad^{r-1}_{A\xi+f^{[2]}(\xi)}(B+g^{[1]}(\xi)) &= (-1)^{r-1}A^{r-1}B + ad^{r-1}_{A\xi}(g^{[1]}(\xi)) \\ &\quad + \sum_{k=0}^{r-2} ad^{r-k-2}_{A\xi}[f^{[2]}(\xi), (-1)^k A^k B] + O(\xi)^2. \end{aligned} \tag{4.4}$$

Proof. (i) It holds that

$$CA^{t-1}[X(\xi), Y(\xi)] = CA^{t-1}\left(\frac{\partial Y}{\partial \xi}X - \frac{\partial X}{\partial \xi}Y\right)$$
$$= \frac{\partial CA^{t-1}Y}{\partial \xi}X - \frac{\partial CA^{t-1}X}{\partial \xi}Y$$
$$= L_x(CA^{t-1}Y) - L_Y(CA^{t-1}X).$$

(ii) Consider identity (4.4). If $r=2$, then

$$ad_{A\xi+f^{[2]}(\xi)}(B+g^{[1]}(\xi)) = -AB + ad_{A\xi}(g^{[1]}(\xi)) + [f^{[2]}(\xi), B] + O(\xi)^2. \tag{4.5}$$

Therefore identity (4.4) is true for $r=2$. Suppose that (4.4) is correct for $r-1$. Consider that

$$ad^{r-1}_{A\xi+f^{[2]}(\xi)}(B+g^{[1]}(\xi))$$
$$= ad_{A\xi+f^{[2]}(\xi)}\Bigg((-1)^{r-2}A^{r-2}B + ad^{r-2}_{A\xi}(g^{[1]}(\xi)) + \sum_{k=0}^{r-3} ad^{r-k-3}_{A\xi}[f^{[2]}(\xi), (-1)^k A^k]\Bigg) + O(\xi)^2$$
$$= ad_{A\xi}\Bigg((-1)^{r-2}A^{r-2}B + ad^{r-2}_{A\xi}(g^{[1]}(\xi)) + \sum_{k=0}^{r-3} ad^{r-k-3}_{A\xi}[f^{[2]}(\xi), (-1)^k A^k]\Bigg)$$
$$+ ad_{f^{[2]}(\xi)}((-1)^{r-2}A^{r-2}B) + O(\xi)^2 \tag{4.6}$$
$$= (-1)^{r-1}A^{r-1}B + ad^{r-1}_{A\xi}(g^{[1]}(\xi)) + \sum_{k=0}^{r-3} ad^{r-k-2}_{A\xi}[f^{[2]}(\xi), (-1)^k A^k B]$$
$$+ [f^{[2]}(\xi)), (-1)^{r-2}A^{r-2}B] + O(\xi)^2$$
$$= (-1)^{r-1}A^{r-1}B + ad^{r-1}_{A\xi}(g^{[1]}(\xi))$$
$$+ \sum_{k=0}^{r-2} ad^{r-k-2}_{A\xi}[f^{[2]}(\xi), (-1)^k A^k B] + O(\xi)^2.$$

Therefore identity (4.4) is true for any $r \geqq 2$.

LEMMA 2. *The characteristic number a^{tr} is a linear map from V to R; i.e., a^{tr} is a linear function of $f^{[2]}(\xi)$ and $g^{[1]}(\xi)$.*

Proof. By (4.3) and (4.4), we can prove the following identity:

$$a^{tr} = L_{(-1)^{r-1}A^{r-1}B}\Bigg(\sum_{k=0}^{r-3} CA^{t-1}ad^{r-k-3}_{A\xi}[f^{[2]}(\xi), (-1)^k A^k B] + CA^{t-1}ad^{r-2}_{A\xi}(g^{[1]}(\xi))\Bigg)$$
$$- L_{(-1)^{r-2}A^{r-2}B}\Bigg(\sum_{k=0}^{r-2} CA^{t-1}ad^{r-k-1}_{A\xi}[f^{[2]}(\xi), (-1)^k A^k B] + CA^{t-1}ad^{r-1}_{A\xi}(g^{[1]}(\xi))\Bigg).$$

This implies that a^{tr} is a linear function of $f^{[2]}(\xi)$ and $g^{[1]}(\xi)$.

