Stochastic mechanics of reciprocal diffusions

Bernard C. Levy[a]
*Department of Electrical and Computer Engineering, University of California,
Davis, California 95616*

Arthur J. Krener[b]
Department of Mathematics, University of California, Davis, California 95616

The dynamics and kinematics of reciprocal diffusions were examined in a previous paper [J. Math. Phys. **34**, 1846 (1993)], where it was shown that reciprocal diffusions admit a chain of conservation laws, which close after the first two laws for two disjoint subclasses of reciprocal diffusions, the Markov and quantum diffusions. For the case of quantum diffusions, the conservation laws are equivalent to Schrödinger's equation. The Markov diffusions were employed by Schrödinger [Sitzungsber. Preuss. Akad. Wiss. Phys. Math Kl. 144 (1931); Ann. Inst. H. Poincaré **2**, 269 (1932)], Nelson [*Dynamical Theories of Brownian Motion* (Princeton University, Princeton, NJ, 1967); *Quantum Fluctuations* (Princeton University, Princeton, NJ, 1985)], and other researchers to develop stochastic formulations of quantum mechanics, called stochastic mechanics. We propose here an alternative version of stochastic mechanics based on quantum diffusions. A procedure is presented for constructing the quantum diffusion associated to a given wave function. It is shown that quantum diffusions satisfy the uncertainty principle, and have a locality property, whereby given two dynamically uncoupled but statistically correlated particles, the marginal statistics of each particle depend only on the local fields to which the particle is subjected. However, like Wigner's joint probability distribution for the position and momentum of a particle, the finite joint probability densities of quantum diffusions may take negative values. © *1996 American Institute of Physics.* [S0022-2488(96)03402-8]

I. INTRODUCTION

Since the early days of quantum mechanics, many researchers have attempted to formulate quantum mechanics in terms of diffusions processes. These efforts were originally motivated by the observation that as the real time t is converted to imaginary time it, the Schrödinger and Fokker–Planck equations, which describe, respectively, the time evolutions of the wave function in quantum mechanics and the density of a Markov diffusion, are transformed into each other. This is, for example, the correspondence that was exploited by Kac[1] to derive a stochastic interpretation of Feynman path integrals.[2] This analogy has been used in recent years to develop a stochastic formulation of quantum mechanics, called Euclidean quantum mechanics, which relies on Markov diffusions.[3,4] However, since this interpretation is based on the Wick rotation $t \rightarrow it$, the resulting stochastic models can be viewed as evocative analogies, but not as a picture of physical reality.

Other attempts at relating quantum mechanics and diffusion processes are more radical, in the sense that they go beyond analogies and seek to demonstrate that these two theories are, in fact, equivalent. This line of investigation was initiated by Schrödinger[5] in 1931, who focused his attention on Markov diffusions. Unfortunately, Schrödinger's work was somewhat premature, since descriptions of Markov diffusions in terms of stochastic differential equations were not yet

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[a]Electronic mail: levy@ece.ucdavis.edu
[b]Electronic mail: krener@ucdmath.ucdavis.edu

available, and as a consequence his results were not immediately exploited, and were taken up only later by Zambrini[6] and Nagasawa.[7] Inspired by the early work of Fenyes,[8] a slightly different approach was proposed by Nelson[9,10] and other researchers[11] to formulate quantum mechanics in terms of Markov diffusions. The key aspect of this theory, which is usually called Markovian stochastic mechanics, is that it employs the calculus of stochastic differential equations to give a precise meaning to kinematic quantities such as velocity and acceleration. Given the wave function of a quantum process, one associates to it a Markov diffusion with an identical probability density, where the gradient of the wave function phase specifies the mean velocity obtained by averaging the forward and backward drifts of the diffusion. However, like Bohm's hidden variables interpretation of quantum mechanics,[12] both the Schrödinger and Nelson versions of stochastic mechanics are *nonlocal*, in the sense that a "quantum potential" needs to be introduced to relate the forward and backward Fokker–Planck equations of the Markov diffusions with the Schrödinger equation of the matching wave function. This potential has the feature that for two dynamically uncoupled and widely separated particles, changes in their joint probability density affect immediately the forces acting on each particle. This lack of locality gives rise to significant differences[10,13] between Markovian stochastic mechanics and standard quantum mechanics.

We propose here an alternative form of stochastic mechanics based on a subclass of reciprocal diffusions, called the quantum diffusions. Reciprocal processes were introduced by Bernstein in 1932,[14] who was influenced by Schrödinger's above mentioned attempt at a stochastic formulation of quantum mechanics. Reciprocal processes were subsequently studied by Jamison,[15] who showed that over a finite interval, they could be constructed from Markov processes by applying a change of measure to the joint probability distribution of the end-point values of the process. This procedure can be used to construct reciprocal diffusions directly from Markov diffusions. In Refs. 16 and 17, Krener showed that reciprocal diffusions satisfy locally a stochastic form of Newton's law, which in the Gaussian case,[18] can be used to express reciprocal diffusions as the solutions of second-order stochastic differential equations. In Ref. 19, a stochastic quantization procedure was introduced to associate a class of reciprocal diffusions to a dynamical system satisfied by its Hamiltonian. This construction was used to characterize the kinematics and dynamics of reciprocal diffusions. It was shown that reciprocal diffusions satisfy a chain of conservation laws, which is generally infinite, but closes after the first two laws for two disjoint subclasses, the Markov and quantum diffusions. The quantum diffusions derive their name from the equivalence existing between their conservation laws and Schrödinger's equation for the associated Hamiltonian.

The stochastic mechanics described here associates to the wave function of a quantum system a matching quantum diffusion. This construction selects, in the equivalence class of reciprocal diffusions associated to a Hamiltonian, the diffusion which models the corresponding quantum process. The quantum diffusions have the feature that the closure rule satisfied by their conservation laws is essentially equivalent to Heisenberg's uncertainty principle for the position and momentum variables. Unlike Markovian stochastic mechanics,[10,13] the stochastic mechanics of quantum diffusions is local, in the sense that given two dynamically uncoupled but statistically correlated particles, the marginal probability density of each particle does not depend on the parameters of the potentials acting on the other particle. However, one interesting feature of this new stochastic mechanics is that the end-point densities that must be applied to model certain quantum processes, such as the excited states of the harmonic oscillator, can take negative values. The appearance of such negative densities should not come as a true surprise if one considers that the finite joint densities for the positions at successive times obtained here form an extension of the Wigner joint position-momentum distribution,[20,21] which also has the feature of taking negative values. Note that the marginal density for the position at a single time is always positive, so that, as argued in Ref. 22, the negativity of densities is not a real drawback, as long as one considers that observable quantum events have always positive probabilities.

The paper is organized as follows. The stochastic quantization procedure used to associate a

class of reciprocal diffusions to a dynamical system specified by its Hamiltonian is described in Sec. II, where the dynamical properties of reciprocal diffusions are reviewed. In Sec. III, the subclass of quantum diffusions is introduced, and a method is presented for constructing the quantum diffusion matching the wave function of a quantum system. This construction is simplified for the case of Gaussian processes in Sec. IV, and then illustrated by considering the minimum uncertainty wavepacket for a free particle and the ground state of the harmonic oscillator. In Sec. V, we prove that quantum diffusions satisfy Heisenberg's position-momentum uncertainty relation. The locality property of quantum diffusions is demonstrated in Sec. VI. Finally, it is shown in Sec. VII that negative probabilities must be allowed if one seeks to model certain quantum processes, such as the excited states of the harmonic oscillator.

II. DYNAMICAL SYSTEMS AND RECIPROCAL DIFFUSIONS

As a starting point, we recall that a process $x(t) \in \mathbb{R}^n$ defined over $[0,T]$ is reciprocal if for arbitrary subintervals $[s,t]$ of $[0,T]$, the process interior to $[s,t]$ is independent of the process exterior to $[s,t]$, given $x(s)$ and $x(t)$. When $x(\cdot)$ admits finite joint probability densities, it was shown by Jamison[15] that $x(\cdot)$ is completely specified by (i) the joint probability density $p(x_0,0;x_T,T)$ of its values $x(0)$ and $x(T)$ at the ends of the interval $[0,T]$; and (ii) its reciprocal transition density $r(x,s;y,t;z,u)$, which is the conditional density of $x(t)=y$, given that $x(s)=x$ and $x(u)=z$, with $s<t<u$. In order to correspond to a reciprocal density, a function r must satisfy two conditions. First, it must be a probability density in y, i.e.

$$\int r(x,s;y,t;z,u)dy = 1, \qquad (2.1a)$$

and the identity

$$r(w,s;x,t;z,v)r(x,t;y,u;z,v) = r(w,s;y,u;z,v)r(w,s;x,t;y,u), \qquad (2.1b)$$

must hold for all $0 \le s < t < u < v \le T$. This last condition is the analog for reciprocal processes of the Chapman–Kolmogorov equation of Markov processes. From this characterization, one can immediately deduce that if a process is Markov, it is necessarily reciprocal, but the converse does not hold, in general. Also, two reciprocal processes with the same transition density $r(x,s;y,t;z,u)$ are said to belong to the same reciprocal class, since they exhibit the same local stochastic behavior.

Let $\mathbb{R}^n$ be the standard n-dimensional Euclidean space with metric $\delta_{ij} = 1$ for $i = j$, and $=0$ otherwise. Consider a dynamical system with Hamiltonian

$$H(x,p,t) = \tfrac{1}{2}(p^i - A^i(x,t))(p_i - A_i(x,t)) + \phi(x,t), \qquad (2.2)$$

where $\{\phi, A_i\}$ denotes a scalar and vector potential pair, and where we employ the standard tensor contraction convention with repeated upper and lower indices corresponding to a summation. Under certain smoothness conditions for the potentials $\{\phi, A_i\}$, a stochastic quantization procedure was proposed in Ref. 19, which associates a class of reciprocal diffusions to the system (2.2).

The first step of this quantization procedure consists in replacing the momentum p_j by $-\nabla_j$ inside the Hamiltonian $H(x,p,t)$, where ∇_j denotes the differentiation with respect to x^j. This correspondence rule is the stochastic analog of the quantization rule $p_j \leftrightarrow -i\nabla_j$ of quantum mechanics. This yields the elliptic operator

$$\mathbf{H} = \tfrac{1}{2}(\nabla^i + A^i)(\nabla_i + A_i) + \phi, \qquad (2.3a)$$

$$= \tfrac{1}{2}\Delta + A^i\nabla_i + \tfrac{1}{2}(\nabla^i A_i + A^i A_i) + \phi, \qquad (2.3b)$$

where Δ denotes the Laplacian. Then the generalized heat operator,

$$L = H - \frac{\partial}{\partial t}, \tag{2.4}$$

is the forward operator of a general Markov diffusion with unit diffusion matrix δ^{ij} drift $b^i(x,t) = -A^i(x,t)$ and creation/killing rate

$$c(x,t) \triangleq \tfrac{1}{2}(A^iA_i - \nabla^iA_i)(x,t) + \phi(x,t). \tag{2.5}$$

See Ref. 23 for a study of Markov diffusions with creation or killing. The Green's function $G(x,s;y,t)$ associated to L is given by

$$L_{y,t}G(x,s;y,t) = 0, \quad t \geq s, \tag{2.6a}$$

$$G(x,s;y,s) = \delta(x-y), \tag{2.6b}$$

where the subscripts $\{y,t\}$ specify the variables upon which the operator L is acting. To ensure that $G(x,s;y,t)$ represents the transition density of a general Markov diffusion, $G(x,s;y,t)$ is also required to decay as $|y| \to \infty$.

Then, $x(t)$ is a reciprocal diffusion over $[0,T]$ in the class associated to the Hamiltonian (2.2) if given an arbitrary set of times $t_0 = 0 < t_1 < \cdots < t_N = T$, the joint probability density of $x(t_0), x(t_1), \ldots, x(t_N)$ can be expressed as

$$p(x_0,t_0;x_1,t_1;\ldots;x_N,t_N) = q(x_0,t_0;x_N,t_N)\prod_{k=0}^{N-1} G(x_k,t_k;x_{k+1},t_{k+1}), \tag{2.7}$$

where the end-point density $q(x_0,0;x_T,T)$ is positive and satisfies the normalization condition

$$\int \int q(x_0,0;x_T,T)G(x_0,0;x_T,T)dx_0 \, dx_T = 1. \tag{2.8}$$

Since the Green's function $G(x,s;y,t)$ is completely specified by the Hamiltonian $H(x,p,t)$, the expression (2.7) indicates that all diffusions associated to a given physical system differ only by the choice of end-point density $q(x_0,0;x_T,T)$. To verify that the finite joint densities (2.7) satisfy the Jamison conditions (2.1a)–(2.1b), note that for $s < t < u$, the three-point transition density $r(x,s;y,t;z,u)$ can be expressed as

$$r(x,s;y,t;z,u) = \frac{G(x,s;y,t)G(y,t;z,u)}{G(x,s;z,u)}. \tag{2.9}$$

Then (2.1b) can be verified by inspection, and the fact that $r(x,s;y,t;z,u)$ is a density in y is a consequence of the transition property,

$$G(x,s;z,u) = \int G(x,s;y,t)G(y,t;z,u)dy \tag{2.10}$$

of the heat kernel G.

