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# OPTIMAL MODEL MATCHING CONTROLLERS FOR LINEAR AND NONLINEAR SYSTEMS

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ABSTRACT: Optimal model matching controllers for linear and nonlinear systems are constructed by generalizing Francis' work on linear regulators and Isidori and Byrnes' work on nonlinear regulators. These are combined with optimal control techniques to obtain solutions.

KEYWORDS: Linear Systems, Nonlinear Systems, Model Matching Controller .

## 1. Introduction

There are several possible ways of defining the model matching problem, we shall adopt the following. One is given two systems, called the plant and the model. The output of the plant and the model lie in the same Euclidean space but the input and state dimensions need not. The goal is to design a feedforward and feedback control law so that the output of the plant asymptotically tracks the output of the model. The control law has feedforward from the model input and state to the plant input and feedback from the plant state to the plant input. If either the state of the model or the state of the plant is not directly measurable then observers are needed to estimate these.

Our approach to linear model matching is based on Francis' approach to the linear servo problem [F] and our construction of a linear optimal control problem in the transverse variables [K]. The nonlinear generalization is based on Isidori and Byrnes' nonlinear regulator [I—B], Huang and Rugh's [H—R] term by term solution of the Francis—Byrnes—Isidori (FBI) equations and Al'brecht's [A] term by term solution of the Hamilton—Jacobi—Bellman (HJB) equations.

For other approaches to the linear model matching problem we refer the reader to Moore and Silverman [M—S], Morse [M] and Malabre [Ma] and their references. The nonlinear model matching problem has been treated by Moog, Perdon and Conti [M—P—C], Huijberts and Nijmeijer [H—N], Grizzle and DiBenedetto [G—D] and their references.

## 2. Linear Model Matching

We consider a linear plant described by

$$\dot{x} = Fx + Gu \tag{2.1a}$$

$$y = Hx + Ju \tag{2.1b}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$. To simplify discussion, we shall assume that the plant is square, $m = p$. We also assume that $(F, G)$ is controllable, $(H, F)$ is observable, $\begin{bmatrix} G \\ J \end{bmatrix}$ is of full column rank and $[H \;\; J]$ is of full row rank. These assumptions can be weakened.
We also have a model

$$\dot{x}_m = Ax_m + Bu_m \tag{2.2a}$$

$$y_m = Cx_m + Du_m \tag{2.2b}$$

with dimensions $n_m$, $m_m$, and $p_m = p$.
The goal is to find a feedforward/feedback law of the form

$$u = Kx + Lx_m + Mu_m \tag{2.3a}$$

such that the plant is asymptotically stable around $x_m = 0$, $u_m = 0$ and such that the mismatch error

$$(2.3b) \qquad e = y - y_m$$

goes to zero as $t \to \infty$. If we assume that $B = 0$ and $D = 0$ then this reduces to the linear regulator problem of Francis [F]. If we assume that $C = 0$ and $D = 0$ then this is a linear version of the gain scheduling problem of Guo and Rugh [G—R]. For these reasons, we don't require the model to be controllable or observable. On the other hand it is reasonable to require that the model is stable or at worst neutrally stable. Moreover, if $B \neq 0$ then it is reasonable to require that the model is bounded input, bounded output stable.

In certain contexts it may happen that state of the model is occasionally reinitialized. The goal then is to achieve model matching on a significantly faster time scale than the times between reinitialization of the model.

Following Francis, we say the model matching problem for (2.1) and (2.2) is solvable if there exists a linear mapping from the state and input of the model to the state and input of the plant

$$(2.4a) \qquad \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} T & 0 \\ L & M \end{bmatrix} \begin{bmatrix} x_m \\ u_m \end{bmatrix}$$

such that

$$(2.4b) \qquad \begin{bmatrix} F & G \\ H & J \end{bmatrix} \begin{bmatrix} T & 0 \\ L & M \end{bmatrix} = \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

If we delete the second block column then (2.4b) is called the Francis' equations.