LEMMA 3. *A system of the form (2.8) is quadratically linearizable by state feedback if and only if all the characteristic numbers are zero.*

Proof. Suppose that a system of the form (2.8) is quadratically linearizable by state feedback. From (4.4) we know that the constant part of the vector fields in D^r is linearly generated by $\{B, AB, A^2B, \cdots, A^{r-1}B\}$. From Theorem 6, we know that D^k is first-degree involutive for $k = 1, 2, \cdots, n-1$. Therefore

$$[X_{r-1}, X_{r-2}] = \sum_{i=1}^{r} c_i A^{i-1} B + O(\xi)^1. \tag{4.7}$$

So

$$a^{tr} = CA^{t-1}[X_{r-1}, X_{r-2}]|_{\xi=0} = 0, \qquad 2 \leqq r \leqq n-1, \quad 1 \leqq t \leqq n-r \tag{4.8}$$

because

$$CA^{t-1}A^{k-1}B = 0, \qquad 1 \leqq k \leqq r, \quad 1 \leqq t \leqq n-r. \tag{4.9}$$

On the other hand, suppose that

$$a^{tr} = 0, \qquad 2 \leqq r \leqq n-1, \quad 1 \leqq t \leqq n-r; \tag{4.10}$$

i.e.,

$$CA^{t-1}[X_{r-1}, X_{r-2}] = 0, \qquad 2 \leqq r \leq n-1, \quad 1 \leqq t \leqq n-r. \tag{4.11}$$

So

$$[X_{r-1}, X_{r-2}] = \sum_{i=1}^{r} c_i A^{i-1} B + O(\xi)^1 \tag{4.12}$$

for some constants c_i. If D^r is not first-degree involutive, and if D^s is first-degree involutive for any $s < r \leqq n-1$, then there exists X_t, $t < r-1$ such that

$$[X_{r-1}, X_t] \neq \sum_{i=1}^{r} d_i A^{i-1} B + O(\xi)^1 \tag{4.13}$$

for any real numbers $d_1, d_2, \cdots, d_r$. By (4.12), we know that

$$t < r-2. \tag{4.14}$$

From the Jacobi identity of Lie bracket, we have that

$$[X_{r-1}, X_t] = ad_{A\xi + f^{[2]}(\xi)}([X_{r-2}, X_t]) - [X_{r-2}, X_{t+1}]. \tag{4.15}$$

Since D^{r-1} is first-degree involutive and $t+1 \leqq r-2$, we know that

$$[X_{t+1}, X_{r-2}] = \sum_{i=1}^{r-1} \bar{c}_i A^{i-1} B + O(\xi)^1, \tag{4.16a}$$

$$[X_{r-2}, X_t] = \sum_{i=1}^{r-1} \tilde{c}_i A^{i-1} B + O(\xi)^1. \tag{4.16b}$$

This implies that

$$[X_{r-1}, X_t] = \sum_{i=1}^{r} c_i A^{i-1} B + O(\xi)^1. \tag{4.17}$$

It is a contradiction. So the distribution D^k is first-degree involutive for any $1 \leqq k \leqq n-1$. This means that the system is quadratically linearizable by state feedback. □

THEOREM 7. *Two systems of the form* (2.8) *are quadratically state feedback equivalent if and only if the corresponding characteristics numbers are equal.*

Proof. Consider two systems

$$\dot{\xi}_1 = A\xi_1 + B\mu_1 + f_1^{[2]}(\xi_1) + g_1^{[1]}(\xi_1)\mu_1 + O(\xi_1, \mu_1)^3, \tag{4.18a}$$

$$\dot{\xi}_2 = A\xi_2 + B\mu_2 + f_2^{[2]}(\xi_2) + g_2^{[1]}(\xi_2)\mu_2 + O(\xi_2, \mu_2)^3. \tag{4.18b}$$

Let a_1^{tr} and a_2^{tr} be the characteristic numbers of (4.18a) and (4.18b), respectively.

