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# Nonlinear Stabilizability and Detectability

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## 1 Introduction

Since the early 1970's a considerable effort has been devoted to development of a state-space theory of nonlinear systems which parallels and generalizes the state space theory of linear systems that had been developed during the 1960's. Using Lie theoretic tools, many of the synthetic results of linear systems theory were successfully generalized to nonlinear systems including concepts such as controllability, observability, minimality and decoupling. We refer the reader to the monograph of Isidori [I] for a description of these developments. But the Lie theoretic effort was not completely successful in at least two important areas, the design of asymptotically stabilizing state feedback control laws and the design of observers and/or filters for nonlinear systems.

In the late 70's and early 80's there was a revival of interest in Lyapunov techniques for nonlinear stabilization. Such techniques have a long tradition in both the Western and Soviet literatures on nonlinear control. When combined with Lie techniques, they proved to be quite fruitful and substantial progress was made on nonlinear stabilizability [Ba]. In 1983, Brockett gave his well-known three necessary conditions for stabilizability [Br]. In the next section, we review these and other results in this area.

In contrast to nonlinear stabilizability, the question of nonlinear detectability has received much less attention. Loosely speaking, a nonlinear system is detectable if one can construct an observer which asymptotically estimates the state of the system. In the late 1970's and earlier 1980's considerable effort was devoted to the closely related question of nonlinear filtering. Unfortunately, this did not lead to methods for the construction of computationally feasible nonlinear filters. See [H-W] for more on nonlinear filtering. In Section 3, we propose several alternative definitions of nonlinear detectability and give necessary conditions dual to Brockett's for one of these.

## 2 Nonlinear Stabilizability.

We consider the nonlinear system

$$\dot{x} = f(x, u) \tag{1}$$

around a critical value of $x$ and $u$ which we conveniently take to be $0, 0$,

$$0 = f(0, 0) \tag{2}$$

The system is said to be (locally) asymptotically stabilizable around $x = 0$ if there exists a state feedback

$$u = k(x) \tag{3}$$

with

$$0 = k(0) \tag{4}$$

such that the closed loop system

$$\dot{x} = f(x, k(x)) \tag{5}$$

is (locally) asymptotically stable to $x = 0$.

This definition can be refined in several ways. Suppose $f(x, u)$ is $C^r$ where $r = 0, 1, \cdots, \infty$, or $\omega$ then one can ask that $k(x)$ also be

(i) $C^r$ or $C^s, s \leq r$ or

(ii) $C^0$ and piecewise $C^r$ or

(iii) piecewise $C^r$ or

(iv) $C^r$ except at $x = 0$ where it is $C^0$,

(v) etc.

One can require that $k(x)$ be such that closed loop system (5) have unique solutions and that there exists a Lyapunov function for the closed loop system (5).

In [Br], three necessary conditions are given for the local asymptotic stabilizability of (1,2). They are

($S_1$) If a $C^1$ system (1,2) can be locally asymptotically stabilized by a $C^1$ control law (3,4) then the linear approximation to (1,2) at $x = 0, u = 0$

$$\dot{x}= \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial u}(0,0)u \tag{6}$$

has no uncontrollable, unstable modes.

($S_2$) If a $C^0$ system (1,2) can be locally asymptotically stabilized by a $C^0$ control law (3,4) then for every initial condition $x^0$ sufficiently close to 0, there exists an open loop control law $u(t)$ such that the trajectory of (1,2) originating at $x^0$ asymptotically converges to $x = 0$.

($S_3$) If a $C^0$ system (1,2) can be locally asymptotically stabilized by a $C^0$ control law (3,4) with unique solutions to the closed loop dynamics (5) then the map

$$f : \mathbf{R}^n \times \mathbf{R}^m \rightarrow \mathbf{R}^n$$

carries neighborhoods of $(0,0)$ onto neighborhoods of 0.