The end-point density q appearing in (2.7) is related to the joint probability density of $x(0)$ and $x(T)$ through the relation

$$p(x_0,0;x_T,T) = q(x_0,0;x_T,T)G(x_0,0;x_T,T). \tag{2.11}$$

From the structure (2.7) for the finite joint densities, we deduce that when the interval of definition of a reciprocal diffusion is restricted from $[0,T]$ to a subinterval $[s,t]$ with $0<s<t<T$, the density $q(x,s;y,t)$ that needs to be applied to the new end points (s,t) is given by

$$q(x,s;y,t) = \int \int G(x_0,0;x,s)G(y,t;x_T,T)q(x_0,0;x_T,T)dx_0 \, dx_T. \qquad (2.12)$$

The expression (2.12) implies that $q(x,s;y,t)$ with $s \leq t$ satisfies the two evolution equations,

$$\mathbf{L}_{x,s}q(x,s;y,t) = 0, \qquad (2.13a)$$

$$\mathbf{L}^*_{y,t}q(x,s;y,t) = 0, \qquad (2.13b)$$

where $\mathbf{L}^*$ denotes the adjoint operator of $\mathbf{L}$.

Within the general class of reciprocal diffusions specified by the Hamiltonian $H(x,p,t)$, it was shown in Theorem 2.1 of Ref. 19 that the subclass of Markov diffusions has the feature that for any subinterval $[t,s]$ of $[0,T]$, the end-point density $q(x,s;y,t)$ admits the separable structure

$$q(x,s;y,t) = q_f(x,s)q_b(y,t), \qquad (2.14)$$

where $q_f(x,t)$ and $q_b(y,t)$ obey, respectively, the forward and backward heat quations,

$$Lq_f(x,t) = 0, \quad L^*q_b(x,t) = 0, \qquad (2.15)$$

with initial conditions $q_f(x_0,0)$ and $q_b(x_T,T)$, respectively.

The identities (2.7) and (2.12) imply that the density $\rho(x,t)$ of a reciprocal diffusion $x(t)$ can be expressed as

$$\rho(x,t) = q(x,t;x,t) = \int \int G(x_0,0;x,t)G(x,t;x_T,T)q(x_0,0;x_T,T)dx_0 \, dx_T, \qquad (2.16)$$

so that it is completely fixed by q. In the Markov case, this expression reduces to

$$\rho(x,t) = q_f(x,t)q_b(x,t), \qquad (2.17)$$

which represents a modification due to Schrödinger of the standard representation for the density of a Markov diffusion, where instead of viewing ρ as the solution of either a forward or a backward heat equation, it is expressed as a product of components q_f and q_b propagating in both time directions. These two components describe the information about the diffusion process specified at each end of the interval $[0,T]$. In Schrödinger's attempt at reformulating quantum mechanics in terms of Markov diffusions, which is further elaborated in Refs. 6 and 7, the identity (2.17) is employed as a substitute for the usual representation,

$$\rho(x,t) = \psi(x,t)\psi^*(x,t), \qquad (2.18)$$

for the density of a quantum process in terms of the wave function $\psi(x,t)$ generated by Schrödinger's equation. Unfortunately, as we shall see below, the potentials $\{A_i,\phi\}$ used to construct the wave function ψ are inconsistent with those employed to generate the matching Markov diffusion.

In Ref. 19, several important properties of reciprocal diffusions were derived, which are now summarized. We denote the mean position and the centered first- and second-order differences of the process $x(t)$ as

$$\bar{x}(t,h) = \tfrac{1}{2}(x(t+h) + x(t-h)), \qquad (2.19a)$$

$$d^1 x(t,h) = \tfrac{1}{2}(x(t+h) - x(t-h)), \qquad (2.19b)$$

$$d^2 x(t,h) = x(t+h) + x(t-h) - 2x(t). \qquad (2.19c)$$

Then, it was shown in Ref. 19 that the three-point transition density $r(\bar{x} - uh, t - h; x, t; \bar{x} + uh, t + h)$ of $x(t) = x$ given $x(t \pm h) = \bar{x} \pm uh$ can be approximated locally by a Gaussian distribution for $d^2 x(t,h)$ with mean

$$E[d^2 x^i | x(t \pm h) = \bar{x} \pm uh] = F^i(\bar{x}, u, t) h^2 + O(h^{5/2}), \qquad (2.20)$$

and variance

$$E[d^2 x^i \, d^2 x^j | x(t \pm h) = \bar{x} \pm uh] = 2h \delta^{ij} + O(h^{5/2}). \qquad (2.21)$$

In relation (2.20), if

$$dA_{ij} = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} \qquad (2.22)$$

denotes the exterior derivative of A_i,

$$F_i(x,u,t) = dA_{ij}(x,t) u^j - \left(\frac{\partial \phi}{\partial x^i} + \frac{\partial A_i}{\partial t} \right)(x,t), \qquad (2.23)$$

represents the force applied to a particle with position x and velocity u due to the potentials $\{A^i, \phi\}$. Thus (2.20) can be viewed as a stochastic form of Newton's law, since it states that the conditional mean acceleration,

$$a^i = E\left[\frac{d^2 x^i}{h^2} \middle| x(t \pm h) = \bar{x} \pm uh \right], \qquad (2.24)$$

for the process at time t equals the force based on the mean position $\bar{x}(t,h)$ and empirical velocity $u(t,h) = d^1 x(t,h)/h$ estimated from the positions at times $t \pm h$. Note that this form of Newton's law is *noncausal*, since the conditioning is taken with respect to positions at times $t - h$ and $t + h$. From a physical point of view, we see that the local motion of a reciprocal diffusion is obtained by superposing the classical motion specified by the force F_i with some random fluctuations, which, according to the expression (2.21) for the conditional variance, have a size proportional to $h^{1/2}$. Note also that the definition (2.24) of the acceleration differs from the one employed by Nelson[9,10] in his derivation of Newton's law for Markov diffusions. See Ref. 24 for a detailed comparison of the two accelerations.

Let

$$M(a,b,t) \triangleq \ln q(a - b, t; a + b, t) \qquad (2.25)$$

and

$$w_i(a,t) = \frac{1}{2} \frac{\partial M}{\partial b^i}(a,0,t), \qquad (2.26a)$$

$$\pi_{ij}(a,t) = \frac{1}{4} \frac{\partial^2 M}{\partial b^i \partial b^j}(a,0,t), \qquad (2.26b)$$

$$\tau_{ijk}(a,t) = \frac{1}{8} \frac{\partial^3 M}{\partial b^i \, \partial b^j \, \partial b^k} (a,0,t).$$

(2.26c)

Then, in addition to the above characterization for the mean acceleration, it was shown in Ref. 19 that the conditional density of the first difference $d^1x(t,h)$ given the mean position $\bar{x}(t,h)$, is locally Gaussian with mean

$$E[d^1x(t,h)|\bar{x}(t,h)=x] = v^i(x,t)h + O(h^2),$$

(2.27)

where

$$v^i(x,t) \triangleq -A^i(x,t) + w^i(x,t),$$

(2.28)

represents the *mean velocity* of the process, and with covariance

$$E[(d^1x^i - v^i(x,t)h)(d^1x^j - v^j(x,t)h)|\bar{x}(t,h)=x] = \delta^{ij}h/2 + \pi^{ij}(x,t)h^2 + o(h^2).$$

(2.29)

By analogy with the kinetic theory of gases, for which the $1/h$ term of the covariance expansion (2.29) is not present, we call $\pi^{ij}(x,t)$ the *stress tensor* of the Gaussian velocity distribution specified by (2.27) and (2.28).

An important difference between the stochastic Newton law and velocity distribution obtained above is that whereas the conditional distribution for d^2x specified by (2.20)–(2.21) is the same for all diffusions in the same reciprocal class, the velocity distribution given by (2.27)–(2.29) depends on the density function $q(x,s;y,t)$ through the functions $w^i(x,t)$ and $\pi^{ij}(x,t)$, so that different diffusions within a same reciprocal class will admit different conditional velocity distributions.

Finally, it was shown in Ref. 19 that reciprocal diffusions admit an infinite chain of conservation laws, which can be generated by considering the function $q(x,t;y,t)$ obtained by letting $s \to t$ in the end-point density of the reciprocal diffusion over $[s,t]$. The identities (2.13a)–(2.13b) imply $q(x,t;y,t)$ obeys the evolution equation

$$\frac{\partial q}{\partial t}(x,t;y,t) = (\mathbf{H}_{x,t} - \mathbf{H}^*_{y,t})q(x,t;y,t).$$

(2.30)

Then, the function

$$m(a,b,t) \triangleq q(a-b,t;a+b,t),$$

(2.31)

plays the role of generating function for the conservation laws. Specifically, by performing a Taylor series expansion of $m(a,b,t)$ in the vicinity of $b=0$ and taking into account the definitions (2.26a)–(2.26b) of w_i and π_{ij}, we find $m(a,b,t)$ admits the power series representation,

$$m(a,b,t) = \rho(a,t)\left[1 + 2w_i(a,t)b^i + 4(\pi_{ij} + w_iw_j)(a,t) \frac{b^ib^j}{2} \right.$$

$$\left. + 8(\tau_{ijk} + \pi_{ij}w_k + \pi_{ki}w_j + \pi_{jk}w_i + w_iw_jw_k)(a,t) \frac{b^ib^jb^k}{6} + \cdots \right],$$

(2.32)

for small b. The chain of conservation laws of reciprocal diffusions is then obtained by performing the change of coordinates $x=a-b$ and $y=a+b$ and matching successive powers of b on both sides of (2.30). The constant term yields the law of mass conservation,

$$\frac{\partial \rho}{\partial t} + \nabla^i(\rho v_i) = 0. \tag{2.33}$$

The linear terms in b^j give the conservation of momentum,

$$\frac{\partial}{\partial t}(\rho v_j) + \nabla^i(\rho P_{ij}) = \rho F_j(x,v,t), \tag{2.34}$$

where F_j is the force defined in (2.23) and where, by analogy with fluid dynamics,

$$P_{ij}(x,t) \triangleq (\pi_{ij} + v_i v_j)(x,t), \tag{2.35}$$

is called the *flux of momentum tensor*. Finally, matching the quadratic terms in $b^j b^k$, we obtain the tensor form

$$\frac{\partial}{\partial t}(\rho P_{jk}) + \nabla^i(\rho S_{ijk}) = \rho(dA_{ji} \ P^i_{\ k} + dA_{ki} \ P^i_{\ j} + f_j v_k + f_k v_j) \tag{2.36}$$

of the conservation of energy. In this expression,

$$f_j(x,t) = -\left(\frac{\partial A_j}{\partial t} + \frac{\partial \phi}{\partial x^j}\right)(x,t), \tag{2.37}$$

corresponds to the electric component (the part independent of v) of the force F_j, and if we introduce the tensor

$$\sigma_{ijk}(x,t) \triangleq \left(\tau_{ijk} - \frac{1}{4}\frac{\partial^2 A_i}{\partial x^j \ \partial x^k}\right)(x,t), \tag{2.38a}$$

$$S_{ijk}(x,t) \triangleq (\sigma_{ijk} + \pi_{ij}v_k + \pi_{ki}v_j + \pi_{jk}v_i + v_i v_j v_k)(x,t), \tag{2.38b}$$

represents the *flux of energy tensor*. Then, if we denote the internal energy by $E = \pi_j^{\ j}/2$,, and take the trace of (2.36) by using the skew symmetry of the tensor dA_{ij}, we obtain the scalar form

$$\frac{\partial}{\partial t}\left(\rho\left(E + \frac{1}{2}v_j v^j\right)\right) + \frac{1}{2}\nabla^i(\rho S_{ij}^{\ j}) = \rho f_j v^j \tag{2.39}$$

of the energy conservation law of fluid mechanics, where the term $v_j v^j/2$ represents the kinetic energy.