Suppose that (4.18a) and (4.18b) are quadratically state feedback equivalent. From Theorem 5′, we know that

$$(f_1^{[2]}, g_1^{[1]}) \in (f_2^{[2]}, g_2^{[1]}) + V_0; \tag{4.19}$$

i.e.,

$$(f_1^{[2]}, g_1^{[1]}) = (f_2^{[2]}, g_2^{[1]}) + (f^{[2]}, g^{[1]}) \tag{4.20a}$$

and

$$(f^{[2]}, g^{[1]}) \in V_0. \tag{4.20b}$$

Let a^{tr} be the characteristic numbers of $(f^{[2]}, g^{[1]})$. Since the characteristic numbers are linear functions of $f^{[2]}$ and $g^{[1]}$ (Lemma 2), we have that

$$a_1^{tr} = a_2^{tr} + a^{tr}. \tag{4.21}$$

From Lemma 3 and (4.20b), we know that $a^{tr} = 0$. So

$$a_1^{tr} = a_2^{tr}, \qquad 2 \leqq r \leqq n, \quad 1 \leqq t \leqq n - r. \tag{4.22}$$

On the other hand, suppose that all the corresponding characteristic numbers are the same. Then

$$a_1^{tr} - a_2^{tr} = 0, \qquad 2 \leqq r \leqq n, \quad 1 \leqq t \leqq n - r. \tag{4.23}$$

So

$$(f_2^{[2]} - f_1^{[2]}, g_2^{[1]} - g_1^{[1]}) \in V_0. \tag{4.24}$$

Theorem 5′ and (4.24) imply that systems (4.18a) and (4.18b) are quadratically state feedback equivalent.

5. The proofs of the theorems in § 2.

Proof of Theorem 2. Consider the following special kind of $\tilde{f}^{[2]}(x)$:

$$\tilde{f}^{[2]}(x) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \tilde{f}_i(x) \\ \vdots \\ 0 \end{bmatrix} \quad \text{for some } 1 \leqq i \leqq n; \tag{5.1a}$$

here

$$\tilde{f}_i(x) = a_{ij} x_j^2 \quad \text{for some } j \geqq i + 2. \tag{5.1b}$$

Then

$$ad_{\tilde{f}^{[2]}(x)}(A^{r-1}B)=0 \quad \text{for } r\neq n-j+1; \tag{5.2a}$$

$$ad_{\tilde{f}^{[2]}(x)}(A^{n-j}B)=\begin{bmatrix} 0\\ 0\\ \vdots\\ -2a_{ij}x_j\\ \vdots\\ 0\end{bmatrix}. \tag{5.2b}$$

Therefore

(5.3)

$$ad^{r-1}_{Ax+\tilde{f}^{[2]}(x)}(B)=\begin{cases} (-1)^{r-1}A^{r-1}B & r\leqq n-j+1,\\ (-1)^{n-j+1}A^{n-j+1}B+\begin{bmatrix} 0\\ 0\\ \vdots\\ (-1)^r 2a_{ij}x_j\\ \vdots\\ 0\end{bmatrix}+O(x)^2 & r=n-j+2,\\ (-1)^{r-1}A^{r-1}B+\begin{bmatrix} *\\ \vdots\\ *\\ *\\ 2a_{ij}x_{2j+r-n-2}\\ \vdots\\ 0\end{bmatrix}+O(x)^2 & n-j+2<r\leqq 2(n-j+1),\\ (-1)^{r-1}A^{r-1}B+O(x)^2 & r\geqq 2(n-j+1).\end{cases}$$

In (5.3), $*$ denotes a linear polynomial of $(x_j, x_{j+1},\cdots, x_n)$. So

$$[ad^{r-1}_{A\xi+\tilde{f}^{[2]}(\xi)}(B),\ ad^{r-2}_{A\xi+\tilde{f}^{[2]}(\xi)}(B)]=0, \qquad r\neq n-j+2; \tag{5.4a}$$

$$[ad^{r-1}_{A\xi+\tilde{f}^{[2]}(\xi)}(B),\ ad^{r-2}_{A\xi+\tilde{f}^{[2]}(\xi)}(B)]=\begin{bmatrix} 0\\ 0\\ \vdots\\ 2a_{ij}\\ \vdots\\ 0\end{bmatrix}, \qquad r=n-j+2. \tag{5.4b}$$