Clearly ($S_1$) is necessary for $C^1$ system stabilized by $C^1$ feedback, what is surprising is that some $C^1$ systems which fail to satisfy ($S_1$) can be stabilized by $C^0$ feedbacks. The following example is by to M. Kawski [Ka1].

Example 2.1

$$\dot{x}_1= f_1(x_1, x_2)$$

$$\dot{x}_2= u$$

Suppose there exists a function $V(x_1, x_2)$ such that

$$(i) V(x) \geq 0 and = 0 iff x = 0$$

$$(ii) \frac{\partial V}{\partial x_2} = -f$$

then let

$$u = k(x) = \frac{\partial V}{\partial x_1} - \frac{\partial V}{\partial x_2}$$

so that

$$\frac{d}{dt}V = -\left(\frac{\partial V}{\partial x_2}\right)^2$$

In particular, if

$$f_1(x_1, x_2) = x_1 - x_2^3$$

then set

$$V = x_1^{4/3} - x_1 x_2 + \frac{x_2^4}{4}$$

and

$$k(x) = \frac{4}{3}x_1^{1/3} - x_2 - x_1 - x_2^3$$

See [Ba, pp. 28-30] for more details.

Clearly condition ($S_2$) is necessary as the open loop control $u(t)$ is readily obtained from the feedback $u = k(t)$ by setting

$$u(t) = k(x(t)). \tag{7}$$

Sussmann [Su] has shown that for $C^\omega$ systems, ($S_2$) is almost sufficient in that it implies there exists a piecewise $C^\omega$ feedback (3,4) such that for every $x^0$ sufficiently close to 0 there exists an open loop control $u(t)$ driving $x^0$ to 0 and such that $u(t)$ satisfies (7).

This is almost sufficient because there may exist another open loop control $u(t)$ satisfying (7) which fails to drive $x^0$ to 0. In other words, solutions to the piecewise analytic closed loop system (5) may fail to be unique and some solutions may fail to go to 0 .

Brockett's third condition ($S_3$) is based on a result of Krasnosel'skií and Zabreíko [K-Z]. They proved that if the $C^0$ vector field

$$\dot{x} = f(x) \tag{8}$$

$$0 = f(0) \tag{9}$$

is locally asymptotically stable and admits unique solutions in some neighborhood of $x = 0$ then the degree of $f$ at 0 is $(-1)^n$ where $n$ is the dimension of $x$ . They proved this by constructing a homotopy between $f$ and $-x$ in $\mathbf{R}^n - \{0\}$. Hence, if the closed loop system (5) is locally asymptotically stable with unique solutions then the map $x \longmapsto f(x, k(x))$ is locally onto and so is $(x, u) \longmapsto f(x, u)$ .

Unfortunately, the Krasnosel'skií - Zabreĭko necessary condition is relatively weak and hence so is the Brockett condition. If $n$ is even, then the vector fields $f(x) = x$ and $f(x) = -x$ have the same degree but not the same stability. However, at present the Brockett conditions are perhaps the most useful rough and ready test that we have for checking stabilizability. Coron [C] has presented some interesting extensions of ($S_3$).

One weakness of these necessary conditions is that they speak to feedback laws which are at least $C^0$. Many nonlinear systems can only be stabilized by feedback laws which are discontinuous somewhere. While such laws may not lend themselves to topological analysis, they are readily implementable by digital computers. Consider the following example from [Su].

