DIDO'S PROBLEM WITH A FIXED CENTER OF MASS

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Abstract. Given an area and coordinates of the center of mass we compute the minimal length curve passing through the origin which envelopes a region having the given area and the prescribed center of mass.

Key Words. Nonlinear system, optimal control; calculus of variations; nilpotent Lie algebras

1. INTRODUCTION.

Let us begin with the plot of the legend as it is narrated in "The Aeneid" by the Roman poet P. Virgilius Maro:

Mercatique solum, facti de nomine Byrsam Taurino quantum possent circumdare tergo...

"They bought a space of ground, which (Byrsa called, from the bull's hide) they first enclosed..." translated by J. Dryden (*The works of Virgil* (1961), p.144,).

The Phoenician princess Dido fled from her brother, the tyrant Pygmalion. Dido and her companions chose a good place on the north coast of Africa (at present the shore of the Gulf of Tunis) and wanted to found a settlement there. Among the natives there was not much enthusiasm for this idea. However, Dido managed to persuade their chieftain Hiarabas to give her as much land as she could enclose with the hide of a bull. Only later did the simple hearted Hiarabas understand how cunning and artful Dido was: she then cut the bull's hide into thin strips, tied them together to form an extremely long thong, and surrounded with it a large extent of territory and founded the city of Carthage there. In commemoration of this event the citadel of Carthage was called Byrsa. According to the legend, all these events occurred in 825 (or 814) B.C.[1]

The situation described in the legend can be stated as the following optimization problem:

-find the optimal form of a lot of land of the maximum area S for a given perimeter L.

Clearly, its solution is circle. Several other possibilities of stating optimization Dido's problems are described in Alekseev *et al.* (1987)

In this paper we will solve Dido's problem with a fixed center of mass. This problem is formulated as follows:

-find the minimal length curve through the origin which envelopes the given area with a fixed center of mass.

Let us look at this problem in slightly more general way. Consider the following control system

$$\dot{x} = b_1(x)u_1 + b_2(x)u_2, \tag{1}$$

where $x \in \mathrm{R}^n$; $b_1(x), b_2(x)$ are smooth vector fields on R^n and u_1, u_2 are controls. We denote by $Lie(b_1, b_2)$ Lie algebra generated by the vector fields b_1 and b_2, i.e., $Lie(b_1, b_2)$ is the smallest algebra which contains b_1 and b_2. Let L be any Lie algebra. Then we use the following notations:

$$L_1 = L,\ L_k = \mathrm{span}\{\oplus_{\nu+\mu=k}[L_\nu, L_\mu]\}$$

$$(k = 1, 2, ...),\ \text{and}\ (L)^k = L/L_{k+1},$$

where $[L_\nu, L_\mu] = \mathrm{span}\{[\xi, \eta];\quad \eta \in L_\nu,\quad \xi \in L_\mu\}$ and $[\xi, \eta]$ denotes the Lie bracket of vector fields ξ and η (for a definition, see e.g. Sternberg (1983)). It is clear that $(L)^k$ is a nilpotent algebra of order k.

We can replace the system (1) by its nilpotent

[1] the description of the legend is taken from [1]

approximation of order k, i.e.,

$$\dot{\bar{x}} = \beta_1(\bar{x})u_1 + \beta_2(\bar{x})u_2, \tag{2}$$

where $\bar{x} \in \mathrm{R}^{\bar{n}}$ $(\bar{n} \geq n)$; $\beta_1(\bar{x}), \beta_2(\bar{x})$ are such polynomial vector fields that

$$rank_{\bar{x}}\{Lie(\beta_1, \beta_2)\} = \bar{n} \quad \forall \bar{x} \in \mathrm{R}^{\bar{n}}$$

and there is a homomorphism

$$\varphi: \ Lie(\beta_1, \beta_2) \longrightarrow (Lie(b_1, b_2))^k,$$

of Lie algebras; $rank_{\bar{x}}\{Lie(\beta_1, \beta_2)\}$ denotes the *rank* of $Lie(\beta_1, \beta_2)$ at the point $\bar{x}$, i.e., the maximal number of linearly independent vectors from $Lie(\beta_1, \beta_2)$ evaluated at $\bar{x}$.