Therefore

$$\begin{aligned} a^{tr} &= CA^{t-1}[ad^{r-1}_{A\xi+\tilde{f}^{[2]}(\xi)}(B),\ ad^{r-2}_{A\xi+\tilde{f}^{[2]}(\xi)}(B)]\\ &=\begin{cases} 2a_{ij} & r=n-j+2 \text{ and } t=i,\\ 0 & \text{others}.\end{cases}\end{aligned} \tag{5.5}$$

As we know, any $\tilde{f}^{[2]}(x)$ of the form (2.10) is a linear combination of the vector fields given in (5.1). Given any system in the extended quadratic controller form (2.10), from

(5.5) and Lemma 2, we can find the characteristic numbers

$$a^{tr} = 2a_{tn-r+2}. \tag{5.6}$$

This implies that, given a set of characteristic numbers, there exists one and only one system in the extended quadratic controller form that has the given characteristic numbers. Theorem 2 follows this fact and Theorem 7.

Proof of Theorem 3. We prove this theorem with the following two steps:

(i) Any two systems given by (2.11) are not quadratically state feedback equivalent to each other.

(ii) Any system is quadratically state feedback equivalent to a system of the form of (2.11).

To prove (i), let us consider the following two systems of the form (2.11):

$$(0, \tilde{g}^{[1]}(x)) \quad \text{and} \quad (0, \tilde{\tilde{g}}^{[1]}(x)). \tag{5.7}$$

They are quadratically state feedback equivalent to each other if and only if

$$(0, \tilde{g}^{[1]}(x) - \tilde{\tilde{g}}^{[1]}(x)) \tag{5.8}$$

is quadratically linearizable by state feedback (Theorem 5'). However, (5.8) is also a system of the form (2.1). So proving that the result in part (i) is equivalent to proving that any system (2.11) is not quadratically linearizable by state feedback if $\tilde{g}^{[1]}(x)$ is not zero. Assume that

$$\tilde{g}^{[1]}(x) \neq 0 \tag{5.1}$$

and that $\tilde{g}^{[1]}_{t_0}(x)$ is the first entry of $\tilde{g}^{[1]}(x)$ such that $\tilde{g}^{[1]}_t(x) \neq 0$; i.e.,

$$\begin{aligned} &\tilde{g}^{[1]}_t(x) = 0 \quad \text{if } t < t_0, \\ &\tilde{g}^{[1]}_{t_0}(x) \neq 0. \end{aligned} \tag{5.10}$$

Assume that

$$\tilde{g}^{[1]}_{t_0}(x) = a_{n-r+2}x_{n-r+2} + a_{n-r+3}x_{n-r+3} + \cdots + a_n x_n, \tag{5.11a}$$

where

$$a_{n-r+2} \neq 0 \quad \text{and} \quad 2 \leqq r \leqq t_0. \tag{5.11b}$$

Then we have that

$$X_{r-2} = ad^{r-2}_{Ax}(B + \tilde{g}^{[1]})$$

$$= (-1)^{r-2}A^{r-2}B + (-1)^{r-2} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \tilde{g}^{[1]}_{t_0}(x) \\ * \\ \vdots \\ * \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} 0 \\ \vdots \\ 0 \\ \tilde{g} \end{matrix}}\right\} t_0 - r + 2 \\ \\ \\ \end{matrix} \quad ; \tag{5.12a}$$

$$X_{r-1} = ad^{r-1}_{Ax}(B + \tilde{g}^{[1]})$$

$$= (-1)^{r-1}A^{r-1}B + (-1)^{r-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \tilde{g}^{[1]}_{t_0}(x) \\ * \\ \vdots \\ * \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} 0 \\ \vdots \\ 0 \\ \tilde{g} \end{matrix}}\right\} t_0 - r + 1 \\ \\ \\ \end{matrix} \quad . \tag{5.12b}$$

So

$$CA^{t_0-r}[X_{r-1}, X_{r-2}] = a_{n-r+2} \neq 0; \tag{5.13}$$

i.e., the characteristic number $a^{t_0\ rr}$ is not zero. Therefore $(0, \tilde{g}^{[1]}(x))$ is not quadratically linearizable by state feedback. Part (i) is proved.