The above procedure for generating the conservation laws of reciprocal diffusions makes clear that they usually form an infinite chain, since each successive law contains a divergence term involving the next conserved quantity in the chain. However for the Gaussian case, i.e., when

$$A_i(x,t) = A_{ij}(t)x^j, \qquad \phi(x,t) = \tfrac{1}{2}\Phi_{ij}(t)x^i x^j, \tag{2.40}$$

are, respectively, linear and quadratic in x, and $\ln q(x_0,0;x_T,T)$ is a quadratic form of x_0 and x_T, the tensor $\sigma_{ijk}(x,t)$ given by (2.38a) is identically zero, so that S_{ijk} depends only on the previous conserved quantities in the chain. In this case, the chain closes after the first three laws. Specifically, in the Gaussian case, $\rho(x,t)$ and $v(x,t)$ admit the parametrization

$$\rho(x,t) = N(x_C(t), K_x(t)), \tag{2.41a}$$

$$v(x,t) = \dot{x}_C(t) + V(t)(x - x_C(t)), \tag{2.41b}$$

where $N(m,K)$ denotes a normal distribution with mean vector and covariance matrix K. Here $x_C(t)$ represents the classical trajectory in the absence of random fluctuations and $K_x(t)$ denotes the covariance matrix of $x(t)$. The stress tensor π_{ij} does not depend on x, and can be represented by an $n \times n$ matrix $\pi(t)$. Let

$$L_1(t) = A^T(t) - A(t), \tag{2.42a}$$

$$L_2(t) = -\Phi(t) - \frac{dA}{dt}(t), \tag{2.42b}$$

where $A(t)$ and $\Phi(t)$ are the $n \times n$ matrices representing the tensors A_{ij} and Φ_{ij} in the parametrization (2.40) of the covector and scalar potentials, and T denotes the matrix transpose. Then, the conservation laws (2.33), (2.34), and (2.36) can be expressed compactly[25] as

$$\frac{d\Omega}{dt} = \Lambda\Omega + \Omega\Lambda^T, \tag{2.43}$$

with

$$\Omega = \begin{bmatrix} K_x & K_x V \\ V K_x & \pi + V K_x V^T \end{bmatrix}, \tag{2.44a}$$

$$\Lambda = \begin{bmatrix} 0 & I_n \\ L_2 & L_1 \end{bmatrix}, \tag{2.44b}$$

which obviously forms a closed system.

III. CONSTRUCTION OF QUANTUM DIFFUSIONS

The general family of reciprocal diffusions contains two interesting and disjoint subclasses, the Markov and quantum diffusions, for which the chain of conservation laws closes after the first two. These two classes are characterized by the requirement that the functions $w_i(x,t)$ and $\pi_{ij}(x,t)$ given by (2.26a)–(2.26b) must satisfy the closure rules,

$$w_i(x,t) = \nabla_i S(x,t), \tag{3.1a}$$

$$\pi_{ij}(x,t) = \frac{\epsilon}{4} \nabla_i \nabla_j \ln \rho(x,t), \tag{3.1b}$$

where $S(x,t)$ is an arbitrary function, and where $\epsilon = 1$ for Markov diffusions and $\epsilon = -1$ in the quantum case. Note that (3.1a) is satisfied whenever the exterior derivative of $w_i = v_i + A_i$ equals zero. Using this last observation, it is easy to verify that for the Gaussian case, the closure rules reduce to

$$L_1(t) = V(t) - V^T(t), \tag{3.2a}$$

$$\pi(t) = -\frac{\epsilon}{4} K_x^{-1}(t), \tag{3.2b}$$

where K_x, V, and L_1 are as defined in (2.41)–(2.42). By using the representation (2.43) of the conservation laws, we can also verify that if the closure rules (3.2) hold at one instant in time, at that instant we have

$$\frac{d}{dt}(L_1 + V^T - V) = \frac{d}{dt}\left(\pi + \frac{\epsilon}{4}K_x^{-1}\right) = 0,$$

(3.3)

so that the closure rules continue to hold for the complete time interval over which (2.43) admits a solution. In other words, once closed, the conservation laws remain closed.

For the non-Gaussian case, to discuss the consequences of the closure rules (3.1), it is convenient to rewrite the conservation laws (2.33), (2.34), and (2.36), which are expressed in Eulerian form, in the equivalent Lagrangian form

$$\frac{\partial \rho}{\partial t} + v_i \nabla^i \rho + \rho \nabla^i v_i = 0,$$

(3.4a)

$$\rho \frac{\partial v_j}{\partial t} + \rho v_i \nabla^i v_j + \nabla^i (\rho \pi_{ij}) - \rho F_j = 0,$$

(3.4b)

$$\rho \frac{\partial \pi_{jk}}{\partial t} + \rho v_i \nabla^i \pi_{jk} + \nabla^i (\rho \sigma_{ijk}) = \rho[(dA_{ji} - \nabla_i v_j) \pi^i{}_k + (dA_{ki} - \nabla_i v_k) \pi^i{}_j].$$

(3.4c)

Then, if we introduce the function

$$R(x,t) = \tfrac{1}{2} \ln \rho(x,t),$$

(3.5)

and use (3.1a) to specify $S(x,t)$, it is easy to verify that under the closure rules (3.1a)–(3.1b), the first two conservation laws can be expressed in terms of R and S as

$$\frac{\partial R}{\partial t} + (\nabla^i S - A^i)\nabla_i R + \frac{1}{2}\nabla^i(\nabla_i S - A_i) = 0,$$

(3.6)

$$\nabla_j I(x,t) = 0,$$

(3.7a)

with

$$I(x,t) \triangleq \frac{\partial S}{\partial t} + \frac{1}{2}(\nabla^i S - A^i)(\nabla_i S - A_i) + \phi + \frac{\epsilon}{2}(\nabla^i R\,\nabla_i R + \Delta R).$$

(3.7b)

The identity (3.7a) implies that $I(x,t)$ depends on t only, i.e., $I(x,t) = I(t)$. At this point, it is useful to note that for a fixed w_i, the relation (3.1a) specifies $S(x,t)$ only up to a function of t, say $f(t)$. This function contributes a term equal to $\dot{f}(t)$ to $I(t)$, which can be used to set $I(t) \equiv 0$. Thus, under the closure rules (3.1), we have shown that the first two conservation laws are equivalent to the coupled evolution equations (3.6) and (3.7b), with $I = 0$, for R and S. Note that except for the addition of the term $-\epsilon\kappa/2$, with

$$\kappa \triangleq -(\nabla^i R\,\nabla_i R + \Delta R) = -\frac{\Delta \rho^{1/2}}{\rho^{1/2}},$$

(3.8)

the equation $I(x,t) = 0$ is identical to the Hamilton–Jacobi equation of classical mechanics.

In the Markov case, for which $\epsilon = 1$, if we denote

$$q_f(x,t) = \exp(R - S)(x,t),$$

(3.9a)

$$q_b(x,t) = \exp(R + S)(x,t),$$

(3.9b)

it is easy to verify that Eq. (3.6) and (3.7b) can be rewritten as the decoupled forward and backward heat equations (2.15) for q_f and q_b. With this choice, the product

$$q_f(x,t)q_b(x,t) = \exp 2R(x,t) = \rho(x,t) \tag{3.10}$$

corresponds precisely to the identity (2.14) for the probability density of a Markov diffusion.

Similarly, for the quantum case, for which $\epsilon = -1$, if we introduce the wave function

$$\psi(x,t) = \exp(R + iS)(x,t), \tag{3.11}$$

the coupled equations (3.6) and (3.7b) correspond, respectively, to the real and imaginary parts of Schrodinger's equation,

$$i\frac{\partial \psi}{\partial t} = H_Q \psi(x,t), \tag{3.12}$$

where we have set Planck's constant $\hbar = 1$, and where the Hermitian operator,

$$H_Q = \tfrac{1}{2}(-i\nabla^i - A^j)(-i\nabla_j - A_j) + \phi, \tag{3.13}$$

is obtained by applying the correspondence principle $p_j \leftrightarrow -i\nabla_j$ of quantum mechanics to the Hamiltonian $H(x,p,t)$. The wave function $\psi(x,t)$ obtained in this manner satisfies

$$|\psi(x,t)|^2 = \exp 2R(x,t) = \rho(x,t), \tag{3.14}$$

and is thus consistent with the probability density of the quantum diffusion we are considering.

Comparing the coupled evolution equations (3.6) and (3.7b) for $\epsilon = 1$ and $\epsilon = -1$, one finds that the probability density ρ of a quantum diffusion with potentials $\{A_i, \phi\}$ is consistent with the density of a Markov diffusion with potentials $\{A_i, \phi'\}$, where

$$\phi' = \phi + \kappa, \tag{3.15}$$

with κ given by (3.8). This identification forms the basis for the reinterpretation of quantum mechanics based on Markov diffusions proposed by Schrödinger[5] and later refined by Zambrini[9] and Nagasawa.[10] In this respect, it is worth noting that the correction term κ relating the physical potential ϕ to the potential ϕ' of the matching Markov diffusion is identical, except for a factor of 2, to the "quantum potential" introduced by Bohm[12] in his causal formulation of quantum mechanics in terms of hidden variables (see Ref. 26 for comprehensive accounts of Bohm's theory). The factor of 2 arises because the correcting potential that must be applied to the stochastic Hamilton–Jacobi equation (3.7b) to transform the Markov motion ($\epsilon = 1$) into a quantum one ($\epsilon = -1$) is twice as large as the correction needed to go from the classical motion ($\epsilon = 0$) to the quantum motion. One problem associated with the introduction of a quantum potential is, of course, that it implies an action at a distance whereby two widely separated particles can affect each other instantaneously, thus violating the locality of classical physics, according to which the dynamics of each particle should be governed only by local force fields.

Nelson's stochastic mechanics[9–11] relies also on Markov diffusions. However, unlike Schrödinger's approach, which redefines the force fields to incorporate a quantum potential, it redefines the acceleration. Specifically, for a Markov diffusion $x(t)$, Nelson's stochastic acceleration takes the form

$$a_N^i(t) = \tfrac{1}{2}(D_-D_+ + D_+D_-)x^i(t), \tag{3.16}$$

where D_+ and D_- denote the mean forward and backward derivatives corresponding to $x(t)$. In flat space, and for an arbitrary tensor $T(x,t)$, these derivatives are defined as

$$D_+ T(x(t),t) = \lim_{h \to 0} \frac{1}{h} E_t[T(x(t+h),t+h) - T(x(t),t)],$$ (3.17a)

$$D_- T(x(t),t) = \lim_{h \to 0} \frac{1}{h} E_t[T(x(t),t) - T(x(t-h),t-h)],$$ (3.17b)

where E_t denotes the conditional expectation given $x(t)$. It is shown in Ref. 24 [see Eq. (7.4)] that for a standard Markov diffusion with forward drift $b^i(x,t)$, which corresponds to setting $A^i = -b^i$ and $\phi = -(b^i b_i + \nabla^i b_i)/2$, and diffusion metric δ_{ij}, the acceleration $a_N^i(t)$ satisfies

$$a_N^i(t) = F^i(x(t), v(x(t),t), t) + \nabla^i \kappa(x(t),t),$$ (3.18)

where $F(x,v,t)$ is the force defined in (2.23),

$$v(x(t),t) = \tfrac{1}{2}(D_+ + D_-)x(t),$$ (3.19)

coincides with the mean velocity specified by (2.27), and κ is the quantum potential given in (3.8). Consequently, when Nelson's acceleration is evaluated for a Markov diffusion with modified potentials $\{A_i, \phi'\}$, we obtain

$$a_N^i(t) = F^i(x(t), v(x(t),t), t),$$ (3.20)

so that the acceleration equals the force associated to the unmodified potentials $\{A_i, \phi\}$. Thus, even though Nelson's stochastic mechanics employs exactly the same Markov diffusions as Schrödinger's theory, in the Newton law $ma = F$, where we have set $m = 1$, the correction term associated to the quantum potential is shifted from the right to the left-hand side through a redefinition of the acceleration. Not surprisingly, it was discovered that Nelson's stochastic mechanics is nonlocal, a feature that, although it is viewed as a virtue by advocates of Bohm's theory (see Ref. 27 for a discussion of stochastic mechanics from a Bohmian viewpoint), led Nelson (Ref. 10, p. 127) to write "But the whole point (of the Markovian stochastic mechanics) was to construct a physically realistic picture of microprocesses, and a theory that violates locality is untenable."

The problem is, of course, that the conservation laws of Markov diffusions are not equivalent to Schrödinger's equation, and our objective here is to develop a stochastic mechanics, which instead of focusing on Markov diffusions, will apply to the quantum diffusions obtained by setting $\epsilon = -1$ in the closure rules (3.1). As a first step, we need to prove that such diffusions exist. Specifically, in the specification (2.7) for the reciprocal diffusions associated to a Hamiltonian with potentials $\{A_i, \phi\}$, the only element that does not depend on the physics of the problem is the end-point density $q(x,s; y, t)$. We need therefore to construct densities q such that the closure rules (3.1) hold with $\epsilon = -1$. In addition, since we seek to model quantum phenomena, it would be nice, if given a wave function $\psi(x,t)$ satisfying Schrödinger's equation, we could construct the matching q directly from ψ.