According to the results obtained in Krener (1975) there is a smooth mapping under which the solutions for (2) become the approximation to order $O(t^{k+1})$ for the solutions of (1) governed by the same controls. Thus if we are able to solve an optimal control problem or motion planning problem for (2) then for small periods of time we have a suboptimal solution or suboptimal motion planning for (1), respectively. This idea inspired some recent investigations on optimal control and motion planning Brockett (1981), Brockett and L.Dai (1992), Jurdjevic (1990). Several interesting results were obtained in the frame work of Sub-Riemannian geometry Brockett (1981). In Brockett (1981) the optimal control problem

$$\frac{1}{2}\int_0^T (u_1^2 + u_2^2)d\tau \longrightarrow inf, x(0) = x(T) = 0 \tag{3}$$

was completely solved for the system (2), with the nilpotent Lie algebra of order k=2. The corresponding problem for the system (2) with the nilpotent Lie algebra of order $k = 3$ was not thoroughly studied. It was only shown in Brockett and L.Dai (1992) that the extremals can be described in terms of elliptic functions when the system generates a nilpotent *Lie* algebra of order $k = 3$.

On the other hand, if $k = 2$, then one can treat the optimization problem (2),(3) as a classical isoparametric problem (see, e.g.,Alekseev *et al.* (1987)): find the minimal length curve which envelopes a region having a given area. This problem was solved, e.g. by princess Dido, many years ago. Therefore the results obtained in Brockett (1981) are not very surprising.

It turns out that for the system (2) with the nilpotent Lie algebra of order $k = 3$ the problem (2),(3) is Dido's problem with a fixed center of mass. In this paper we will describe all the extremals of this problem.

2. PROBLEM STATEMENT.

Consider the system

$$\begin{aligned} \dot{x} &= u, \\ \dot{y} &= \det(x, u)x, \\ \dot{z} &= \det(x, u), \end{aligned} \tag{4}$$

where $x, y \in \mathrm{R}^2$ $z \in \mathrm{R}$ and

$$\det(x, u) = x_1u_2 - x_2u_1.$$

We are interested in solving the following optimal control problem:

$$J(u) = \int_0^T ((u_1(\tau))^2 + (u_2(\tau))^2)d\tau \longrightarrow inf \tag{5}$$

$$(x(0) = x(T) = 0, y(T) = Y, z(T) = Z),$$

where $Y \in \mathrm{R}^2$, $Z \in \mathrm{R}$ are given. In other words, one needs to design a control which minimizes the value $J(u)$ and steers the system (4) from the origin into the point with coordinates $x(T) = 0$, $y(T) = Y$, $z(T) = Z$. It is easy to see that if $y(T) = 2Z \cdot Y$ and $Z(T) = 2 \cdot Z$, then the optimal curve $\gamma = \{x(t); 0 \leq t \leq T, \frac{d}{dt}x(t) = u(t)\}$ has the minimal length among all the closed curves through the origin each of which envelopes a region having the area Z and the center of mass at the point Y.

On the other hand, let

$$b_1 = \begin{pmatrix} 1 \\ 0 \\ -x_2x_1 \\ -x_2^2 \\ -x_2 \end{pmatrix} \quad b_2 = \begin{pmatrix} 0 \\ 1 \\ x_1^2 \\ x_1x_2 \\ x_1 \end{pmatrix} \tag{6}$$

Then $Lie(b_1, b_2)$ is a nilpotent Lie algebra of order 3; and

$$rank_x(Lie(b_1, b_2)) = 5 \quad \forall x \in \mathrm{R}^2, y \in \mathrm{R}^2, z \in \mathrm{R}.$$