The intuitive content of (2.4) is that the linear submanifold $x = Tx_m$ of combined state spaces is invariant under the closed loop dynamics with $u = Lx_m + Mu_m$ for all $u_m(t)$. Also, on this linear submanifold the mismatch error $e(t)$ is zero. Although there is no mention of stability, if (2.4) is solvable then we shall show that it is easy to choose $K$ so that linear submanifold $x = Tx_m$ is asymptotically stable under the feedforward and feedback

$$(2.5) \qquad u = K(x - Tx_m) + Lx_m + Mu_m.$$

A frequency $s \in \mathbb{C}$ is a <u>zero frequency</u> if

$$(2.6a) \qquad \operatorname{rank} \begin{bmatrix} F - sI & G \\ H & T \end{bmatrix} < n + p .$$

If $s$ is a zero frequency then there exists a $(\zeta, \psi) \in \mathbb{C}^{1\times n} \times \mathbb{C}^{1\times p}$ such that $\psi \neq 0$ and

$$(2.6b) \qquad [\zeta \;\; \psi] \begin{bmatrix} F - sI & G \\ H & J \end{bmatrix} = [0 \;\; 0] .$$

The triple $(s, \zeta, \psi)$ is called an output zero triple.

The output zero structure need not be semisimple, there may exist sequences of triples $(s, \zeta_i, \psi_i)$ for $i = 1, \cdots, d$ such that $\psi_1, \cdots, \psi_d$ are linearly independent and

$$(2.6c) \qquad \begin{bmatrix} \zeta_1 & \psi_1 \\ \zeta_2 & \psi_2 \\ \vdots & \vdots \\ \zeta_d & \psi_d \end{bmatrix} \begin{bmatrix} F-sI & G \\ H & J \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \zeta_1 & 0 \\ \vdots & \vdots \\ \zeta_{d-1} & 0 \end{bmatrix}$$

The relevance of these concepts for the model matching problem is as follows. Suppose (2.5b) holds and $(s, \zeta, \psi)$ is an output triple of the plant. We subtract the identity

$$\begin{bmatrix} sI & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T & 0 \\ L & M \end{bmatrix} = \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} sI & 0 \\ 0 & 0 \end{bmatrix}$$

from (2.5b) to obtain

$$(2.7) \qquad \begin{bmatrix} F - sI & G \\ H & J \end{bmatrix} \begin{bmatrix} T & 0 \\ L & M \end{bmatrix} = \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}$$

and we multiply (2.7) by $[\zeta \;\; \psi]$ to obtain

$$[0 \;\; 0] = [\zeta T \;\; \psi] \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}$$

so $(s, \zeta, \psi)$ is an output zero triple of the model where $\xi = \zeta T$. Hence if (2.4b) is solvable then corresponding to each output triple $(s, \zeta, \psi)$ of the plant, there is an output triple $(s, \xi, \psi)$ of the model. In a similar way we see from (2.6c) and (2.7) that corresponding to each sequence

$(s, \zeta_i, \psi_i)$, $i = 1, \cdots, d$ of generalized output triples of the plant is a sequence $(s, \xi_i, \psi_i)$, $i = 1, \cdots, d$, of generalized output zero triples of the model where $\xi_i = \zeta_i T$. Moreover, for corresponding output triples both $\zeta$ and $\xi$ are determined by $\psi$,

$$\zeta = \psi H(sI - F)^{-1} \tag{2.8a}$$

$$\xi = \psi C(sI - A)^{-1} \tag{2.8b}$$

and for corresponding generalized output zero triples both $\zeta_i$ and $\xi_i$ are determined by $\psi_i$ and $\zeta_{i-1}, \xi_{i-1}$.

$$\zeta_i = (\psi_i H - \zeta_{i-1})(sI - F)^{-1} \tag{2.9a}$$

$$\zeta_i T = (\psi_i C - \zeta_{i-1} T)(sI - A)^{-1} \tag{2.9b}$$

Hence, if there are n linearly independent triples then T is uniquely determined by (2.8) and (2.9).