To elucidate the structure of the density $q(x,s; y, t)$ for quantum diffusions, consider the function $M(a,b,t)$ defined in (2.25). Taking into account the definition (2.26b) of $\pi_{ij}(a,t)$ and observing that

$$M(a,0,t) = \ln \rho(a,t) = 2R(a,t),$$ (3.21)

the closure rule (3.1b) can be expressed as

$$\left(\frac{\partial^2 M}{\partial a^k \, \partial a^l} + \frac{\partial^2 M}{\partial b^k \, \partial b^l} \right)(a,0,t) = 0.$$ (3.22)

This constraint is, of course, satisfied if

$$\left(\frac{\partial^2 M}{\partial a^k \, \partial a^l} + \frac{\partial^2 M}{\partial b^k \, \partial b^l}\right)(a,b,t)=0, \tag{3.23}$$

for all a and b in $\mathbb{R}^n$, where we recognize the Cauchy–Riemann conditions for an analytic function of the n complex variables $z^k=a^k+ib^k$.

Specifically, $M(a,b,t)$ can be viewed as the real part of an analytic function,

$$F(z,t)=M(a,b,t)+iN(a,b,t), \tag{3.24}$$

with $z=a+ib\in\mathbb{C}^n$, for which the Cauchy–Riemann conditions take the form

$$\frac{\partial M}{\partial a^k}=\frac{\partial N}{\partial b^k}, \tag{3.25a}$$

$$\frac{\partial M}{\partial b^k}=-\frac{\partial N}{\partial a^k}. \tag{3.25b}$$

These conditions imply (3.23). Furthermore, substituting (3.25b) inside the expression (2.26a) for w_k gives

$$2w_k(a,t)=\frac{\partial M}{\partial b^k}(a,0,t)=-\frac{\partial N}{\partial a^k}(a,0,t), \tag{3.26}$$

so that the closure rule (3.1a) is also satisfied with

$$2S(a,t)=-N(a,0,t). \tag{3.27}$$

To summarize, the end-point density $q(x,t;y,t)$ obtained by shrinking to zero the interval of definition $[s,t]$ of a reciprocal diffusion corresponds to a quantum diffusion, provided

$$q(x,t;y,t)=\exp\, M\left(\frac{y+x}{2},\frac{y-x}{2},t\right), \tag{3.28}$$

where $M(a,b,t)=\mathrm{Re}\, F(z,t)$ is the real part of an analytic function of z. Furthermore, from (3.21) and (3.27), the evaluation of $F(z,t)$ for $z=a+i0\in\mathbb{R}^n$ gives

$$F(a+i0,t)=M(a,0,t)+iN(a,0,t)=2(R(a,t)-iS(a,t))=2\ln\,\psi^*(a,t), \tag{3.29}$$

i.e., $F(z,t)$ can be viewed as obtained by analytical continuation of the function $2\ln\,\psi^*(a,t)$ defined over $\mathbb{R}^n$. Note that this is only possible as long as $\rho(a,t)=|\psi(a,t)|^2$ does not admit nodes, i.e., values of a for which the density is zero. When nodes are present, $\ln\,\psi^*(a,t)$ has singularities, so that a straightforward analytic continuation is not possible, although $F(z,t)$ can still be defined as a meromorphic function.

To analyze the effect of the nodes of $\rho(x,t)$ on the end-point density, let $\ln\,\psi(z,t)$ with $z=a+ib$ denote the analytical continuation of $\ln\,\psi(a,t)$ to $\mathbb{C}^n$. Then the expression (3.28) for the end-point density can be rewritten as

$$q(x,t;y,t)=\left|\psi\left(\frac{y+x}{2}-i\,\frac{y-x}{2},t\right)\right|^2, \tag{3.30}$$

so that whenever $\rho(x,t)$ has a node at $x=x_0$, $q(x,t;y,t)$ has a node at $x=y=x_0$. Note also that the structure (3.30) of the end-point density is the analog for quantum diffusions of the separable structure (2.14) of Markov diffusions.

Assuming at this point that a function $q(x,t;y,t)$ of the form (3.28) has been constructed, to obtain the end-point density $q(x,s;y,t)$ for an interval $[s,t]$ of nonzero length, we can either propagate the evolution equation (2.13a) backward in time, with initial condition $q(x,t;y,t)$, or propagate (2.13b) forward in time starting from $q(x,s;y,s)$. In both cases, this corresponds to propagating a heat equation in an unstable direction, i.e., we are trying to push back the heat toward its source. Consequently, solutions will usually exist only over a finite time interval, thus suggesting that quantum diffusions have generally a finite lifetime. In addition, although $q(x, t;y,t)$ given by (3.28) is always non-negative, for $s<t$, the solution $q(x,s;y,t)$ of (2.13a) or (2.13b) may become negative for some values of x and y, as will be shown in Sec. VII.

IV. GAUSSIAN PROCESSES

The construction procedure we have just outlined can be simplified further for the case when the potentials $\{A_i,\phi\}$ have the structure (2.40) and the wave function $\psi(x,t)$ is Gaussian, in which case the corresponding quantum diffusion is a Gaussian process. Specifically, following Ref. 10, Sec. 16, assume that

$$\ln \psi(x,t) = R(x,t) + iS(x,t),$$ (4.1a)

with

$$R(x,t) = -\tfrac{1}{2}(x-x_C(t))^T R(t)(x-x_C(t)) + f_R(t),$$ (4.1b)

$$S(x,t) = \tfrac{1}{2}(x-x_C(t))^T S(t)(x-x_C(t)) + x^T p_C(t) + f_S(t),$$ (4.1c)

where $f_R(t), f_S(t)$ are functions of t only. $x_C(t)$ denotes the classical trajectory of the particle, and $p_C^i(t) = \dot{x}_C^i(t) + A^i(x_C,t)$ represents the momentum along this trajectory. In this case,

$$\rho(x,t) = |\psi(x,t)|^2$$ (4.2)

is a Gaussian distribution with mean $x_C(t)$ and covariance matrix

$$K_x(t) = R^{-1}(t)/2.$$ (4.3)

Since $2\ln \psi^*(a,t)$ is quadratic in a, its analytical continuation is obtained by replacing a with $z = a + ib$, which gives

$$F(z,t) = -(z-x_C(t))^T R(t)(z-x_C(t)) - i(z-x_C(t))^T S(t)(z-x_C(t)) - 2iz^T p_C(t) + f_F(t),$$ (4.4)

where $f_F(t)$ depends again on t only. Taking the real part, we find

$$M(a,b,t) = -(a-b-x_C(t))^T R(t)(a+b-x_C(t)) + 2b^T S(t)(a-x_C(t)) + 2b^T p_C(t) + f_M(t).$$ (4.5)

Substituting this expression inside the identities (2.26a)–(2.26b) for $w(a,t)$ and the stress tensor $\pi(a,t)$ gives

$$w(a,t) = p_C(t) + S(t)(a-x_C(t)),$$ (4.6a)

$$\pi(a,t) = \frac{R(t)}{2}.$$ (4.6b)

Clearly, π satisfies the closure rule (3.1b), and (4.6a) can be rewritten in terms of the mean velocity $v^i = w^i - A^i$ in the form (2.41a) with

$$V(t) = S(t) - A(t),$$

(4.7)

where $A(t)$ denotes the matrix representing the tensor A_{ij} in (2.40).

Next, the expression (3.28) for $q(x,t;y,t)$ yields

$$\ln q(x,t;y,t) = -\frac{1}{2}[(x-x_C(t))^T(y-x_C(t))^T]Q(t,t)\begin{bmatrix} x-x_C(t) \\ y-x_C(t) \end{bmatrix}$$

$$-[(x-x_C(t))^T(y-x_C(t))^T]\begin{bmatrix} p_C(t) \\ -p_C(t) \end{bmatrix} + f_q(t),$$

(4.8)

with

$$Q(t,t) = \begin{bmatrix} S(t) & R(t) \\ R(t) & -S(t) \end{bmatrix}.$$

(4.9)

Assume now that $q(x,s;y,t)$ has the structure

$$\ln q(x,s;y,t) = -\frac{1}{2}[(x-x_C(s))^T(y-x_C(t))^T]Q(s,t)\begin{bmatrix} x-x_C(s) \\ y-x_C(t) \end{bmatrix}$$

$$-[(x-x_C(s))^T(y-x_C(t))^T]p(s,t) + f_q(s,t),$$

(4.10a)

with

$$Q(s,t) = \begin{bmatrix} Q_{xx}(s,t) & Q_{xy}(s,t) \\ Q_{yx}(s,t) & Q_{yy}(s,t) \end{bmatrix},$$

(4.10b)

$$p(s,t) = \begin{bmatrix} p_x(s,t) \\ p_y(s,t) \end{bmatrix}.$$

(4.10c)

By substituting this expression and the representation (2.40) of the potentials $\{A_i, \phi\}$ inside the heat equation (2.13a) for $q(x,s;y,t)$, we find it reduces to the ordinary differential equations,

$$-\frac{dQ}{ds}(s,t) = \begin{bmatrix} Q_{xx} - A^T(s) \\ Q_{yx} \end{bmatrix}[Q_{xx} - A(s) \quad Q_{xy}] + \begin{bmatrix} \Phi(s) & 0 \\ 0 & 0 \end{bmatrix},$$

(4.11)

$$-\frac{dp}{ds}(s,t) = \begin{bmatrix} Q_{xx} \\ Q_{yx} \end{bmatrix}(p_x - A(s)x_C(s) - \dot{x}_C(s)) - \begin{bmatrix} A^T(s)(p_x - A(s)x_C(s)) - \Phi(s)x_C(s) \\ 0 \end{bmatrix},$$

(4.12)

where Φ is the matrix representing the tensor Φ_{ij} in (2.40). By observing that the position x_C and momentum p_C for the classical trajectory, satisfy Hamilton's equations,

$$\dot{x}_C = \frac{\partial H}{\partial p} = p_C - Ax_C,$$

(4.13a)

$$\dot{p}_C = -\frac{\partial H}{\partial x} = A^T(p_C - Ax_C) - \Phi x_C,$$

(4.13b)

it is easy to verify that the differential equation (4.12) is satisfied by

$$p_x(s,t) = p_C(s), \quad p_y(s,t) = -p_C(t).$$

(4.14)

Thus, once the classical trajectory has been evaluated, to specify the end-point density $q(x,s;y,t)$, we only need to solve the matrix Riccati equation (4.11) for $s \leq t$, with initial condition (4.9).

To illustrate the above results, we consider two simple examples: the minimum uncertainty wave packet and the coherent state of the harmonic oscillator.

Example 4.1: For a *free particle*, we have

$$A(x,t) = \phi(x,t) = 0, \tag{4.15}$$

so that the Green's function,

$$G(x,s;y,t) = \frac{1}{(2\pi(t-s))^{1/2}} \exp{-\frac{(y-x)^2}{2(t-s)}} \tag{4.16}$$

is the standard heat kernel. The wave function corresponding to a minimum uncertainty wave packet centered about the classical trajectory $x_C(t) = x_0 + v_0 t$ is given by

$$\psi(x,t) = \frac{1}{\pi^{1/4}(r+it/r)^{1/2}} \exp\left[-\frac{(x-x_0-ir^2 v_0)^2}{2(r^2+it)} - \frac{r^2 v_0^2}{2} \right]. \tag{4.17}$$

The real and imaginary parts R and S of $\ln \psi$ have the form (4.1b)–(4.1c) with

$$R(t) = \frac{r^2}{r^4+t^2}, \quad S(t) = \frac{t}{r^4+t^2}, \tag{4.18}$$

and where the momentum $p_C(t) = v_0$ along the classical trajectory remains constant. The density $\rho(x,t)$ is therefore Gaussian, centered about $x_C(t)$, with standard deviation,

$$\sigma_x(t) = \frac{1}{\sqrt{2}} \left(r^2 + \frac{t^2}{r^2} \right)^{1/2}. \tag{4.19}$$

The momentum density, which is obtained by Fourier transforming $\psi(x,t)$, and squaring the resulting transform, is also Gaussian with mean v_0 and standard deviation,

$$\sigma_p(t) = \frac{1}{r\sqrt{2}}. \tag{4.20}$$

The position-momentum uncertainty product,

$$\sigma_x(t)\sigma_p(t) = \tfrac{1}{2}(1+t^2/r^4)^{1/2}, \tag{4.21}$$

equals the Heisenberg lower bound of $\tfrac{1}{2}$ at $t=0$, which explains why this process is called a minimum uncertainty wave packet. The factor r^2 represents the ratio of the position and momentum standard deviations, i.e., their relative spreading, at $t=0$.