Example 2.2

$$\dot{x}_1 = u_1$$

$$\dot{x}_2 = u_2$$

$$\dot{x}_3 = (x_1 u_2 - x_2 u_1)/2.$$

This is $C^\omega$ satisfying ($S_1$) and ($S_2$) but not ($S_3$). We can try to find open loop controls which drive any $x^0$ at $t = 0$ to the origin at $t = 1$ while minimizing

$$\int_0^1 u_1^2 + u_2^2 dt.$$

Using the maximal principle, it is not hard to see that the optimal trajectories are generated by sinusoidal controls of the form

$$u_1(t) = r\cos(\omega t + \phi)$$

$$u_2(t) = r\sin(\omega t + \phi).$$

When projected onto the $x_1 - x_2$ plane the optimal trajectories are arcs of circles beginning at $(x_1^0, x_2^0)$ and ending at $(0, 0)$. The area enclosed by the arc and the straight line from $(x_1^0, x_2^0)$ to $(0, 0)$ is equal to the magnitude of $x_3^0$ and the orientation of the circular arc is determined by the sign of $x_3^0$. This determines $(r, \omega, \phi)$ as a function of $(x_1^0, x_2^0, x_3^0)$ where $|\omega| \leq 2\pi$ and $0 \leq \phi \leq 2\pi$. If $x_1^0 \neq 0$ or $x_2^0 \neq 0$ then $|\omega| < 2\pi$ and $(r, \omega, \phi)$ is unique. If $x_1^0 = x_2^0 = 0$ then $|\omega| = 2\pi$ and $\phi$

is arbitrary within $0 \leq \phi \leq 2\pi$. Hence optimal controls are unique except along the $x_3$ axis. They yield a feedback $u = k(x)$ which asymptotically stabilizes the system. This feedback is $C^0$ except on the $x_3$ axis. By ($S_3$) we know there is no $C^0$ asymptotically stabilizing feedback.

This example suggests the need for further research on necessary conditions for asymptotically stabilizing feedbacks that are only piecewise $C^0$ in some nice fashion.

We briefly review the methods for constructing stabilizing feedback laws. We have already seen an example of perhaps the most useful approach, namely, set up an optimal control problem where the desired feedback is the optimal solution. In general, we choose a lagrangian $\ell(x, u) \geq 0$ and consider the problem of minimizing

$$\int_0^\infty \ell(x, u) ds \tag{10}$$

subject to the dynamics (1). We are led to the Hamilton-Jacobi-Bellman partial differential equation for the optimal cost $V(x)$ and optimal feedback $u = k(x)$ given that $x(t) = x$,

$$\frac{\partial V}{\partial x}(x) f(x, k(x)) + \ell(x, k(x)) = 0 \tag{11}$$

$$\frac{\partial V}{\partial x}(x) \frac{\partial f}{\partial u}(x, k(x)) + \frac{\partial \ell}{\partial u}(x, k(x)) = 0 \tag{12}$$

A typical choice of lagrangian $\ell(x, u)$ is a quadratic form

$$\ell(x, u) = \frac{1}{2}(x'Qx + u'Ru)$$

when $Q$ is nonnegative definite matrix and $R$ is positive definite matrix.

If the linearization (6) of a $C^\omega$ (1) is stabilizable and $y = Qx$ is a detectable output for (6) then the HJB equations (11,12) can be solved term by term following Al'brecht [Al] and Lukes [L].

Other stabilization methods start with a positive definite function $V(x)$ that is a candidate Lyapunov function for the closed loop system. Jurdjevic and Quinn [J-Q] assume the existence of a positive definite $V(x)$ which demonstrates that the undriven system ($u = 0$) is at least neutrally stable, i.e.

$$\frac{d}{dt} V = \frac{\partial V}{\partial x}(x) f(x, 0) \leq 0 \tag{13}$$

and attempt to find $u = k(x)$ such that

$$\frac{d}{dt}V = \frac{\partial V}{\partial x}(x)f(x, k(x)) \leq 0. \tag{14}$$

If (14) is a strict inequality for all $x \neq 0$ then asymptotic stability is guaranteed, if not then a more detailed analysis using La Salle's invariance principle might guarantee asymptotic stability.

Artstein [A], Tsinias [Ts1],[Ts2] and Sontag [So] assume the existence of a control Lyapunov function $V(x)$ , namely a positive-definite functions such that for every $x$,

$$u\inf \frac{\partial V}{\partial x}(x)f(x, u) \leq 0 \tag{15}$$

They construct selections $u = k(x)$ so that (14) is satisfied.