3. EXTREMALS.

To derive the equations describing the extremals of the problem (4),(5) we use Pontryagin maximum principle. It is possible also to obtain the same equations in the framework of classical theory of variations (see Brockett and L.Dai (1992))

Following the procedure stated in Pontryagin *et al.* (1962) we take the hamiltonian

$$H(\tilde{x}, p, u) = < p, b_1 > u_1 + < p, b_2 > u_2 - \frac{p_0}{2}|u|^2,$$

where $\tilde{x}$ denotes (x, y, z); b_1, b_2 are defined in (6). The extremals are governed by the equations:

$$\begin{aligned} 0 &= \frac{\partial H(\tilde{x}, p, u)}{\partial u}, \\ \dot{\tilde{x}} &= \frac{\partial H(\tilde{x}, p, u)}{\partial p}, \\ \dot{p} &= -\frac{\partial H(\tilde{x}, p, u)}{\partial \tilde{x}}. \end{aligned} \tag{7}$$

Instead of $\dot{p} = -\frac{\partial H(\tilde{x},p,u)}{\partial \tilde{x}}$ it is more convenient to exploit the equations for $w_i =< p, b_i > \quad i = 1, 2$. Differentiation of $w_i \quad (i = 1, 2)$ along the solutions of (7) yields

$$\frac{d}{dt} w_1 = -w_{12} u_2,$$

$$\frac{d}{dt} w_2 = w_{12} u_1,$$

where w_{12} denotes $< p, [b_1, b_2] >$. Calculating the derivative of $w_{12} =< p, [b_1, b_2] >$ along the solutions of (7) we obtain

$$\frac{d}{dt} w_{12} = w_{112} u_1 + w_{212} u_2,$$

where

$$\frac{d}{dt} w_{112} = 0,$$

$$\frac{d}{dt} w_{212} = 0,$$

since $Lie(b_1, b_2)$ is a nilpotent algebra of order 3, and hence

$$[b_1, [b_2, b]] = 0, \quad [b_2, [b_1, b_2]] = 0.$$

Thus we can rewrite the equations (7) in the following form

$$\begin{aligned} \dot{x} &= u, \quad w_1 = p_0 u_1, \quad w_2 = p_0 u_2, \\ \dot{y} &= \det(x, u) x, \\ \dot{z} &= \det(x, u), \\ \dot{w}_1 &= -w_{12} u_2, \\ \dot{w}_2 &= w_{12} u_1, \\ \dot{w}_{12} &= w_{112} u_1 + w_{212} u_2, \\ \dot{w}_{112} &= 0, \\ \dot{w}_{212} &= 0, \end{aligned} \tag{8}$$

where p_0 is some real number.

Notice that (8) provides necessary conditions for the curve

$$\gamma_T = \{x(t); \ t \in [0, T], \ x(0) = x(T) = 0\}$$

to be the minimal length curve passing through the origin and enveloping a region having a given area and fixed center of mass. Thus if γ_T is a projection of an extremal onto the plane $\mathrm{R}^2_{\mathrm{x}}$, then the curve

$$g\gamma_T + q = \{gx(t) + q; \ t \in [0, T]\}$$

obtained from γ_T by means of the transformation

$$gx + q = r \cdot \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} x + \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

$$\varphi \in \mathrm{R} \quad r > 0, r \in \mathrm{R}, q \in \mathrm{R}^2$$

is also a projection onto $\mathrm{R}^2_{\mathrm{x}}$ of an extremal, satisfying (8). But $g\gamma_T + q$ passes already through the point q and its center of mass is shifted along the vector $q \in \mathrm{R}^2$.

Let $\tilde{x}(t) \in \mathrm{C}$ and $\tilde{x}(t) = x_1(t) + i x_2(t)$, where $i^2 = -1$. Then two dimensional complex space C^2 acts on C as follows

$$(\tilde{g}, \tilde{q}) : \tilde{x} \longrightarrow \tilde{g}\tilde{x} + \tilde{q},$$

where $\tilde{g} = e^{\ln r}(\cos\varphi + i \sin\varphi) \quad (r > 0, \quad r \in \mathrm{R}, \quad \varphi \in \mathrm{R})$ and $\tilde{q} = q_1 + i q_2$.