According to (4.7) and (4.6b), the mean velocity and stress tensor of the quantum diffusion modeling this process are given by

$$v(x,t) = v_0 + \frac{t(x-x_C(t))}{r^4+t^2}, \tag{4.22a}$$

$$\pi(x,t) = \frac{r^2}{2(r^2+t^2)}. \tag{4.22b}$$

Note that as $t \to \infty$, the mean velocity,

$$v(x,t) \approx v_0 + (x - x_C(t))/t = (x - x_0)/t,$$ (4.23)

is obtained by modeling trajectories passing through x at time t as straight lines originating from x_0. Solving the Riccati equation (4.11) yields

$$Q(s,t) = \frac{1}{r^4 + st} \begin{bmatrix} t & r^2 \\ r^2 & -s \end{bmatrix},$$ (4.24)

which specifies $q(x,s;y,t)$.

Since both $q(x,s;y,t)$ and the Green's function $G(x,s;y,t)$ are Gaussian, the process $x(t)$ is Gaussian, so that it is entirely described by its mean and autocovariance functions. By combining the expressions (4.10) and (4.16) for q and G, we find that for $s < t$, the joint density of $x(s)$ and $x(t)$ is given by

$$p(x,s;y,t) = G(x,s;y,t)q(x,s;y,t) = N\left(\begin{bmatrix} x_C(s) \\ x_C(t) \end{bmatrix}, \mathbf{P}(s,t) \right),$$ (4.25a)

where

$$\mathbf{P}(s,t) = \frac{1}{2} \begin{bmatrix} r^2 + s^2/r^2 & r^2 + st/r^2 - (t-s) \\ r^2 + st/r^2 - (t-s) & r^2 + t^2/r^2 \end{bmatrix},$$ (4.25b)

denotes the covariance matrix of $x(s)$ and $x(t)$. Thus, if $z(t) = x(t) - x_C(t)$ represents the deviation of $x(t)$ with respect to the classical trajectory, $z(t)$ is Gaussian, with zero mean, and autocorrelation function

$$K(t,s) = E[z(t)z(s)] = \tfrac{1}{2}(r^2 + ts/r^2 - |t-s|).$$ (4.26)

Taking into account Newton's law and the characterization of Gaussian reciprocal diffusions given in Refs. 18 and 25, we find that over an arbitrary interval $[0,T]$, $z(t)$ satisfies the second-order stochastic differential equation,

$$\mathscr{L}_F z(t) = \xi(t),$$ (4.27a)

$$\mathscr{L}_F \triangleq -\frac{d^2}{dt^2},$$ (4.27b)

with Dirichlet conditions

$$\begin{bmatrix} z(0) \\ z(T) \end{bmatrix} \sim N(0, \mathbf{P}(0,T)).$$ (4.28)

The driving noise $\xi(t)$, which is usually called the dual or conjugate process of $z(t)$, is a generalized Gaussian process independent of $z(0)$ and $z(T)$, with zero mean and autocorrelation

$$E[\xi(t)\xi(s)] = \mathscr{L}_F \delta(t-s).$$ (4.29)

The Green's function of $\mathscr{L}_F$ can be used to decompose the solution of (4.27)–(4.29) into a component depending only on the noise $\xi(t)$ and a component representing the effect of the boundary conditions, so that over $[0,T]$, $z(t)$ can be represented as

$$z(t) = B(t) + \frac{T-t}{T} z(0) + \frac{t}{T} z(T), \tag{4.30}$$

where $B(t) = W(t) - tW(T)/T$ is a Brownian bridge process, independent of $z(0)$ and $z(T)$. Since the interval length T is arbitrary, the quantum diffusion corresponding to a minimum uncertainty wave packet has an infinite lifetime. This can also be seen by noting that a stochastic model equivalent to the one obtained above is given by

$$z(t) = W(t) + z(0)(1 - t/r^2), \tag{4.31}$$

where $W(t)$ is a Wiener process independent of $z(0)$. See Ref. 28, Sec. 5 for general results on the representation of scalar Gaussian reciprocal processes in terms of the Wiener process.

Example 4.2: The *harmonic oscillator* has for potentials,

$$A(x,t) = 0, \quad \phi(x,t) = \frac{(\omega x)^2}{2}. \tag{4.32}$$

Since the Lagrangian $L(x,p,t) = p\dot{x} - H(x,p,t)$ is quadratic, the Green's function can be expressed (see Ref. 2, Sec. 3.5) as

$$G(x,s;y,t) = C(t-s)\exp - S_C(x,s;y,t), \tag{4.33a}$$

where

$$S_C(x,s;y,t) = \int L(x_C, \dot{x}_C, u) du \tag{4.33b}$$

denotes the action for the classical path linking $x(s) = x$ to $x(t) = y$, and $C(t-s)$ is a function of $t-s$. Evaluating S_C yields

$$S_C = \frac{\omega}{2 \sin(\omega(t-s))} [(x^2 + y^2)\cos(\omega(t-s)) - 2xy], \tag{4.34a}$$

and substituting (4.32a) inside expression (2.10) for the transition property of the Green's function, we can identify

$$C(t-s) = \left(\frac{\omega}{2\pi \sin(\omega(t-s))} \right)^{1/2}. \tag{4.34b}$$

An important property of the Green's function $G(x,s;y,t)$ is that, viewed as a function of y, it decays as $|y| \to \infty$ for $t - s < \pi/2\omega$, but for $(t-s) > \pi/2\omega$, it grows exponentially with y, so that the Green's function does not exist for intervals of length larger than $T = \pi/2\omega$, which represents one-quarter of the period of the harmonic oscillator.

The wave function corresponding to a coherent state centered about the classical trajectory $x_C(t) = l \cos(\omega t)$, where l denotes the oscillation amplitude, takes the form (4.1) with

$$R(t) = \omega, \quad S(t) = 0, \tag{4.35}$$

and where the momentum along the classical trajectory is given by $p_C(t) = \dot{x}_C(t) = -l\omega \sin(\omega t)$. According to (4.7) and (4.6b), the mean velocity and stress tensor of the quantum diffusion modeling this process are given by

$$v(x,t) = p_C(t), \tag{4.36a}$$

$$\pi(x,t) = \omega/2. \tag{4.36b}$$

Note that since the mean velocity $v(x,t)$ does not depend on x, the wave packet does not undergo any deformation with time and retains its coherence, hence the name of the process. The solution of the Riccati equation (4.11) is given by

$$Q(s,t) = \frac{\omega}{\cos(\omega(t-s))} \begin{bmatrix} \sin(\omega(t-s)) & 1 \\ 1 & \sin(\omega(t-s)) \end{bmatrix}. \tag{4.37}$$

Both $q(x,s;y,t)$ and $G(x,s;y,t)$ are again Gaussian, so $x(t)$ is a Gaussian process. The joint density of $x(s)$ and $x(t)$ takes again the form (4.25a), with

$$\mathbf{P}(s,t) = \frac{1}{2\omega} \begin{bmatrix} 1 & \gamma(t-s) \\ \gamma(t-s) & 1 \end{bmatrix}, \tag{4.38a}$$

where

$$\gamma(t-s) = \cos(\omega(t-s)) - \sin(\omega(t-s)) = \frac{\cos(\omega(t-s) + \pi/4)}{\cos(\pi/4)}, \tag{4.38b}$$

denotes the correlation coefficient of $x(t)$ and $x(s)$. Note that for $\gamma(t-s)$ to be a correlation coefficient, its magnitude must be less than unity, which requires $t-s < \pi/2\omega$. Thus, the deviation $z(t) = x(t) - x_C(t)$ of $x(t)$ with respect to the classical trajectory $x_C(t)$ is Gaussian, with zero mean and autocorrelation function

$$K(t,s) = E[z(t)z(s)] = \frac{1}{2\omega} \frac{\cos(\omega(t-s) + \pi/4)}{\cos(\pi/4)}. \tag{4.39}$$

This process, which is called the *shifted cosine process*, was introduced by Carmichael, Massé, and Theodorescu,[29] while completing a classification of scalar stationary Gaussian reciprocal processes proposed earlier by Jamison.[30] In Ref. 18, it is shown that this process is defined over a finite interval of length $T = \pi/2\omega$, and satisfies the second-order stochastic differential equation,

$$\mathscr{L}_H z(t) = \xi(t), \tag{4.40a}$$

$$\mathscr{L}_H \triangleq -\frac{d^2}{dt^2} - \omega^2, \tag{4.40b}$$

with boundary condition

$$z(0) = -z(T) \sim N(0, 1/(2\omega)), \tag{4.41}$$

where the driving noise $\xi(t)$ is a generalized Gaussian process independent of $z(0)$ and $z(T)$, with zero mean and autocorrelation

$$E[\xi(t)\xi(s)] = \mathscr{L}_H \delta(t-s). \tag{4.42}$$

The relation (4.40a) shows that the dynamics of the shifted cosine process are those of an oscillator subjected to random fluctuations $\xi(t)$. The noise $\xi(t)$ is not white but has a local correlation structure depending on the oscillator dynamics $\mathscr{L}_H$. The lifetime $\pi/(2\omega)$ of the shifted cosine process corresponds to only one-quarter of the period of the harmonic oscillator. To con-

struct a model valid for longer periods of time, since the ground state of the harmonic oscillator is stationary, it is natural to expect that its model should not depend on the choice of time origin. The boundary condition $z(\pi/(2\omega)) = -z(0)$ therefore suggests the rule

$$z(t + \pi/(2\omega)) = -z(t), \tag{4.43}$$

for extending $z(\cdot)$ to all times. However, this choice implies $z(t + n\pi/\omega) = z(t)$ for n integer, whereas Heisenberg's position operator satisfies $X(t + n\pi/\omega) = (-1)^n X(t)$. This difference, and in particular its effect on the evaluation of multitime quantum correlations, requires further analysis.   □

V. POSITION-MOMENTUM UNCERTAINTY

Although the position-momentum uncertainty relations form a cornerstone of standard quantum mechanics, their role has been somewhat diminished in the development of Markovian stochastic mechanics.[31] This difference arises in part from difficulties in giving an operational definition to the momentum process.[32] Indeed, unlike the Hilbert space formulation of quantum mechanics, where position and momentum play completely symmetric roles, stochastic mechanics privileges position variables. While this last feature is retained by the stochastic mechanics of quantum diffusions described here, the position-momentum uncertainty relations will regain a key role. Specifically, we show they are a consequence of the closure rule (3.1b) with $\epsilon = -1$, and are thus characteristic of quantum diffusions.

As a starting point, consider the empirical momentum process,

$$p^k(t;h) = u^k(t;h) + A^k(x(t),t), \tag{5.1}$$

where $u(t;h) = d^1 x(t;h)/h$ is the empirical velocity of the diffusion. The expression (2.27) for the conditional mean of $d^1 x(t;h)$ given the mean position $\bar{x}(t;h)$ implies

$$E[p^k(t;h)|\bar{x}(t;h) = x] = v^k(x,t) + A^k(x,t) + o(1) = w^k(x,t) + o(1), \tag{5.2}$$

so that as $h \to 0$, the conditional mean of the abstract momentum process $p(t)$ given the position $x(t)$ can be defined as

$$E[p^k(t)|x(t) = x] \triangleq \lim_{h \to 0} E[p^k(t;h)|\bar{x}(t;h) = x] = w^k(x,t). \tag{5.3}$$

Taking into account the closure rule (3.1a) gives

$$E[p_k(t)] = \int w_k(x,t)\rho(x,t)dx = \int \psi^*(x,t)\nabla_k S(x,t)\psi(x,t)dx = \int \psi^*(x,t)(-i\nabla_k \psi(x,t))dx, \tag{5.4}$$

so that the usual correspondence principle,

$$p_k \leftrightarrow -i\nabla_k, \tag{5.5}$$

holds for this momentum definition.