These necessary conditions are not sufficient. To obtain necessary and sufficient conditions we must examine the infinite output zeros of the plant and model. Consider a pair of vectors $[\zeta(1/s) \;\; \psi(1/s)]$ which are polynomial in $1/s$ such that as $s \to \infty$

$$\psi(1/s) \to \psi_\infty \neq 0 \tag{2.9a}$$

and

$$[\zeta(1/s) \;\; \psi(1/s)] \begin{bmatrix} F - sI & G \\ H & J \end{bmatrix} = O(1/s)^d . \tag{2.9b}$$

A triple $(s, \zeta(1/s), \psi(1/s))$ satisfying (2.9) is called an infinite output zero triple of degree d of the plant. Once again from (2.7) we see that for each infinite output zero triple $(s, \zeta(1/s), \psi(1/s))$ of degree d of the plant, there must be an finite output zero triple $(s, \zeta(1/s)T, \psi(1/s))$ of degree d of the model. Each infinite output zero triple satisfies

$$\zeta(1/s) = \psi(1/s)H(sI - F)^{-1} \tag{2.10a}$$

$$= \frac{1}{s} \psi(s)H(I + \frac{F}{s} + \frac{F^2}{s^2} + \cdots )$$

$$\xi(1/s) = \psi(1/s)C(sI - A)^{-1} \tag{2.10b}$$

$$= \frac{1}{s} \psi(s)H(I + \frac{A}{s} + \frac{A^2}{s^2} + \cdots )$$

so $\zeta(1/s)$ and $\xi(1/s)$ are $O(1/s)$ as $s \to \infty$.

We have proven one direction of the following theorem. Because of space limitations, the other half of the proof is omitted.

<u>Theorem 2.1</u>. The linear model matching equations (2.5) are solvable iff the following three conditions hold: (i) for every output zero triple $(s, \zeta, \psi)$ of the plant there is an output zero triple $(s, \xi, \psi)$ of the model, (ii) for every sequence of generalized output zero triples $(s, \zeta_i, \psi_i)$ $i = 1, \cdots, d$, of the plant, there is a sequence of generalized output zero triples $(s, \xi_i, \psi_i)$ of the model, (iii) for every infinite output zero triple $(s, \zeta(1/s), \psi(1/s))$ of degree d of the plant there is an infinite output zero triple $(s, \xi(1/s), \psi(1/s))$ of degree d of the model.

If we have a solution of the model matching equation (2.5), then we can define new coordinates on the combined system consisting of the plant and the model

$$z = x - Tx_m \tag{2.11a}$$

$$v = u - Lx_m - Mu_m \tag{2.11b}$$

$$e = y - y_m . \tag{2.11c}$$

In these coordinates the combined system is

$$\dot{z} = Fz + Gv \tag{2.12a}$$

$$\dot{x}_m = Ax_m + Bu_m \tag{2.12b}$$

$$e = Hz + Jv . \tag{2.12c}$$

To find a feedback control, $v = Kz$, that will drive z and e to zero we set up a standard linear quadratic control problem of minimizing

(2.13)
$$\frac{1}{2}\int_0^\infty z^* Qz + 2z^* Sv + v^* Rv \, dt$$

subject to (2.12). The optimal cost

(2.14a)
$$\frac{1}{2} z^* Pz$$

and optimal feedback

(2.14b)
$$v = Kz$$

are found be solving the familiar Riccati equation. The resulting feedforward/feedback is

(2.15)
$$v = Kz = K(x - Tz) .$$

## 3. Nonlinear Model Matching

Suppose we are given a nonlinear plant

(3.1a)
$$\dot{x} = f(x, u)$$

(3.1b)
$$y = h(x, u)$$

and nonlinear model

(3.2a)
$$\dot{x}_m = a(x_m, u_m)$$

(3.2b)
$$y_m = c(x_m, u_m) .$$

Following Isidori and Byrnes [I—B], we seek a submanifold

(3.3a)
$$x = \theta(x_m)$$

and feedforward/feedback

(3.3b)
$$u = \mu(x_m, u_m)$$

such that under the closed loop dynamics the submanifold (3.3a) is invariant and the mismatch error