Now we can prove part (ii). Since any system is quadratically state feedback equivalent to system (2.10), we must prove that any system (2.10) is quadratically state feedback equivalent to system (2.11). Since any system (2.11) is quadratically state feedback equivalent to exactly one system of the form (2.10) (Theorem 2), and different systems given by (2.11) are quadratically state feedback equivalent to different systems of the form (2.10) (part (i) and Theorem 2), also since the set of systems (2.10) and the set of systems (2.11) are linear space of the same dimension $((n-1)(n-2))/2$, we know that any system (2.10) is quadratically state feedback equivalent to system (2.11). Part (ii) is proved.

Proof of Theorem 4. According to Theorem 2, it is sufficient to prove the result for the systems in the extended quadratic controller form. Let the dynamic state feedback be

$$\begin{aligned} \dot{\omega}_1 &= \omega_2, \\ \dot{\omega}_2 &= \omega_3, \\ &\vdots \\ \dot{\omega}_{n-1} &= \tilde{v}, \\ v &= \omega_1 + \gamma^{[2]}(x, \omega), \end{aligned} \tag{5.14}$$

where $\gamma^{[2]}$ is a homogeneous polynomial of second degree in (x, ω). The extended system is

$$\begin{bmatrix} \dot{x} \\ \dot{\omega} \end{bmatrix} = A_1 \begin{bmatrix} x \\ \omega \end{bmatrix} + B_1 \tilde{v} + \begin{bmatrix} \tilde{f}^{[2]}(x) \\ 0 \end{bmatrix} + \begin{bmatrix} B\gamma^{[2]}(x, \omega) \\ 0 \end{bmatrix} \tilde{v}. \tag{5.15}$$

Here (A_1, B_1) is in the form of (2.6a) of dimension $2n-1$. We define the change of coordinates as follows:

$$z_1 = x_1, \tag{5.16a}$$

$$z_k = \text{linear and quadratic part of } \dot{z}_{k-1}, \qquad 2 \leq k \leq n, \tag{5.16b}$$

$$z_{n+p} = \omega_p, \qquad 1 \leq p \leq n-1. \tag{5.16c}$$

We claim that

$$z_k = x_k + \psi_k(x, \omega_1, \cdots, \omega_{k-2}), \qquad 2 \leq k \leq n, \tag{5.17}$$

where $\psi_k(x, \omega_1, \cdots, \omega_{k-2})$ is a homogeneous polynomial of second degree. For $k=2$, we have that

$$z_2 = \text{linear and quadratic part of } \dot{z}_1 = x_2 + \sum_{j \geq 3}^{n} a_{2j} x_j^2. \tag{5.18}$$

So (5.17) is true. Assume that (5.17) is true for $k-1$; then

$$\begin{aligned} \dot{z}_{k-1} &= \dot{x}_{k-1} + \dot{\psi}_{k-1}(x, \omega_1, \cdots, \omega_{k-3}) \\ &= x_k + \sum_{j \geq k+1} a_{k-1j} x_j^2 + \dot{\psi}_{k-1}(x, \omega_1, \cdots, \omega_{k-3}) \\ &= x_k + \psi_k(x, \omega_1, \cdots, \omega_{k-2}) + O(x, \omega_1, \cdots, \omega_{k-2})^3. \end{aligned} \tag{5.19}$$

The last equality is true because

$$\dot{x}_i = x_{i+1} + \tilde{f}^{[2]}(x) + O(x)^3, \qquad \dot{x}_n = \omega_1 + \gamma^{[2]}(x, \omega), \tag{5.20}$$

and $\dot{\omega}_1, \cdots, \dot{\omega}_{k-3}$ are related only to $\omega_1, \cdots, \omega_{k-2}$. Therefore $z_k = x_k + \psi_k(x, \omega_1, \cdots, \omega_{k-2})$. So (5.17) is true for any $2 \leqq k \leqq n$. By (5.16) and (5.17), we have that

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= z_3 + O(z, \tilde{v})^3, \\ &\vdots \\ \dot{z}_{n-1} &= z_n + O(z, \tilde{v})^3, \end{aligned} \tag{5.21a}$$

and

$$\begin{aligned} \dot{z}_n &= \dot{x}_n + \dot{\psi}_n(x, \omega_1, \cdots, \omega_{n-2}) \\ &= \omega_1 + \gamma^{[2]}(x, \omega) + \dot{\psi}_n(x, \omega_1, \cdots, \omega_{n-2}). \end{aligned} \tag{5.21b}$$