The correlation matrix of the abstract momentum $p(t)$ is harder to define, since according to the characterization (2.27), (2.29) of the velocity distribution, the empirical momentum $p^k(t;h)$ has a size proportional to $h^{-1/2}$. However, following Ref. 2, p. 179, we can define it as the correlation of the empirical momentum evaluated for two successive infinitesimal time intervals. Specifically, we consider

$$p^k\left(t+\frac{h}{2};\frac{h}{2}\right)p^l\left(t-\frac{h}{2},\frac{h}{2}\right)=\left(\frac{dx_+^k}{h}+A^k\right)\left(\frac{dx_-^l}{h}+A^l\right),\tag{5.6}$$

where $dx_\pm=\pm(x(t\pm h)-x(t))/h$. Observing that the cross-product of the forward and backward differences d^+x and d^-x can be expressed in terms of the centered first- and second-order differences d^1x and d^2x as

$$dx_+^k\ dx_-^l=d^1x^k\ d^1x^l-\tfrac{1}{4}d^2x^k\ d^2x^l,\tag{5.7}$$

and using expressions (2.21) and (2.29) to evaluate the conditional covariances of d^2x and d^1x given the mean position $\bar{x}(t;h)$, we find

$$E[p^k(t+h/2;h/2)p^l(t-h/2,h/2)|\bar{x}(t;h)=x]=\pi^{kl}(x,t)+u^k(x,t)w^l(x,t)+o(1).\tag{5.8}$$

This implies

$$E[p^k(t)p^l(t)|x(t)=x]\triangleq\lim_{h\to 0}E[p^k(t+h/2;h/2)p^l(t-h/2,h/2)|\bar{x}(t;h)=x]$$

$$=\pi^{kl}(x,t)+w^k(x,t)w^l(x,t).\tag{5.9}$$

Taking into account the closure rules (3.1a)–(3.1b), it can then be checked that

$$E[p_k(t)p_l(t)]=\int\ (\pi_{kl}+w_kw_l)(x,t)\rho(x,t)dx$$

$$=\int\ \psi^*(x,t)(-\tfrac{1}{2}\nabla_k\nabla_l R(x,t)+\nabla_k S(x,t)\nabla_l S(x,t))\psi(x,t)dx$$

$$=\int\ \psi^*(x,t)(-\nabla_k\nabla_l\psi(x,t))dx,\tag{5.10}$$

so that we have again the usual correspondence principle,

$$p_kp_l\leftrightarrow(-i\nabla_k)(-i\nabla_l).\tag{5.11}$$

Note, however, that this correspondence depends on the unusual rule employed in (5.9) to evaluate the conditional correlation matrix of the momentum given the position.

From expressions (5.9) and (5.3) we see that the entries of the conditional covariance matrix $K_{p|x}(t)$ of the momentum $p(t)$ given the position $x(t)$ take the form

$$K_{p|x,kl}(x,t)=\text{cov}(p_k(t),p_l(t)|x(t)=x)=\pi_{kl}(x,t).\tag{5.12}$$

Taking expectations, this implies

$$E[K_{p|x,kl}(x(t),t)]=\int\ \pi_{kl}(x,t)\rho(x,t)dx.\tag{5.13}$$

On the other hand, the covariance matrix $K_p(t)$ of $p(t)$ is given by

$$K_{p,kl}(t)=\text{cov}(p_k(t),p_l(t))=\int\ [\pi_{kl}(x,t)+(\psi_k(x,t)-E[p_k(t)])(w_l(x,t)-E[p_l(t)])]^T\rho(x,t)dx.\tag{5.14}$$

Comparing (5.13) and (5.14), we obtain

$$K_p(t) \geq E[K_{p|x}(t)],$$ (5.15)

which just expresses the fact that the knowledge of $x(t)$ reduces the momentum variance.

Our derivation of the position-momentum uncertainty relations relies on the following lemma, which is an adaptation of standard results of Bayesian estimation theory.

Lemma 5.1: Let X be a random vector of $\mathbf{R}^n$ with probability density $p(x)$. Its mean vector and covariance matrix are denoted by m and K, respectively, i.e.,

$$E[X^k] = m^k, \quad E[(X^k - m^k)(X^l - m^l)] = K^{kl}.$$ (5.16)

Then, if we consider the Fisher information matrix J with entries

$$J_{kl} = -E[\nabla_k \nabla_l \ln p(X)],$$ (5.17)

the matrix

$$\mathbf{P} = \begin{bmatrix} K & -I \\ -I & J \end{bmatrix}$$ (5.18)

is non-negative definite. If K is positive definite, this implies

$$J \geq K^{-1}.$$ (5.19)

Proof: Consider the random vectors $A = X - m$ and $B = \nabla \ln p(X)$. Both have zero mean, A has covariance matrix K, and

$$E[A^k B_l] = E\left[(X^k - m^k) \frac{\partial}{\partial x^l} \ln p(X)\right] = \int (x^k - m^k)\left(\frac{\partial}{\partial x^l} \ln p(x)\right) p(x) dx$$

$$= -\int \frac{\partial}{\partial x^l} (x^k - m^k) p(x) dx = -\delta_l^k,$$ (5.20a)

$$E[B_k B_l] = E\left[\frac{\partial}{\partial x^k} \ln p(X) \frac{\partial}{\partial x^l} \ln p(X)\right] = \int \frac{\partial}{\partial x^k} \ln p(x) \frac{\partial}{\partial x^l} \ln p(x) p(x) dx$$

$$= -\int \frac{\partial^2}{\partial x^k \partial x^l} \ln p(x) p(x) dx = J_{kl}.$$ (5.20b)

The identities (5.20) indicate that the matrix $\mathbf{P}$ given by (5.18) is just the covariance matrix of A and B, and thus must be non-negative. In this covariance matrix, $J - K^{-1}$ is the Schur complement of the (2,2) block K, so that it is non-negative. □

The Lemma 5.1 can be combined with the quantum closure rule (3.1b) to derive the following result.

Theorem 5.1: If $x(t)$ is a quantum diffusion with position covariance matrix $K_x(t)$, it satisfies the matrix position-momentum uncertainty relation,

$$K_p(t) \geq E[K_{p|x}(t)] \geq \tfrac{1}{4} K_x^{-1}(t),$$ (5.21)

where the conditional covariance matrix $K_{p|x}(t)$ and the covariance matrix $K_p(t)$ are specified by (5.12) and (5.14), respectively.

Proof: Let $J(t)$ denote the Fisher information matrix corresponding to the quantum diffusion $x(t)$ with density $\rho(x,t)$. Taking into account (5.13), the closure rule (3.1b) implies

$$E[K_{p|x}(t)] = \tfrac{1}{4}J(t),\tag{5.22}$$

which, when combined with inequalities (5.15) and (5.19), yields (5.21).   ☐

The key step in the above derivation was the use of the quantum closure rule to derive the equality (5.22). This implies that the uncertainty relation (5.21) is specific to quantum diffusions and may not hold for other reciprocal diffusions. Also, as expressed in (5.21), the position-momentum uncertainty relation is slightly stronger than the usual version, since it holds for the averaged conditional covariance matrix $E[K_{p|x}(x(t),t)]$ of the momentum, given the position.

VI. LOCALITY

An important test for evaluating the realism of stochastic formulations of quantum mechanics is whether such formulations preserve locality. Indeed, in spite of the existence of interpretations of quantum mechanics, such as Bohm's theory,[26] which are apparently nonlocal, when viewed as a set of computational rules for evaluating statistical averages, quantum mechanics is inherently local, in a sense that will be described below. Unfortunately, as was demonstrated in Ref. 10, Markovian stochastic mechanics does not have this property, and must therefore be rejected as a proper model of quantum phenomena. In contrast, we show that the new form of stochastic mechanics described here is local.

To demonstrate this fact, consider a general reciprocal diffusion,

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},\tag{6.1}$$

not necessarily of quantum type, but where the components $x_1(t)$ and $x_2(t)$ are dynamically uncoupled. This means that the Hamiltonian (2.2) admits the decomposition

$$H(x,p,t) = H_1(x_1,p_1,t) + H_2(x_2,p_2,t),\tag{6.2}$$

where $\{A_1(x_1,t),\phi_1(x_1,t)\}$ and $\{A_2(x_2,t),\phi_2(x_2,t)\}$ are the potentials affecting each of the diffusion components. From the form (6.2) of the Hamiltonian, we can deduce that the Green's function $G(x,s;y,t)$ corresponding to **H** can be factored as

$$G(x,s;y,t) = G_1(x_1,s;y_1,t)G_2(x_2,s;y_2,t),\tag{6.3}$$

where G_1 and G_2 are the Green's functions corresponding to $\mathbf{H}_1$ and $\mathbf{H}_2$, respectively. This structure can be used to derive the following result.

Lemma 6.1: Let $x(t)$ be a reciprocal diffusion with two dynamically uncoupled components $x_1(t)$ and $x_2(t)$. Then each component $x_1(t)$ or $x_2(t)$ considered separately is a reciprocal diffusion. Furthermore, if $x(t)$ is a quantum diffusion, its component $x_1(t)$ is of quantum type if and only if the function R and S given by (3.5) and (3.1a) admit the additive decomposition,

$$R(x,t) = R_1(x_1,t) + R_2(x_2,t),\tag{6.4a}$$

$$S(x,t) = S_1(x_1,t) + S_2(x_2,t),\tag{6.4b}$$

or equivalently if the wave function $\psi(s,t)$ decomposes multiplicatively as

$$\psi(x,t) = \psi_1(x_1,t)\psi_2(x_2,t).\tag{6.5}$$

Proof: Substituting (6.3) inside expression (2.7) for the joint density of $x(t_0)$, $x(t_1),\dots,x(t_N)$ and integrating over the x_2 components, we find that the marginal joint density of $x_1(t_0),x_1(t_1),\dots,x_1(t_N)$ can be expressed as

$$p_1(x_{10},t_0;x_{11},t_1;\ldots;x_{1N},t_N) = q_1(x_{10},t_0;x_{1N},t_N) \prod_{k=0}^{N-1} G_1(x_{1k},t_k;x_{1k+1},t_{k+1}), \qquad (6.6)$$

where the end-point density $q_1(x_1,s;y_1,t)$ is given by

$$q_1(x_1,s;y_1,t) = \int G_2(x_2,s;y_2,t)q(x,s;y,t)dx_2\,dy_2, \qquad (6.7)$$

with $x^T = [x_1^T, x_2^T]$ and $y^T = [y_1^T, y_2^T]$. The structure (6.6) of the finite joint densities of $x_1(t)$ indicates it is a reciprocal diffusion. However, if $x(t)$ is a quantum diffusion, there is no guarantee that $x_1(t)$ will also be of the quantum type. Specifically, consider the partition

$$v(x,t) = \begin{bmatrix} v_1(x,t) \\ v_2(x,t) \end{bmatrix}, \quad \pi(x,t) = \begin{bmatrix} \pi_{11}(x,t) & \pi_{12}(x,t) \\ \pi_{21}(x,t) & \pi_{22}(x,t) \end{bmatrix} \qquad (6.8)$$

of the mean velocity and stress tensor in terms of their x_1 and x_2 components. According to the characterization (2.27) and (2.29) of the velocity distribution, the mean velocity $v_1^m(x_1,t)$ and stress tensor $\pi_{11}^m(x_1,t)$ for the marginal probability distribution of $x_1(\cdot)$ can be expressed as

$$v_1^m(x_1,t) = \int v_1(x,t)\rho_{2|1}(x_2,t|x_1,t)dx_2, \qquad (6.9a)$$

$$\pi_1^m(x_1,t) = \int \pi_{11}(x,t)\rho_{2|1}(x_2,t|x_1,t)dx_2 + \int (v_1(x,t) - v_1^m(x_1,t))$$

$$\times (v_1(x,t) - v_1^m(x_1,t))^T \rho_{2|1}(x_2,t|x_1,t)dx_2, \qquad (6.9b)$$

where $\rho_{2|1}(x_2,t|x_1,t)$ denotes the conditional probability density of $x_2(t)$ given $x_1(t)$. Note that since the covector potential A_1 depends only on x_1,

$$v_1(x,t) - v_1^m(x_1,t) = w_1(x,t) - w_1^m(x_1,t). \qquad (6.10)$$

Then if $x_1(t)$ is a quantum diffusion, w_1^m and π_1^m must satisfy the closure rules,

$$w_1^m(x_1,t) = \nabla_1 S_1(x_1,t), \qquad (6.11a)$$

$$\pi_1^m(x_1,t) = \frac{-1}{4} \nabla_1 \nabla_1^T \ln \rho_1(x_1,t), \qquad (6.11b)$$

where $\rho_1(x_1,t)$ denotes the marginal probability density of $x_1(t)$ and ∇_1 is the gradient with respect to x_1. Substituting these closure rules and the closure rules for $x(t)$, we find that after integration by parts, (6.9b) can be rewritten as

$$0 = \frac{1}{4} \int \nabla_1 \ln \rho_{2|1}(\nabla_1 \ln \rho_{2|1})^T \rho_{2|1}\,dx_2 + \int (w_1 - w_1^m)(w_1 - w_1^m)^T \rho_{2|1}\,dx_2, \qquad (6.12)$$

which implies

$$0 = \nabla_1 \ln \rho_{2|1} = \nabla_1(R(x,t) - R_1(x_1,t)), \qquad (6.13a)$$

$$0 = w_1 - w_1^m = \nabla_1(S(x,t) - S_1(x_1,t)), \qquad (6.13b)$$

so that R and S admit the decomposition (6.4).   □

Note that the conclusion that dynamically uncoupled components of quantum diffusions need not be quantum diffusions is consistent with the standard interpretation of quantum mechanics, which holds that statistically coupled components of a quantum process cannot be analyzed separately. It is also worth observing that the multiplicative structure of the decomposition (3.5) is preserved by the Schrödinger equation (or the equivalent conservation laws of quantum diffusions), so that if (3.5) holds for one t, it holds for all t.