Hermes [H1],[H2] and Kawski [Ka1], [Ka2], [Ka3], [Ka4] have used homogeneous approximations to construct stabilizing feedbacks. Byrnes-Isidori [B-I], Dayawansa [DM1], [DM2], [DM3], [DM4] and others have used the Center Manifold Theorem [Ca] while Abed-Fu [AF1], [AF2] have employed bifurcation theory. More recently Kokotovic and co-workers [Ko] have employed backstepping techniques based on earlier work of Tsinias and Byrnes-Isidori. Dayawansa-Martin-Knowles [DM1], [DM2], [DM3], [DM4], [DMK] have exhaustively studied the stabilizability of low dimensional systems. For a fuller discussion of these and other approaches we recommend [Ba] and [CTAT].

Before closing this section, we briefly discuss the $H^\infty$ approach to nonlinear stabilization. See [B-B], [I-A], [vdS2] and their references for more on $H^\infty$ control. We assume that the basic model (1) is affine in the control and we add a driving noise $w(t)$

$$\dot{x} = f(x) + g(x)u + b(x)w \tag{16}$$

We assume that it is critical that the control and certain functions of the state

$$z = h(x)u \qquad h(0) = 0 \tag{17}$$

be kept as small as possible. More precisely, we wish to choose a feedback (3) so as to minimize the $L^2$ gain between $w$ and $z$ . This is too hard in general so instead we choose a gain level $\gamma$ and seek to find a feedback so that the closed loop system has gain at most $\gamma$. We shall not go into detail because of space limitations, but refer the reader to [vdS2], [I-A], [B-B], [Kr2]. The solution of this problem involves finding nonnegative solutions $V(x)$ to inequalities such as

$$\frac{\partial V}{\partial x}f + \frac{1}{2}\frac{\partial V}{\partial x}\left(\frac{1}{\gamma^2}bb' - gg'\right)\frac{\partial V'}{\partial x} + \frac{1}{2}hh' \leq 0. \tag{18}$$

This is a Hamilton Jacobi Issaccs inequality generalizing the HJB equation (11,12) and under suitable observability hypothesis its solution is a Lyapunov function for the closed loop system when the driving noise is zero.

# 3 Nonlinear Detectability

For simplicity we restrict our attention to systems with observations but without controls,

$$\dot{x} = f(x) \tag{19}$$

$$y = h(x). \tag{20}$$

Suppose $\Omega_x$ is a region of $x$ space where solutions to (19) are unique and where the system is positively invariant, i.e., if $x(0) \in \Omega$ then $x(t) \in \Omega$ for all $t \geq 0$. We would like to construct a method for estimating the current state $x(t)$ from the past observations, $\{y(s), 0 \leq s \leq t\}$. Ideally we would like the method to be another dynamical system driven by the observations $y(t)$ whose output is the estimate $\hat{x}(t)$ of $x(t)$, i.e.

$$\dot{z} = g(z, y) \tag{21}$$

$$\hat{x} = k(z, y). \tag{22}$$

Such a system is called an observer and it should have certain properties. At the very least it should be well defined, in other words, there should exist a region $\Omega_z$ of $z$ space such that the combined system (19, 20, 21, 22) has unique solutions and is positively invariant on $\Omega_x \times \Omega_z$. In addition, the error

$$e(t) = x(t) - \hat{x}(t) \tag{23}$$

should go to zero as $t \to \infty$. We might also require that when the error is zero, it remains zero. This is equivalent to

$$\dot{e} = 0 \tag{24}$$

when $e = 0$. In some cases this last condition is too much to expect.

When a suitable observer exists, we say the nonlinear system (19,20) is detectable.