Let

$$\gamma^{[2]} = \text{the quadratic part of } -\dot{\psi}_n(x, \omega_1, \cdots, \omega_{n-2}); \tag{5.22}$$

then

$$\dot{z}_n = \omega_1 = z_{n+1} + O(z, \tilde{v})^3. \tag{5.23}$$

Therefore, by the change of coordinates (5.16) and (5.22), system (5.15) is transformed into

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= z_3 + O(z, \tilde{v})^3, \\ &\vdots \\ \dot{z}_{n-1} &= z_n + O(z, \tilde{v})^3, \\ \dot{z}_n &= z_{n+1} + O(z, \tilde{v})^3, \\ \dot{z}_{n+1} &= z_{n+2}, \\ &\vdots \\ \dot{z}_{2n-1} &= \tilde{v}. \end{aligned} \tag{5.24}$$

It is linearly controllable system without quadratic terms. Theorem 4 is proved.

Remark 5. Sometimes the dimension of the dynamic state feedback used in Theorem 2 can be less than $n-1$. Suppose that a system is in extended quadratic controller form (2.10). Let

$$q = \max\{j - i;\ a_{ij} \neq 0, j \geqq i+2, 1 \leqq i \leqq n-2\}. \tag{5.25}$$

To quadratically linearize the system, a q-dimensional dynamic state feedback is sufficient. The proof is almost the same as above, except that (5.17) is changed to

$$z_k = x_k + \psi_k(x, \omega_1, \cdots, \omega_{k-1-n+q}), \qquad n-q+1 \leqq k \leqq n. \tag{5.26}$$

Remark 6. From this proof, we find that the dynamic state feedback is chosen to be in Brunovsky form, as follows:

$$\dot{\omega} = A\omega + Bv, \qquad \mu = \omega_1 + \gamma^{[2]}(x, \omega). \tag{5.27}$$

Furthermore, (5.16) and (5.17) imply that the change of coordinates in the extended state space is

$$\begin{bmatrix} x \\ \omega \end{bmatrix} = \begin{bmatrix} z \\ \omega \end{bmatrix} + \begin{bmatrix} \phi^{[2]}(z, \omega_1, \cdots, \omega_{n-2}) \\ 0 \end{bmatrix}; \tag{5.28}$$

i.e., ω is not changed, and the quadratic part is independent of ω_{n-1}.

Proof of Corollary 1. By Theorem 1, there exists a linear change of coordinates and state feedback

$$\xi_1 = T\xi, \qquad \mu_1 = \alpha(\xi_1) + \beta\mu, \tag{5.29}$$

where $\alpha(\xi_1)$ is a linear function and $\beta \neq 0$ is a constant, such that system (2.4) is transformed into

$$\begin{aligned} \dot{\xi}_1 &= Tf(T^{-1}\xi_1) + Tg(T^{-1}\xi_1)\left(\frac{1}{\beta}\mu_1 - \frac{\alpha(\xi_1)}{\beta}\right) \\ &= A\xi_1 + B\mu_1 + f^{[2]}(\xi_1) + g^{[1]}(\xi_1)\mu_1 + O(\xi_1, \mu_1)^3, \end{aligned} \tag{5.30}$$

where (A, B) is in Brunovsky form. By Theorem 4, we can find a dynamic state feedback such that the extended system can be linearized to the second degree by a change of coordinates. This extended system is

$$\begin{aligned} \dot{\xi}_1 &= Tf(T^{-1}\xi_1) + Tg(T^{-1}\xi_1)\left(\frac{1}{\beta}\mu_1 - \frac{\alpha(\xi_1)}{\beta}\right), \\ \dot{\omega} &= a(\xi_1, \omega) + b(\xi_1, \omega)v, \\ \mu_1 &= c(\xi_1, \omega) + d(\xi_1, \omega)v. \end{aligned} \tag{5.31}$$

It is linearly controllable. Under the old coordinates ξ, this system is

$$\begin{aligned} \dot{\xi} &= f(\xi) + g(\xi)\left\{\frac{1}{\beta}(c(T^{-1}\xi, \omega) + d(T^{-1}\xi, \omega)v) - \frac{\alpha(T^{-1}\xi)}{\beta}\right\}, \\ \dot{\omega} &= a(T^{-1}\xi, \omega) + b(T^{-1}\xi, \omega)v. \end{aligned} \tag{5.32}$$