Nevertheless, as indicated by Lemma 6.1, dynamically uncoupled components of quantum diffusions are reciprocal diffusions. This result is now employed to demonstrate locality. We use the fact that if $\psi(x,t)$ is the wave function of two dynamically decoupled, but possibly statistically coupled particles, quantum mechanics has the following locality property. Let O_1 be a Hermitian operator involving only the position and momentum operators of $x_1(t)$. Then, since the Heisenberg representation $O_1(t)$ of O_1 depends only on the Hamiltonian H_1, the inner product,

$$\langle \psi(x,t), O_1 \psi(x,t) \rangle, \tag{6.14}$$

is independent of the potentials $\{A_2, \phi_2\}$.

Theorem 6.1: Let $x(t)$ be a quantum diffusion constructed from a wave function with two dynamically uncoupled components $x_1(t)$ and $x_2(t)$. Then, the end-point density $q_1(x_1,s;y_1,t)$ of $x_1(t)$ does not depend on the potentials $\{A_2, \phi_2\}$ acting on $x_2(t)$.

Proof: The argument is patterned after that of Ref. 10, Theorem 23.1. Setting $s=t$ in (6.7), the end-point density of x_1 takes the form

$$q_1(a_1-b_1,t;a_1+b_1,t) = \int \exp M(a_1,a_2,b_1,0,t) da_2$$

$$= \int \exp[M(a_1,a_2,b_1,0,t) - M(a_1,a_2,0,0,t)]\rho(a_1,a_2,t) da_2, \tag{6.15}$$

where $\rho(x_1,x_2,t)$ is the joint density of $x_1(t)$ and $x_2(t)$. But in (6.15),

$$M(a_1,a_2,b_1,0,t) - M(a_1,a_2,0,0,t) = \mathrm{Re}[F(z_1,a_2,t) - 2 \ln \psi^*(a_1,a_2,t)], \tag{6.16}$$

where $F(z_1,z_2,t)$ is the analytical continuation of $2 \ln \psi^*(a_1,a_2,t)$. Since in expression (6.16), $z_2 = a_2 + i0$ is real, the analytic continuation needs to be performed only with respect to the z_1 variables, which requires taking derivatives of $2 \ln \psi^*(a_1,a_2,t)$ with respect to the entries of a_1 only. Noting that $q_1(a_1-b_1,t;a_1+b_1,t)$ is non-negative real, by integrating (6.15) against an arbitrary function $f(a_1)$, we find

$$\int f(a_1) q_1(a_1-b_1,t;a_1+b_1,t) da_1 = \langle \psi(a_1,a_2,t), O_1(b_1) \psi(a_1,a_2,t) \rangle, \tag{6.17}$$

where $O_1(b_1)$ is Hermitian, so that the left-hand side of (6.17) does not depend on $\{A_2,\phi_2\}$. Since $f(a_1)$ is arbitrary, this implies that the end-point density $q_1(x_1,t;y_1,t)$ does not depend on the potentials $\{A_2,\phi_2\}$. $\square$

To illustrate this result, we consider an example used by Nelson in Ref. 10, pp. 125–126 and Ref. 13 to demonstrate that his version of stochastic mechanics is nonlocal.

Example 6.1: Consider a quantum diffusion $x(t)$ with components $x_1(t)$ and $x_2(t)$, which correspond to the positions of two correlated but dynamically uncoupled particles, where the first particle is free and the second is a harmonic oscillator with frequency ω and rest position x_{20}. The initial wave function is given by

$$\psi(x,0) = \frac{1}{(2\pi)^{1/2}} \exp-\frac{1}{4}(x-x_0)^T K_x^{-1}(0)(x-x_0),$$

(6.18a)

with

$$K_x(0) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

(6.18b)

and

$$x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix},$$

(6.18c)

where the rest positions x_{10} and x_{20} of the two particles are assumed to be widely separated. Then, although $x_1(t)$ and $x_2(t)$ are statistically correlated, the locality property requires that the statistics of the $x_1(t)$ process should not depend on the frequency ω. This is due to the fact that, in the absence of any instantaneous action at a distance, the first particle does not know which frequency ω has been selected for the harmonic oscillator.

To verify that this is the case, note that since the Hamiltonian is quadratic and the initial wave function is Gaussian, the wave function remains Gaussian and can be expressed as

$$\psi(x,t) = \exp[-\tfrac{1}{2}(x-x_0)^T \Psi^{-1}(t)(x-x_0) + \alpha(t)],$$

(6.19)

where $\alpha(t)$ denotes a normalizing constant. Then the Schrödinger equation reduces to the Riccati equation,

$$i\frac{d\Psi}{dt} = -I_2 + \Psi\Phi\Psi,$$

(6.20)

where

$$\Phi = \begin{bmatrix} 0 & 0 \\ 0 & \omega^2 \end{bmatrix}$$

(6.21)

is the matrix representing the scalar potentials of the free particle and harmonic oscillator. Solving this equation with initial condition $2K_x(0)$ gives

$$\Psi(t) = \frac{1}{\beta(t)}\begin{bmatrix} (4+it)\beta(t)-4i\omega\,\sin(\omega t) & 2 \\ 2 & 2\cos(\omega t)+\dfrac{i}{\omega}\sin(\omega t) \end{bmatrix},$$

(6.22a)

with

$$\beta(t) = \cos(\omega t) + 2i\omega\,\sin(\omega t).$$

(6.22b)

Then the real and imaginary parts $R(x,t)$ and $S(x,t)$ of $\ln\psi(x,t)$ take the form (4.1b)–(4.1c) with $x_C(t)=x_0$, $p_C(t)=0$, and

$$(2R(t))^{-1} = K_x(t) = \begin{bmatrix} 2+\dfrac{t^2}{4} & \cos(\omega t)-\dfrac{t}{4\omega}\sin(\omega t) \\ \cos(\omega t)-\dfrac{t}{4\omega}\sin(\omega t) & \cos^2\omega t+\dfrac{1}{2\omega^2}\sin^2(\omega t) \end{bmatrix},$$

(6.23)

$$K_x(t)S(t) = \begin{bmatrix} \dfrac{t}{4} & -\omega \sin(\omega t) - \dfrac{t}{4}\cos(\omega t) \\ -\dfrac{1}{4\omega}\sin(\omega t) & \left(\dfrac{1}{2\omega} - \omega\right)\sin(\omega t)\cos(\omega t) \end{bmatrix}. \tag{6.24}$$

Fourier transforming $\psi(x,t)$ and squaring the magnitude of the resulting transform, we also find that the momentum process $p(t)$ has a zero-mean Gaussian density with covariance matrix

$$K_p(t) = \frac{K_x^{-1}(t)}{4} + S(t)K_x(t)S(t). \tag{6.25}$$

Since $S(0)=0$, the process $x(t)$ has minimum uncertainty at $t=0$.

The end-point density $q(x,t;y,t)$ takes the form (4.8)–(4.9), and to complete the construction of the quantum diffusion corresponding to the wave function (6.19), we would need, in principle, to evaluate $q(x,s;y,t)$ by propagating the backward heat equation (2.13a). However, this computation is rather tedious. A simpler approach consists in observing from the characterization of Gaussian reciprocal diffusions in terms of their Newton law given in Ref. 18, that for an interval $[0,T]$ over which it is defined, $z(t)=x(t)-x_0$ is a solution of the second-order stochastic differential equation,

$$\mathscr{L}_F z_1(t) = \xi_1(t) \tag{6.26a}$$

$$\mathscr{L}_H z_2(t) = \xi_2(t), \tag{6.26b}$$

with Dirichlet boundary conditions

$$\begin{bmatrix} z(0) \\ z(T) \end{bmatrix} \sim N(0,\mathbf{P}), \tag{6.27a}$$

where

$$\mathbf{P} = \begin{bmatrix} K_x(0) & K(0,T) \\ K(T,0) & K_x(T) \end{bmatrix}. \tag{6.27b}$$

In this equation, the operators $\mathscr{L}_F$ and $\mathscr{L}_H$ are given by (4.27b) and (4.40b), respectively, and the noises $\xi_1(t)$ and $\xi_2(t)$ are two independent generalized Gaussian processes, independent of $z(0)$ and $z(T)$, with zero mean and autocorrelations

$$E[\xi_1(t)\xi_1(s)] = \mathscr{L}_F \delta(t-s), \tag{6.28a}$$

$$E[\xi_2(t)\xi_2(s)] = \mathscr{L}_H \delta(t-s). \tag{6.28b}$$

In (6.27b), $K(t,s) = E[z(t)z^T(s)]$ denotes the matrix autocorrelation of $z(t)$, which still remains to be determined.

In our analysis, the length T of the interval is selected as one-quarter of the period of the harmonic oscillator, i.e., $T = \pi/2\omega$. Then, the Green's functions of the operators $\mathscr{L}_F$ and $\mathscr{L}_H$ with homogeneous Dirichlet boundary conditions at $t=0$ and $t=T$ are given by

$$\Gamma_F(t,s) = \begin{cases} \left(1 - \dfrac{t}{T}\right)s, & \text{for } t \geq s, \\[2mm] t\left(1 - \dfrac{s}{T}\right), & \text{for } s \geq t, \end{cases} \tag{6.29}$$

and

$$\Gamma_H(t,s) = \begin{cases} \dfrac{1}{\omega}\cos(\omega t)\sin(\omega s), & \text{for } t \geq s, \\ \dfrac{1}{\omega}\sin(\omega t)\cos \omega s, & \text{for } s \geq t. \end{cases} \tag{6.30}$$

As shown in Ref. 18, the solution of the stochastic boundary value problem (6.26)–(6.28) can be expressed as

$$z_1(t) = I_1(t) + \frac{T-t}{T} z_1(0) + \frac{t}{T} z_1(T), \tag{6.31a}$$

$$z_2(t) = I_2(t) + \cos(\omega t) z_2(0) + \sin(\omega t) z_2(T), \tag{6.31b}$$

where

$$I_1(t) = \int_0^T \Gamma_F(t,s) \xi_1(s) ds, \tag{6.32a}$$

$$I_2(t) = \int_0^T \Gamma_H(t,s) \xi_2(s) ds, \tag{6.32b}$$

are two independent zero-mean Gaussian processes with covariances $\Gamma_F(t,s)$ and $\Gamma_H(t,s)$, respectively. This implies that $I_1(t)$ is a Brownian bridge process, i.e., $I_1(t) = W_1(t) - tW_1(T)/T$, where $W_1(\cdot)$ is a standard Wiener process. The only element missing in the above specification of $z(t)$ is the correlation matrix $K(0,T)$ of the end-point vectors $z(0)$ and $z(T)$. Evaluating the autocorrelation function of the solution $z(t)$ given by (6.31), and comparing with the expression (6.23) for $K_x(t)$ gives

$$E[z(0)z^T(T)] = K(0,T) = \begin{bmatrix} 2 - \dfrac{T}{2} & 0 \\ 1 & -\dfrac{1}{2\omega} \end{bmatrix}, \tag{6.33}$$

from which we deduce that the autocorrelation of the process $z(t)$ is given by

$$K_1(t,s) = E[z_1(t)z_1(s)] = 2 + \frac{ts}{4} - \frac{|t-s|}{2}, \tag{6.34a}$$

$$K_{12}(t,s) = E[z_1(t)z_2(s)] = \cos(\omega s) - \frac{t}{4\omega} \sin(\omega s), \tag{6.34b}$$

$$K_2(t,s) = E[z_2(t)z_2(s)] = -\frac{1}{2\omega} \sin(\omega|t-s|) + \cos \omega t \cos \omega s + \frac{1}{2\omega^2} \sin \omega t \sin \omega s. \tag{6.34c}$$

From (6.34a), we see that the covariance function of the $x_1(t) = z_1(t) + x_{10}$ process is independent of the frequency ω of the harmonic oscillator, thus demonstrating locality.