There are many possible refinements of the concepts of an observer and of detectability. If the original system is $C^r$ we might ask that the observer also be

$C^r$. A local observer is one such that error $e(t)$ goes to zero whenever the initial error $e(0)$ is sufficiently small.

Detectable linear systems admit observers whose state dimension is no greater than the state of the system but this may be too much to expect for many nonlinear systems. We distinguish three important subcases of observers. The first is a reduced order observer where the dimension of $x$ equals the sum of the dimensions $z$ and $y$ and the map $z, y \longmapsto k(z, y)$ is 1-1 and onto from $\Omega_z \times h(\Omega_x)$ to $\Omega_x$. The second is a full order observer where the dimension of $x$ equals the dimension of $z$ and the estimate does not depend directly on $y$, $\hat{x}= k(z)$ where $k$ is a diffemorphism from $\Omega_z$ to $\Omega_x$. After a change of $z$ coordinates, we have $\hat{x}= z$. The third is an expanded order observer where the dimension of $x$ is less that the dimension of $z$ ($z$ might be infinite dimensional) and again the map $k$ does not depend on $y$, $\hat{x}= k(z)$ where $k$ is a submersion of $\Omega_z$ onto $\Omega_z$. For reduced or full order observers it is reasonable to require (24) but for expanded order observers this is generally not possible.

The above discussion does not fully capture exactly what one would want in an observer. Any asymptotically stable nonlinear system (19,20) is detectable under our current definition even if there is no information in the output, say $y = 0$. For linear systems, this is usually not a problem as detectability implies the existence of an observer with exponentially stable error dynamics, i.e., there exists $\alpha > 0$ and $M > 0$ such that

$$|e(t)| \leq Me^{-\alpha t} |e(0)| . \tag{25}$$

But for asymptotically stable nonlinear system the observer error convergence may be so slow as to be useless. Consider the following:

Example 3.1 Let $x \in \mathbf{R}^2$,

$$f^1(x) = \begin{vmatrix} -sign x_1 \\ \\ 0 \end{vmatrix} \qquad f^2(x) = \begin{vmatrix} -x_1 \\ \\ -x_2 \end{vmatrix},$$

$$\lambda(x_1) \qquad C^\infty function, 0 \leq \lambda(x_1) \leq 1$$

$$\lambda(x_1) = \begin{cases} 1 & x_1 \geq |2\varepsilon| \\ \\ 0 & x_1 \leq |\varepsilon| \end{cases}$$

$$f(x) = \lambda(x_1) f^1(x) + (1 - \lambda(x_1)) f^2(x).$$

$$h(x) = 0.$$

This system is globally asymptotically stable to $x = 0$ . The full order observer

$$\dot{z} = f(z)$$

$$\hat{x} = z$$

is well-defined, satisfies (24) and the error goes to zero. But the error may not change for long periods of time. Suppose the initial values of $x(0)$ and $z(0)$ are both to the far left or far right of the $x_2$ axis. Then the error is constant for $\min \{|x_1(0)|, |z_1(0)|\} - 2\varepsilon$ units of time. This is hardly satisfactory. So we consider imposing stricter requirements for convergence of the observer error.

Following Vidyasagar [V] one might require the existence of function $V(x, z)$ such that there exists class $K$ functions $\alpha_i$ satisfying

$$\alpha_1(|e|) \leq V(x, z) \leq \alpha_2(|e|) \tag{26}$$

$$\dot{V}(x, z) \leq -\alpha_3(|e|). \tag{27}$$

Recall a function $\alpha$ is of class $K$ if it is $C^0$, strictly monotone increasing and $\alpha(0) = 0$. Actually Vidyasagar only considers full order observers but his definition readily generalizes.

In collaboration with M. Zeitz, we have proposed a similar condition [Kr-Ze], namely that there exist a positive definite function $V(e)$ such that $\dot{V}(x, z)$ is negative definitive.