If we define

$$\mu = \frac{1}{\beta}(c(T^{-1}\xi, \omega) + d(T^{-1}\xi, \omega)v) - \frac{\alpha(T^{-1}\xi, \omega)}{\beta} \tag{5.33}$$

as the output of the dynamic state feedback, then (5.32) becomes

$$\begin{aligned} \dot{\xi} &= f(\xi) + g(\xi)\mu, \\ \dot{\omega} &= a(T^{-1}\xi, \omega) + b(T^{-1}\xi, \omega)v, \\ \mu &= \frac{1}{\beta}(c(T^{-1}\xi, \omega) + d(T^{-1}\xi, \omega)v) - \frac{\alpha(T^{-1}\xi)}{\beta}. \end{aligned} \tag{5.34}$$

System (5.34) is quadratically linearizable under a change of coordinates because system (5.32) is quadratically linearizable. This implies that the system

$$\dot{\xi} = f(\xi) + g(\xi)\mu \tag{5.35}$$

is quadratically linearizable by a dynamic state feedback. Corollary 1 is proved.

Proof of Corollary 2. By Remark 6, we know that the dynamic state feedback in (5.31) can be chosen in Brunovsky form, as follows:

$$\dot{\omega} = A\omega + Bv, \qquad \mu_1 = \omega_1 + \gamma^{[2]}(\xi_1, \omega). \tag{5.36}$$

The dimension of A and B is $n-1$. In this case, the dynamic state feedback in (5.34) is

$$\dot{\omega} = A\omega + Bv, \qquad \mu = \frac{1}{\beta}\omega_1 - \frac{\alpha(T^{-1}\xi)}{\beta} + \gamma^{[2]}(T^{-1}\xi, \omega). \tag{5.37}$$

We make the change of coordinates for ω and v and denote $(1/\beta)\omega$ by ω and $(1/\beta)v$ by v; then the dynamic state feedback (5.37) will be changed to a dynamic state feedback that is in the same form as (2.19). Therefore we proved that any system (2.4) is quadratically linearizable by the dynamic state feedback (2.19). By using Taylor's series expansion (2.18), the extended system is

$$\begin{aligned} \dot{\xi} &= F\xi + G(\omega_1 + \gamma^{[1]}(\xi, \omega) + \gamma^{[2]}(\xi, \omega)) \\ &\quad + f^{[2]}(\xi) + g^{[1]}(\xi)(\omega_1 + \gamma^{[1]}(\xi, \omega)) + O(\xi, \omega)^3, \\ \dot{\omega} &= A\omega + Bv. \end{aligned} \tag{5.38}$$

This system is linearizable by a change of coordinates. From Remark 6, we know that the change of coordinates can be chosen in the form of (5.28); i.e.,

$$\begin{bmatrix} \xi \\ \omega \end{bmatrix} = \begin{bmatrix} z \\ \omega \end{bmatrix} + \begin{bmatrix} \phi^{[2]}(z, \omega_1, \cdots, \omega_{n-2}) \\ 0 \end{bmatrix}. \tag{5.39}$$

Substituting this into the equations in the Theorem 5, we have that

$$\begin{aligned} &\left[\begin{pmatrix} Fz + G(\omega_1 + \gamma^{[1]}(z, \omega)) \\ A\omega \end{pmatrix}, \begin{pmatrix} \phi^{[2]}(z, \omega_1, \cdots, \omega_{n-2}) \\ 0 \end{pmatrix}\right] \\ &\quad = \begin{pmatrix} G\gamma^{[2]}(z, \omega) + f^{[2]}(z) + g^{[1]}(z)\omega_1 + g^{[1]}(z)\gamma^{[1]}(z, \omega) \\ 0 \end{pmatrix}, \end{aligned} \tag{5.40a}$$

$$\left[\begin{pmatrix} 0 \\ B \end{pmatrix}, \begin{pmatrix} \phi^{[2]}(z, \omega_1, \cdots, \omega_{n-2}) \\ 0 \end{pmatrix}\right] = 0. \tag{5.40b}$$