Thus the set of all the projections onto $\mathrm{R}^2_{\mathrm{x}}$ of the extremals defined by (8) is invariant under this action of C^2 on C.

Theorem 1 The projections onto $\mathrm{R}^2_{\mathrm{x}}$ of the extremals defined by (8) are obtained from the solutions of the equations

$$\begin{aligned} \dot{\tilde{x}} &= e^{i\omega}, \\ \ddot{\omega} &= -\sin\omega \end{aligned} \tag{9}$$

by means of the action of C^2 on C described above.

Proof: First of all consider the case $p_0 = 0$. Then

$$w_1 = 0 \quad \text{and} \quad w_2 = 0.$$

Therefore we have either $w_{12} = 0$ or $u = 0$. Thus

$$w_{112} u_1 + w_{212} u_2 = 0,$$

and hence the corresponding extremals are points and straight lines. That proves the assertion of Theorem 1 in the abnormal case, $p_0 = 0$.

Suppose $p_0 \neq 0$, then we take $p_0 = 1$ and according to the maximum principle Pontryagin *et al.* (1962).

$$(u_1(t))^2 + (u_2(t))^2 = const$$

for extremals. Therefore we have

$$u_1(x) = \eta \cos(\omega(t) + \psi),$$

$$u_2(x) = \eta \sin(\omega(t) + \psi)$$

for some $\eta \in \mathrm{R}, \eta > 0, \psi \in \mathrm{R}$.

It follows from (8), that

$$\ddot{\omega} = -\xi \sin(\omega(t) + \varphi),$$

where $\xi > 0$, φ, $\xi \in \mathrm{R}$. Thus for $p_0 = 1$ the equations (8) imply that the projections of the extremals onto $\mathrm{R}^2_{\mathrm{x}}$ are given by the solutions of the system

$$\begin{aligned} \dot{\tilde{x}} &= e^{\ln|\eta| + i(\omega+\psi)}, \\ \ddot{\omega} &= -\xi \sin(\omega + \varphi), \end{aligned} \tag{10}$$

with different $\eta, \xi, \varphi \in \mathrm{R}$ $(\xi > 0)$. It is easy to see that all the curves corresponding to the solutions of (10) are obtained from the solutions for (9) by means of the action of C^2 on C. Indeed, taking the new time scale

$$\tau = \sqrt{\xi} t$$

and introducing the notation

$$\tilde{\omega} = \omega + \varphi,$$

we obtain

$$\begin{aligned} \dot{\tilde{x}} &= \frac{1}{\sqrt{\xi}} e^{\ln|\eta| + i(\tilde{\omega}+\psi-\varphi)}, \\ \ddot{\tilde{\omega}} &= -\sin\tilde{\omega}. \end{aligned} \tag{11}$$

After multiplying the equation for $\tilde{x}$ in (11) by $\sqrt{\xi} e^{-\ln|\eta| - i(\psi-\varphi)}$ we obtain (9).

■

The system (9) arises in various branches of mathematics Brockett and L.Dai (1992), Bloch and Crouch (1993), Griffiths (1983), Jurdjevic (1990). Its solutions are represented in terms of elliptic integrals and very well studied.

The solutions of (9) are sketched in Fig.1.

With the help of the theory of elliptic functions (see, e.g.. Lawden (1989)) we can calculate the closed curves in $\mathrm{R}^2_{\mathrm{x}}$ which are the projections of the minimizers, i.e., the solutions for the problem (4),(5). The length minimizers are shown in Fig.2.