Both of these definitions require that nonzero errors are continuously decreasing as measured by $V$. Both imply $\dot{e} = 0$ when $e = 0$. There are several possible variations of these definitions which have more or less the same properties. But do they capture what we want in a nonlinear observer? Consider the following:

<u>Example 3.2</u> Let $x \in \mathbf{R}^1$ and

$$\dot{x} = -x^\alpha$$

$$y = 0$$

where $\alpha = 1, 3$ or $1/3$. Define the observer

$$\dot{z} = -z^\alpha$$

$$\hat{x} = z$$

$$V(e) = e^2/2$$

$$\dot{V}(x,z) = -(x^\alpha - z^\alpha)e.$$

If $\alpha = 1$

$$\dot{V}(x,z) = -e^2.$$

If $\alpha = 3$

$$\dot{V}(x,z) = -\frac{e^2}{2}(x^2 + z^2 + (x+z)^2) \leq \frac{-e^4}{4}.$$

If $\alpha = 1/3$

$$\dot{V}(x,z) = -\frac{e^{10/9}}{2}(x^{2/9} + z^{2/9} + (x^{1/9} + z^{1/9})^2) \leq -()12^{2/9}e^{12/9}.$$

For each $\alpha = 1, 3, 1/3$, the observer error converges in the sense of both Vidyasagar and Krener-Zeitz. Of course this is no surprise as the three systems differ by nondifferentiable changes of coordinates.

It should be noted that neither the definition of Vidyasagar or Krener-Zeitz is coordinate independent. Consider a full order observer where $\hat{x} = z$. Suppose we consider a change of state coordinates

$$\overline{x} = \phi(x) \tag{28}$$

for (19,20) and we make the corresponding change of state coordinates on the observer

$$\overline{z} = \phi(z) \tag{29}$$

with the corresponding estimate

$$\hat{\overline{x}} = \overline{z}. \tag{30}$$

Except for linear systems and linear changes of coordinates, the error does not transform in the same fashion

$$\overline{e} = \overline{x} - \overline{z} = \phi(x) - \phi(z) \neq \phi(e) = \phi(x - z). \tag{31}$$

For this reason both the Vidyasagar and Krener-Zeitz error convergence conditions are coordinate dependent.

We turn now to a discussion of necessary conditions for detectability analogous to those of Brockett. In [Kr-Ze] the following three conditions are presented.

($D_1$) If a $C^1$ system (19,20) admits a $C^1$ observer with locally asymptotically stable error dynamics, then the linear approximation of (19,20) around any critical point $x^0$ in $\Omega_x$, $f(x^0) = 0$

$$\dot{x} = \frac{\partial f}{\partial x}(x^0) \tag{32}$$

$$y = \frac{\partial h}{\partial x}(x^0) \tag{33}$$

must have no unobservable, unstable modes.

($D_2$) If a $C^0$ system (19,20) admits a $C^0$ observer with (locally) asymptotically stable error dynamics and if two (sufficiently close) initial conditions $x^1(0)$ and $x^2(0)$ generate the same output trajectory $y^1(t) = y^2(t)$ then the two state trajectories converge,

$$\left| x^1(t) - x^2(t) \right| \to 0 \tag{34}$$

($D_3$) If a $C^0$ system (19,20) admits a $C^0$ full order observer with (locally) asymptotically stable error dynamics in the sense of Krener-Zeitz then the mapping

$$x \longmapsto \begin{bmatrix} f(x) \\ h(x) \end{bmatrix} \tag{35}$$

is (locally) 1-1.

It is fairly obvious that ($D_1$) is necessary for $C^1$ systems and observers, the following shows that it is not necessary for $C^0$ system and observers.

<u>Example 3.3</u> Let $x \in \mathbf{R}^1$

$$\dot{x} = x$$

$$y = x^3$$

$$\dot{z} = z + 2(y^{1/3} - z)$$

$$\hat{x} = z$$

$$\dot{e} = -e$$

Condition ($D_2$) is reminiscent of a definition of nonlinear (local) observability, namely, that if two (sufficiently close) initial conditions give rise to the same output then they are identical.