Since $(\partial\phi^{[2]}(z, \omega_1, \cdots, \omega_{n-2}))/\partial\omega_{n-1} = 0$, (5.40b) is always true. Equation (5.40a) is equivalent to

$$\begin{aligned} &\begin{bmatrix} \dfrac{\partial\phi^{[2]}}{\partial z}(Fz + G\omega_1 + G\gamma^{[1]}) + \dfrac{\partial\phi^{[2]}}{\partial\omega}A\omega \\ 0 \end{bmatrix} - \begin{bmatrix} F\phi^{[2]} + G\dfrac{\partial\gamma^{[1]}}{\partial z}\phi^{[2]} \\ 0 \end{bmatrix} \\ &\quad = \begin{bmatrix} G\gamma^{[2]}(z, \omega) + f^{[2]}(z) + g^{[1]}(z)(\omega_1 + \gamma^{[1]}(z, \omega)) \\ 0 \end{bmatrix}. \end{aligned} \tag{5.41}$$

It is equivalent to

$$\begin{aligned} &[Fz + G(\omega_1 + \gamma^{[1]}(z, \omega)), \phi^{[2]}(z, \omega_1, \cdots, \omega_{n-2})] + \frac{\partial\phi^{[2]}(z, \omega_1, \cdots, \omega_{n-2})}{\partial\omega}A\omega \\ &\quad = G\gamma^{[2]}(z, \omega) + f^{[2]}(z) + g^{[1]}(z)(\omega_1 + \gamma^{[1]}(z, \omega)). \end{aligned} \tag{5.42}$$

Corollary 2 is proved.

Remark 7. From (5.37) we know that $\gamma^{[1]}(z, \omega)$ can be chosen as $-\alpha(T^{-1}\xi)/\beta$.

6. An example of Theorem 4. Consider that

$$\dot{\xi}_1 = \xi_2 + \xi_3^2, \quad \dot{\xi}_2 = \xi_3, \quad \dot{\xi}_3 = \mu. \tag{6.1}$$

This is a system in extended quadratic controller form, so it is a typical three-dimensional system that is not quadratically linearizable by state feedback. We construct the following dynamic state feedback:

$$\dot{\xi}_4 = \xi_5, \quad \dot{\xi}_5 = v, \quad \mu = \xi_4 + \gamma^{[2]}(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5), \tag{6.2}$$

where $\gamma^{[2]}(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)$ is a quadratic homogeneous polynomial in $(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)$, which will be determined later. The extended system is

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 + \xi_3^2, \\ \dot{\xi}_2 &= \xi_3, \\ \dot{\xi}_3 &= \xi_4 + \gamma^{[2]}(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5), \\ \dot{\xi}_4 &= \xi_5, \\ \dot{\xi}_5 &= v. \end{aligned} \tag{6.3}$$

By Hunt and Su's method of linearization, let us take

$$\begin{aligned} z_1 &= \xi_1, \\ z_2 &= \xi_2 + \xi_3^2 = \dot{z}_1, \\ z_3 &= \xi_3 + 2\xi_3\xi_4 = \text{linear and quadratic part of } \dot{z}_2, \\ z_4 &= \xi_4 + \gamma + 2\xi_4^2 + 2\xi_3\xi_5 = \text{linear and quadratic part of } \dot{z}_3, \\ z_5 &= \xi_5. \end{aligned} \tag{6.4}$$

If we take $\gamma^{[2]} = -2\xi_4^2 - 2\xi_3\xi_5$, then

$$z_4 = \xi_4. \tag{6.5}$$

Therefore we have that

$$\begin{aligned} \dot{z}_1 &= x_2, \\ \dot{z}_2 &= x_3 + O(x, v)^3, \\ \dot{z}_3 &= x_4 + O(x, v)^3, \\ \dot{z}_4 &= x_5, \\ \dot{z}_5 &= v. \end{aligned} \tag{6.6}$$

Therefore system (6.1) is quadratically linearizable by the dynamic state feedback (6.2). This is an example of Theorem 4. In fact, the idea used in the proof of Theorem 4 is similar to the argument in this example.

In this paper, all the results are restricted to the single-input nonlinear systems. In fact, similar results in the multi-input case are also correct, and they will be given in another paper. The idea of finding quadratic normal forms and extending the state space was also successfully used in the problem of finding nonlinear observers.

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