(3.4)
$$e = h(x, u) - c(x_m, u_m)$$

is zero on this submanifold. In other words $\theta$ and $\mu$ must satisfy a generalisation of the Francis—Byrnes—Isidori (FBI) equations, given by

(3.5a)
$$f(\theta(x_m), \mu(x_m, u_m))$$
$$= \frac{\partial \theta}{\partial X_m}(x_m)\, a(x_m, u_m)$$

(3.5b)
$$h(\theta(x_m), \mu(x_m, u_m)) = C(x_m, u_m) .$$

We expand the system and model in a Taylor series

(3.6a)
$$\dot{x} = Fx + Gu + f^{[2]}(x, u) + \cdots$$

(3.6b)
$$y = Hx + Ju + h^{[2]}(x, u) + \cdots$$

(3.7a)
$$\dot{x}_m = Ax_m + Bu_m + a^{[2]}(x_m, u_m) + \cdots$$

(3.7b)
$$y_m = Cx_m + Du_m + c^{[2]}(x_m, u_m) + \cdots$$

where the superscript [2] denotes a homogeneous polynomial of degree two.

We also expand the invariant manifold and feedback/feedforward in a series

(3.8a)
$$\theta(x) = Tx_m + \theta^{[2]}(x_m) + \cdots$$

(3.8b)
$$\mu(x) = Lx_m + Mu_m + \mu^{[2]}(x_m, u_m) + \cdots$$

and plug the expansions into (3.5). Collecting terms of like degree yields at linear level (2.4b) and at the quadratic level .

(3.9a)
$$F\theta^{[2]}(x_m) + G\mu^{[2]}(x_m, u_m) - \frac{\partial \theta^{[2]}}{\partial x_m}(x_m)(Ax_m + Bu_m)$$
$$= Ta^{[2]}(x_m, u_m) - f^{[2]}(Tx_m, Lx_m + Mu_m).$$

$$(3.9b) \quad H\theta^{[2]}(x_m) + J\mu^{[2]}(x_m, u_m) = c^{[2]}(x_m, u_m) - h^{[2]}(Tx_m, Lx_m + Mu_m) .$$

Assuming T, L and M satisfy (2.4b), we break up (3.9) into terms quadratic in $x_m$, bilinear in $x_m$ and $u_m$ and quadratic in $u_m$.

In obvious abuse of notation the term quadratic in $x_m$ yield the equation

$$(3.10) \quad \begin{bmatrix} F & G \\ H & J \end{bmatrix} \begin{bmatrix} \theta^{[2]}(x_m, x_m) \\ \mu^{[2]}(x_m, x_m) \end{bmatrix} - \begin{bmatrix} \frac{\partial\theta^{[2]}}{\partial x_m}(x_m) \\ 0 \end{bmatrix} Ax_m = \begin{bmatrix} Ta^{[2]}(x_m, x_m) \\ c^{[2]}(x_m, x_m) \end{bmatrix} - \begin{bmatrix} a^{[2]}(x_m, x_m) \\ h^{[2]}(x_m, x_m) \end{bmatrix}$$

These are the familiar FBI equations of degree 2, [K] and are solvable if there is no resonance between an output zero of the plant and a pair of poles of the model [H—R, K]. If (3.10) is solvable then we can turn our attention to the other parts of (3.9), namely

$$(3.11) \quad \begin{bmatrix} G \\ J \end{bmatrix} \mu^{[2]}(x_m, u_m) = \begin{bmatrix} \frac{\partial\theta^{[2]}}{\partial x_m}(x_m) \\ 0 \end{bmatrix} Bu_m + \begin{bmatrix} Ta^{[2]}(x_m, u_m) \\ c^{[2]}(x_m, u_m) \end{bmatrix} - \begin{bmatrix} f^{[2]}(x_m, u_m) \\ h^{[2]}(x_m, u_m) \end{bmatrix}$$

$$(3.12) \quad \begin{bmatrix} G \\ J \end{bmatrix} \mu^{[2]}(u_m, u_m) = \begin{bmatrix} Ta^{[2]}(u_m, u_m) \\ c^{[2]}(u_m, u_m) \end{bmatrix} - \begin{bmatrix} f^{[2]}(u_m, u_m) \\ h^{[2]}(u_m, u_m) \end{bmatrix}.$$