Condition $D_3$ is proven as follows. Suppose (19,20) admits a full order observer

$$\dot{\hat{x}} = \hat{f}\,(\hat{x}, y) \tag{36}$$

satisfying (24), This implies that

$$\hat{f}\,(x, h(x)) = f(x).$$

Suppose there exist $x^1 \neq x^2$ such that

$$f(x^1) = f(x^2)$$

$$h(x^1) = h\left(x^2\right)$$

then

$$f(x^1) = f(x^2) = \hat{f}\,(x^2, h(x^2)) = \hat{f}\,(x^2, h(x^1)).$$

Suppose $x = x^1$ and $\hat{x} = x^2$ then

$$e = x^1 - x^2 \neq 0$$

while

$$\dot{e} = f(x^1) - \hat{f}\,(x^2, h(x^1)) = 0.$$

Therefore there cannot exist a positive definite $V(e)$ such that $\dot{V}\,(x, \hat{x})$ is negative definite.

Most methods for constructing nonlinear observers are based on the linear paradigm. Suppose the system (19, 20) has an asymptotically stable critical point, $x^0$, where the output is $y^0 = h\,(x^0)$. Without loss of generality we can assume $x^0 = 0$ and $y^0 = 0$. The linear approximation is

$$\dot{x} = \frac{\partial f}{\partial x}(0)x \tag{37}$$

$$y = \frac{\partial h}{\partial x}(0)x \tag{38}$$

A linear observer for this system will be a local observer for the nonlinear system. Associated with the linear observer is a quadratic Lyapunov function for the linear error convergence. One can try to extend this Lyapunov function and the linear observer to obtain a nonlinear observer. For variations on this theme, see [Th], [K-E-T], [Ts3], [Ts4], [B-P].

The observability of the (19, 20) is equivalent to the state being recoverable from the output and its time derivatives. This suggests the use of a high gain observer which approximates a differentiator to estimate the state [E-K-N-N], [G-H-O].

One way to find a linear observer is to add noises to the model (37,38) and solve the resulting Kalman filtering problem. One can also add noises to the nonlinear model (19,20) and try to solve the resulting nonlinear filtering problem [H-W]. Generally speaking this is computationally intractable as it requires solving in real time the Zakai partial differential equation, a parabolic PDE driven by the observations. Most nonlinear filtering problems are inherently infinite dimensional.

Other approaches include considering the effect of changes of state and output coordinates and output injection on the nonlinear model (19,20). If the system can be transformed into a linear system in this fashion, then an observer or filter can be constructed for the latter and transformed back to obtain an observer for the nonlinear system (19,20). See [B-Z], [K-I], [K-R], [Ph], [X-G1], [X-G2], [X-G3], [L-T].

Recently, a variation on an older approach to nonlinear estimation called maximum likelihood estimation by Mortenson [M] and minimum energy filtering by Hijab [Hi] has been utilized in the nonlinear $H^\infty$ problems [B-B], [vdS2]. In [Kr2], we describe how the linear $H^\infty$ filters of Khargonekar and Nagpal [K-N] can be generalized to a nonlinear setting. We briefly describe the approach and refer the reader to [Kr2] for more details and proofs.

The basic approach is to view the estimation problem as a game against nature similar to the $H^\infty$ control problem [B-B]. To this end we add unknown driving noise, observation noise and initial conditions to the model (16,17).

$$\dot{x} = f(x) + w \tag{39}$$

$$y = h(x) + v \tag{40}$$

$$x(0) = x^0 \tag{41}$$

We seek to estimate the state $x(t)$, or more generally some function $k(x(t))$, from the past observation in such a way as to minimize the $L^2$ norm of the mapping from the unknown $x^0, w(\cdot), v(\cdot)$ to the estimation error. Even in the linear case, this is a difficult problem so instead, we choose an attenuation level $\gamma$ and seek an estimator that achieves this level.