4. ACKNOWLEDGEMENT

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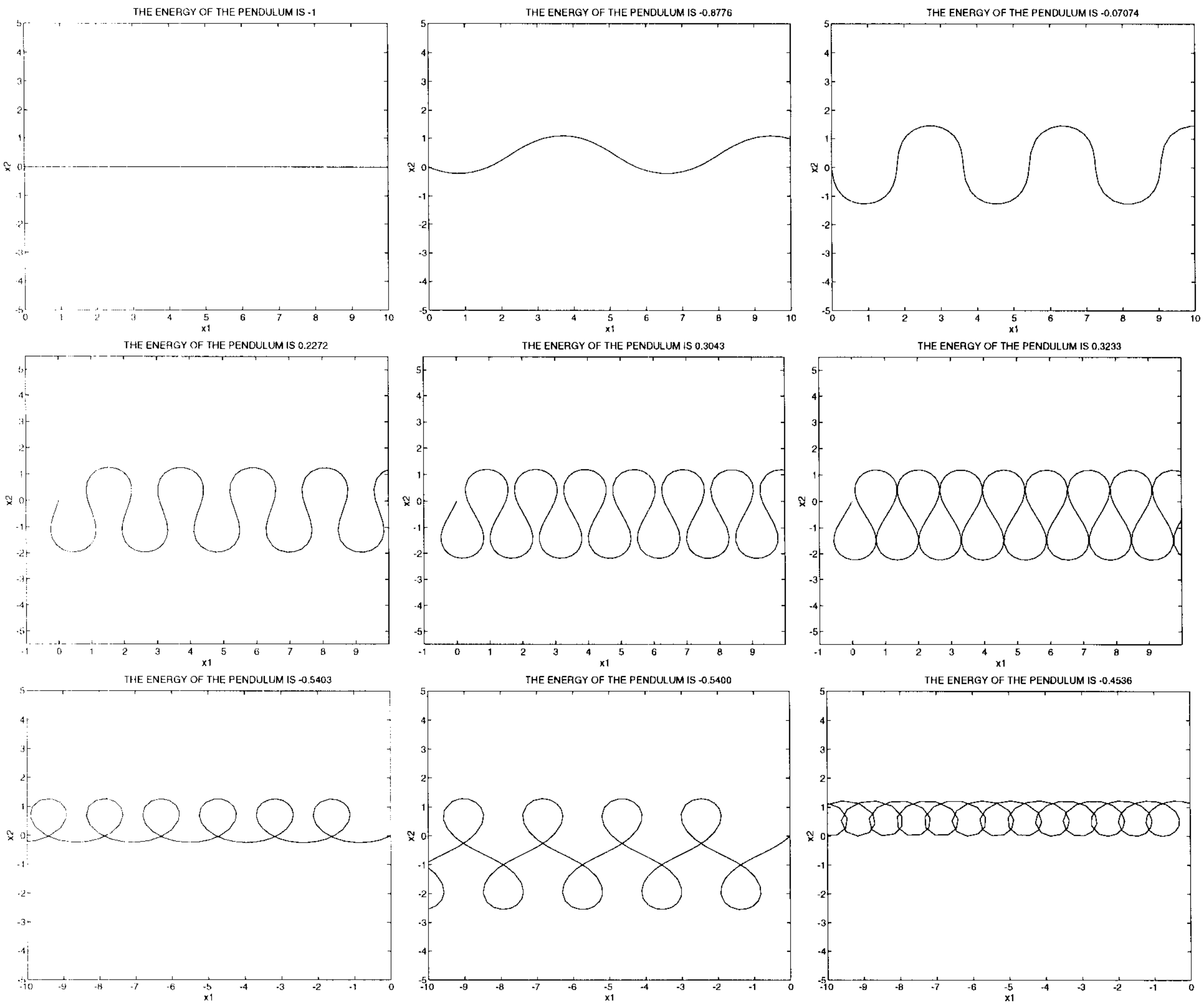

Fig. 1. The projection of the extremals of (8) onto $R^2{}_x$.