If (3.5) is solvable up to degree two, then we consider the effect of the change of coordinates and feedback

$$(3.13a) \quad z = x - Tx_m - \theta^{[2]}(x_m)$$

$$(3.13b) \quad x_m = x_m$$

$$(3.13c) \quad v = u - Lx_m - Mu_m - \mu^{[2]}(x_m, u_m)$$

$$(3.13d) \quad u_m = u_m$$

on the combined system (3.6a) (3.7a) and output (3.4). The result is (3.7a) and

$$(3.14a) \quad \dot{z} = Fz + Gv + \bar{f}^{[2]}(z, x_m, u_m, v) + \cdots$$

$$(3.14b) \quad e = Hz + Gv + \bar{h}^{[2]}(z, x_m, u_m, v) + \cdots$$

where

$$(3.15a) \quad \bar{f}^{[2]}(z, x_m, u_m, v) = f^{[2]}(z + Tx_m, v + Lx_m + Mu_m) - f^{[2]}(Tx_m, Lx_m + Mu_m)$$

$$(3.15b) \quad \bar{h}^{[2]}(z, x_m, u_m, v) = h^{[2]}(z + Tx_m, v + Lx_m + Mu_m) - h^{[2]}(Tx_m, Lx_m + Mu_m) .$$

Notice that

$$(3.16a) \quad \bar{f}^{[2]}(0, x_m, u_m, 0) = 0$$

$$(3.16b) \quad \bar{h}^{[2]}(0, x_m, u_m, 0) = 0$$

We wish to find a feedback

$$u = Kz + \kappa^{[2]}(z, x_m, u_m) \tag{3.17}$$

which drives the combined system (3.14) to the submanifold, $z = 0$. To do so we set up as before a standard linear quadratic optimal control problem of minimizing (2.14) subject to (3.14). We are considering the infinite time problem, and in order to obtain a stationary solution we assume that the model input is constant in time. We expand the optimal cost

$$\pi(z, x_m, u_m) = \frac{1}{2} z^* Px + \pi^{[3]}(z, x_m, u_m) + \cdots . \tag{3.18}$$

The stationary Hamilton–Jacobi–Bellman equations are

$$\frac{\partial\pi(z, x_m, u_m)}{\partial(z, x_m)} \begin{bmatrix} \bar{f}(z, x_m, u_m, v) \\ a(x_m, u_m) \end{bmatrix} + l(z, v) = 0 \tag{3.18a}$$

$$\frac{\partial\pi}{\partial z}(z, x_m, u_m)\frac{\partial\bar{f}}{\partial v}(z, x_m, u_m, v) + \frac{\partial l}{\partial v}(z, v) = 0 \tag{3.18b}$$

where $\bar{f}(z, x_m, u_m, v)$ is the right side of (3.14a) and $l(z, v)$ is the lagrangian of (2.13). Following [A] and [K], we solve these term by term. The lowest terms in (3.18a) are quadratic and those in (3.18b) are linear, and are the familiar equations of a linear quadratic regulator as discussed in the previous section. We plug this solution into (3.18) and collect cubic terms from (3.18a) and quadratic terms from (3.18b). The first yields an equation for $\pi^{[3]}(z, x_m, u_m)$ which can be solved provided a certain nonresonance condition is satisfied. The condition is that no sum of two or three eigenvalues of $F + GK$ be zero and no sum of two eigenvalues of $F + GK$ and one eigenvalue of $A$ equal zero, where $v = Kz$ is the optimal linear feedback. This will be satisfied if $F + GK$ is strictly stable and $A$ is stable.

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