This can be viewed as a game against nature where the payoff is

$$\frac{1}{2}\int_0^t \frac{1}{\gamma^2}\left|k(x(s)) - \hat{k}(s)\right|^2 - |w(s)|^2 - |v(s)|^2\,ds - \frac{1}{2}\left|x^0\right|^2. \tag{42}$$

We seek to minimize this choosing $\hat{k}$, a functional of the past observations. Nature seeks to maximize this by choosing $x^0, w$ and $v$ consistent with the observations. If there exists a choice of $\hat{k}(s)$, a functional of the past observations such that (42) is negative for all $t \geq 0$ and all $x^0, w(s), v(s)$ consistent with $y(s)$ then we have achieved the noise attenuation level $\gamma$. In [Kr2] the following are proven in a slightly more general setting.

**Theorem 1** *A $\hat{k}(t)$ achieves the noise attenuation level $\gamma$ iff there exists $Q(x,t)$, a functional of the past observations, such that*

*(i) $Q(x,t)$ is nonpositive definite and $Q(x^0,0) = -\frac{1}{2}\left|x^0\right|^2$.*

*(ii) Along all trajectories $x(t)$ generated by $x^0, w(t), v(t)$ consistent with $y(t)$*

$$Q(x(t),t)]._{t_1}^{t_2} \geq \frac{1}{2}\int_{t_1}^{t_2} \frac{1}{\gamma^2}\left|z - \hat{z}\right|^2 - |w|^2 - |v|^2 \, dt. \tag{43}$$

**Theorem 2** *Suppose $\hat{k}(t)$ and $Q(x,t)$ satisfy Theorem 1 and suppose $Q$ is $C^1$ then $Q$ satisfies*

$$\frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial x} f - \frac{1}{2\gamma^2}\left|k - \hat{k}\right| - \frac{1}{2}\frac{\partial Q}{\partial x} gg' \frac{\partial Q'}{\partial x} + \frac{1}{2}\left|y - h(x)\right|^2 \geq 0. \tag{44}$$

On the other hand if $\hat{k}(t)$ and $Q(x,t)$ are functionals of the past observations, $Q$ is $C^1$, nonpositive definite, $Q(x^0,0) = -\frac{1}{2}\left|x^0\right|^2$ and satisfies (44) then $\hat{k}(t)$ achieves the attenuation level $\gamma$.

**Theorem 3** *Suppose $Q(x,t)$ is a $C^1$, nonpositive definite solution of*

$$Q(x^0,0) = \frac{1}{2}\left|x^0\right|^2$$

*and*

$$\frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial x} f - \frac{1}{2\gamma^2}\left|k - \hat{k}\right| - \frac{1}{2}\frac{\partial Q}{\partial x} gg' \frac{\partial Q'}{\partial x} + \frac{1}{2}\left|y - h(x)\right|^2 = 0$$

*where $\hat{k}(t) = k(\hat{x}(t))$ and $\hat{x}(t)$ is the assumed unique max of $Q(x,t)$. Then $\hat{k}(t)$ and $\hat{Q}(x,t)$ are functionals of the past observations and $\hat{k}(t)$ achieves the attenuation level $\gamma$. Furthermore the differential game with payoff (42) has a saddle point solution with value $Q(\hat{k}(t),t)$. The function $Q(\hat{x}(t),t) - Q(x,t)$ is an observation dependent Lyapunov function for the undriven system, $w(t) = 0, v(t) = 0$.*

These results can be readily combined with the $H^\infty$ state feedbacks described in Section 2 to obtain nonlinear $H^\infty$ measurement controllers [Kr2]. As with nonlinear filters, nonlinear $H^\infty$ observers are infinite dimensional because the observer state is $Q(x,t)$.

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