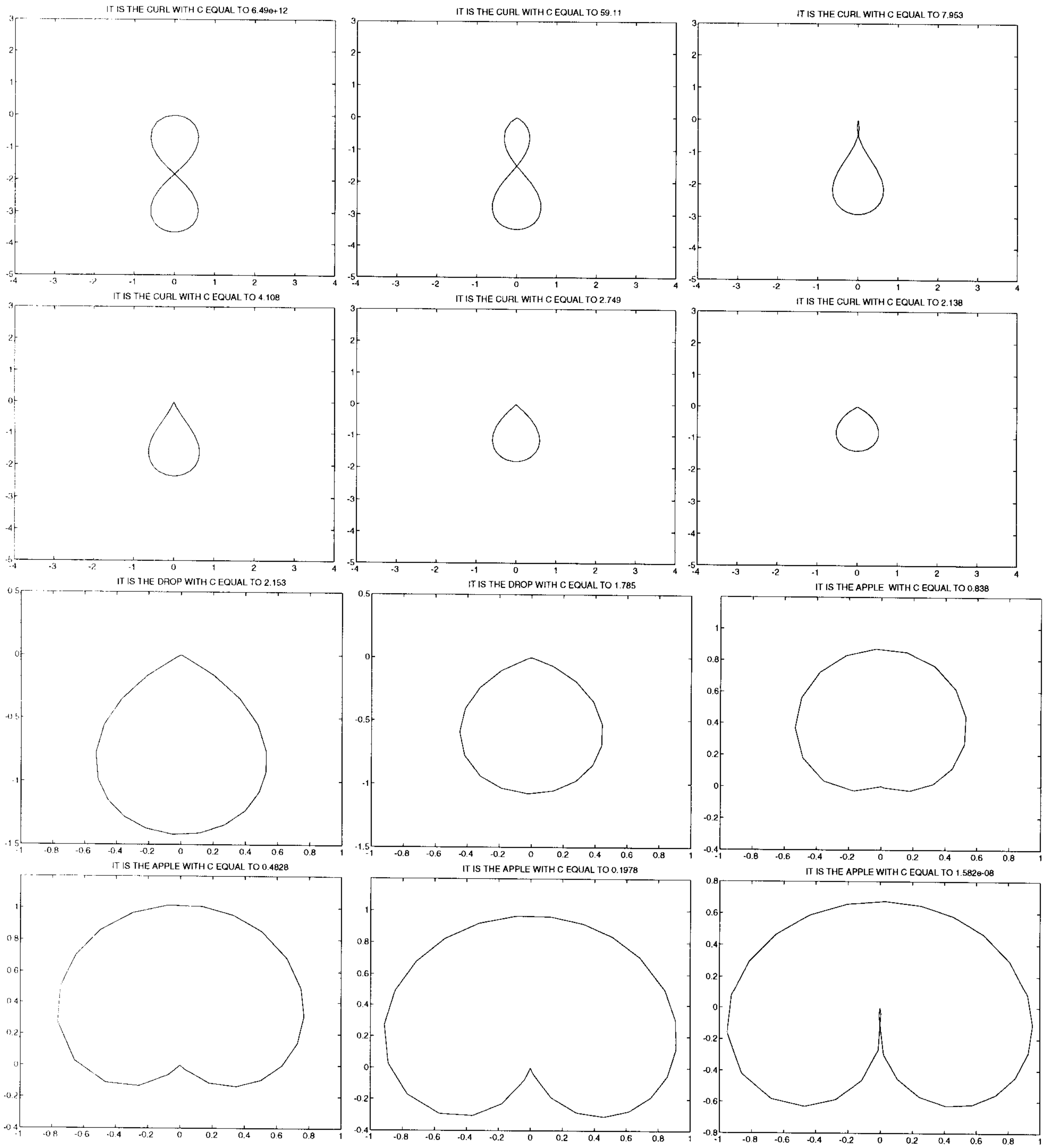

Fig. 2. Length minimizers for different values of $C = (Y_2)^2/(|Z|^3)$, where Y and Z are defined in (5).