It is also worth noting that the components $z_1(t)$ and $z_2(t)$, viewed as isolated processes, are not quantum diffusions. To verify this fact, note that according to Lemma 6.1, z_1 and z_2 are

zero-mean Gaussian reciprocal diffusions. Given such a diffusion with autocorrelation $K(t,s)$, it is shown in Ref. 18 that the matrix $V(t)$ parametrizing the mean velocity in (2.41b), and the stress tensor $\pi(t)$ can be expressed as

$$V(t) = \frac{1}{2} \left(\frac{\partial K}{\partial t}(t^+,t) + \frac{\partial K}{\partial t}(t^-,t) \right) K^{-1}(t,t), \qquad (6.35a)$$

$$\pi(t) = \frac{1}{2} \left(\frac{\partial^2 K}{\partial t\,\partial s}(t^+,t) + \frac{\partial^2 K}{\partial t\,\partial s}(t^-,t) \right) - V(t)K(t,t)V^T(t). \qquad (6.35b)$$

Applying these expressions to the autocorrelation functions K_1 and K_2 in (6.34a) and (6.34c) gives

$$V_1(t) = \frac{t/4}{2+t^2/4}, \qquad \pi_1(t) = \frac{1}{2(2+t^2/4)} = \frac{1}{2} K_{x_1}^{-1}(t), \qquad (6.36a)$$

$$V_2(t)K_2(t,t) = \left(\frac{1}{2\omega} - \omega \right) \sin(\omega t)\cos(\omega t), \qquad (6.36b)$$

$$\pi_2(t) = \left[2 \left(\cos^2(\omega t) + \frac{1}{2\omega^2} \sin^2(\omega t) \right) \right]^{-1} = \frac{1}{2} K_{x_2}^{-1}(t). \qquad (6.36c)$$

Thus, the stress tensors $\pi_i(t)$ with $i=1,2$ are proportional to the inverse covariances $K_{x_i}^{-1}(t)$, but the coefficient of proportionality is $\frac{1}{2}$, instead of $\frac{1}{4}$, as required by the closure rule (3.2b). $\quad\square$

Finally, it is worth noting that the reciprocity property of quantum diffusions can be viewed as locality in time, in the sense that given $x(t-h)$ and $x(t+h)$, the position $x(t)$ is conditionally independent of $x(s)$ for s outside the interval $[t-h,t+h]$, so that, in some sense, the stochastic mechanics of quantum diffusions achieves locality in both space and time.

VII. NEGATIVE PROBABILITIES

For all the quantum processes considered up to this point, such as the Gaussian processes of Sec. IV, the end-point density $q(x,s;y,t)$ of the quantum diffusion associated to the wave function $\psi(x,t)$ was always positive. Unfortunately, this property does not hold when $\rho(x,t) = |\psi(x,t)|^2$ has some nodes. To see this, note that the density $q(x,t;y,t)$ obtained from (3.27) is non-negative but takes zero values whenever $x=y=x_0$, where x_0 denotes an arbitrary node of $\rho(x,t)$. But the heat equation (2.13a) for $q(x,s;y,t)$ with $s<t$ implies that it satisfies the integral equation,

$$q(x,t;y,t) = \int G(z,s;x,t)q(z,s;y,t)dz. \qquad (7.1)$$

In this expression, the Green's function $G(z,s;x,t)$, which is the transition density of a Markov diffusion with creation or killing, is positive for all z and x. Consequently, zero values on the left side of the above identity can only occur if the integrand $q(z,s;y,t)$ takes both negative and positive values. To illustrate this phenomenon, consider the excited states of the harmonic oscillator.

Example 7.1: The wave functions corresponding to the eigenstates of a harmonic oscillator with frequency ω are given by

$$\psi_n(x,t) = \left(\frac{\omega}{\pi} \right)^{1/4} \frac{1}{(2^n n!)^{1/2}} H_n(\omega^{1/2}x)\exp\left[-\frac{\omega x^2}{2} + i\omega\left(n + \frac{1}{2} \right)t \right], \qquad (7.2)$$

where the $H_n(y)s$ are the Hermite polynomials. They satisfy the recursion

$$H_n(y) = 2yH_{n-1}(y) - 2(n-1)H_{n-2}(y), \tag{7.3a}$$

for $n \geq 2$, with

$$H_0(y) = 1, \quad H_1(y) = 2y. \tag{7.3b}$$

Since $\psi_n(x,t)$ depends on t only through its phase, the probability density $\rho_n(x,t) = \rho_n(x) = |\psi_n(x,t)|^2$ of each eigenstate is time invariant. In (7.2), $n=0$ represents the ground state of the harmonic oscillator, and $n \geq 1$ the excited states. Note that the ground state is obtained by setting $x_C(t) = 0$ in the coherent state examined in Example 4.2. The excited states have nodes at the values of x corresponding to zeros of the Hermite polynomials $H_n(\omega^{1/2}x)$. For example, for the first two excited states, $\rho_1(x)$ has a node at $x=0$, and $\rho_2(x)$ has nodes at $x = \pm 1/(2\omega)^{1/2}$. This implies that the function

$$2 \ln \psi_n^*(a,t) = -\omega a^2 + 2 \ln H_n(\omega^{1/2}a) + f_n(t), \tag{7.4a}$$

with

$$f_n(t) = \frac{1}{2} \ln\left(\frac{\omega}{\pi}\right) - \ln(2^2 n!) + i\omega(2n+1), \tag{7.4b}$$

has logarithmic singularities at these nodes. Consequently, the function

$$F_n(z,t) = -\omega z^2 + 2 \ln H_n(\omega^{1/2}z) + f_n(t), \tag{7.5}$$

obtained by analytic continuation of $2 \ln \psi^*(a,t)$ is meromorphic. Its real part,

$$M_n(a,b) = -\omega(a^2 - b^2) + \ln[|H_n(\omega^{1/2}(a+jb))|^2] + f_{M,n}, \tag{7.6a}$$

with

$$f_{M,n} = \frac{1}{2} \ln\left(\frac{\omega}{\pi}\right) - \ln(2^2 n!), \tag{7.6b}$$

has the feature of being time invariant. From (2.26b)–(2.26c), we find that the mean velocity and stress tensor of the quantum diffusions modeling the eigenstates of the harmonic oscillator take the form

$$v_n(a,t) = v_n(a) = 0 \tag{7.7a}$$

$$\pi_n(a,t) = \pi_n(a) = \frac{\omega}{2} - \frac{1}{2}\frac{d^2}{da^2} \ln H_n(\omega^{1/2}a), \tag{7.7b}$$

for values of a that do not correspond to nodes of $\rho_n(a)$. The zero value of the mean velocity just reflects the stationarity of the eigenstates.

Then the expression (2.35) for the end-point density gives

$$q_n(x,t;y,t) = \left(\frac{\omega}{\pi}\right)^{1/2} P_n(x,y)\exp{-\omega xy}, \tag{7.8a}$$

where

$$P_n(x,y) = \frac{1}{2^n n!} \left| H_n\left(\omega^{1/2}\left(\frac{y+x}{2} + i\frac{y-x}{2}\right)\right)\right|^2 \tag{7.8b}$$

is a polynomial of degree $2n$ in x and y. In particular,

$$P_0(x,y) = 1, \quad P_1(x,y) = \omega(x^2 + y^2), \tag{7.9a}$$

$$P_2(x,y) = \tfrac{1}{8}[(4\omega yx - 2)^2 + 4\omega^2(y^2 - x^2)^2]. \tag{7.9b}$$

Note that $q_n(x,t;y,t)$ does not depend on t, and, as expected, $q_1(x,t;y,t)$ has a node at $x = y = 0$ and $q_2(x,t;y,t)$ has nodes at $x = y \pm 1/(2\omega)^{1/2}$.

To solve the backward heat equation (2.13a) for $q_n(x,s;y,t)$, it is convenient to perform the transformation

$$q_n(x,s;y,t) = P_n(x,s;y,t)q_0(x,s;y,t), \tag{7.10}$$

where $q_0(x,s;y,t)$ is the end-point density for the ground state of the harmonic oscillator. It can be evaluated by setting $x_C(t) = p_C(t)$ in the expression obtained for the coherent oscillator of Example 4.2. This gives

$$q_0(x,s;y,t) = D(t-s)\exp\left[-\frac{\omega}{2\cos(\omega(t-s))}(\sin(\omega(t-s)))(x^2 + y^2) + 2xy)\right], \tag{7.11a}$$

with

$$D(t-s) = \left(\frac{\omega}{\pi \cos(\omega(t-s))}\right)^{1/2}. \tag{7.11b}$$

Note that since the initial condition $q_0(x,t;y,t)$ does not depend on t, and the spatial part,

$$\mathbf{H} = \frac{1}{2}\left(\frac{d^2}{dx^2} + (\omega x)^2\right) \tag{7.12}$$

of the heat operator $\mathbf{L}_{x,s}$ is time invariant, $q_0(x,s;y,t)$ depends only on $t-s$. Under the transformation (7.10), the heat equation (2.13a) for q_n is transformed into the equation

$$\left(\frac{\partial}{\partial s} - \frac{1}{2}\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\ln q_0(x,s;y,t)\frac{\partial}{\partial x}\right)P_n(x,s;y,t) = 0, \tag{7.13}$$

for P_n with $s \leq t$. Since

$$\frac{\partial}{\partial x}\ln q_0(x,s;y,t) = \frac{\omega}{\cos(\omega(t-s))}(\sin(\omega(t-s))x + y) \tag{7.14}$$

is linear in x and y, and the initial condition $P_n(x,t;y,t)$ for this equation is a polynomial of degree $2n$, $P_n(x,s;y,t)$ remains a polynomial of degree $2n$ in x and y for all $s \leq t$. Matching the coefficients of like powers of x and y in (7.13), this equation can be transformed into a system of ordinary differential equations for the coefficients of P_n.

For the first excited state of the harmonic oscillator, we obtain

$$P_1(x,s;y,t) = \frac{\omega}{\cos^2(\omega(t-s))}\left[(x^2 + y^2) + 2xy\,\sin(\omega(t-s)) - \frac{1}{2}\sin(2\omega(t-s))\right], \tag{7.15}$$

which even for small values of $t-s$ takes negative values in the vicinity of the node $x = y = 0$ of $q_1(x,t;y,t)$.

Finally, since $q_n(x,t;y,t)$ does not depend on t, and the spatial part $\mathbf{H}$ of the heat operator $\mathbf{L}_{x,s}$ is time invariant, the end-point density $q_n(x,s;y,t)$ depends only on $t-s$. This implies that the

quantum diffusions modeling the excited states of the harmonic oscillator are all stationary, since their finite joint densities (2.7) are invariant under time shifts.   □

The above example shows that, given a quantum process whose wave function $\psi(x,t)$ includes nodes, the end-point density $q(x,s;y,t)$ of the corresponding quantum diffusion must necessarily take negative values. Although this feature may appear troubling at first sight, it is not completely inconsistent with standard probability theory, provided that instead of selecting the Borel cylinder sets as the family of events associated to the finite joint densities (2.7), we select a smaller family of events E whose probability $P(E)$ satisfies $0 \leq P(E) \leq 1$. In other words, as was already argued from a physics perspective in Ref. 22, we can employ negative probabilities as an intermediate bookkeeping step, as long as all "observable" events have a positive probability, and provided the operations we perform respect the axioms of probability theory. In this respect, note that the set of "observable" events is highly restricted in quantum mechanics, since such events must concern quantities represented by commuting operators. For example, for the position and momentum processes, only events concerning the position only, or the momentum only, are observable. Since most operators do not commute, the emphasis in quantum mechanics is usually on marginal densities, such as for the position, or the momentum only. By contrast, the goal of stochastic mechanics in either Markovian or non-Markovian form, is to construct finite joint densities for the position process at successive times $t_1 \leq \cdots \leq t_i \leq \cdots \leq t_N$, even if the position operators $\{X(t_i), 1 \leq i \leq N\}$ in the Heisenberg representation do not commute.

The first attempt at accomplishing an objective of this type, at least in a limited way, dates back to the introduction by Wigner[20] of a joint density for the position and momentum of a particle at time t. Specifically, the Wigner distribution, which is defined as

$$W(x,p,t) = \frac{1}{2\pi} \int \psi\left(x+\frac{y}{2},t\right) \psi^*\left(x-\frac{y}{2},t\right) \exp{-ipy} \ dy, \tag{7.16}$$

has the feature that its marginals with respect x and p correspond to the position and momentum probability densities of quantum mechanics. However, an aspect of the Wigner distribution that some researchers find unappealing is that it takes negative values. For example, the Wigner distribution for the eigenstates of the harmonic oscillator is given by[21]

$$W_n(x,p,t) = \frac{(-1)^n}{\pi} L_n\left(\frac{4H}{\omega}\right) \exp\left(\frac{-2H}{\omega}\right), \tag{7.17a}$$

where

$$H(x,p) = \frac{p^2}{2} + \frac{(\omega x)^2}{2} \tag{7.17b}$$

is the harmonic oscillator's Hamiltonian, and $L_n(y)$ is the nth Laguerre polynomial. From the first few Laguerre polynomials,

$$L_0(y) = 1, \quad L_1(y) = 1 - y, \quad L_2(y) = 1 - 2y + y^2, \tag{7.18}$$

we see that while $W(x,p,t)$ is positive for the oscillator's ground state, it takes negative values for the excited states. The fact that the Wigner distribution can be negative inspired efforts to find distributions that would always be positive. However, it was later shown[20] that if, beyond the requirement that the marginals of $W(x,p,t)$ should coincide with the position and momentum densities, a few additional conditions are imposed, $W(x,p,t)$ is unique.

Note that the Wigner distribution is closely related to the end-point density $q(x,t;y,t)$ of quantum diffusions, since the inverse Fourier transform $w(x,z,t)$ of $W(x,p,t)$ with respect to p takes the form

$$w(x,z,t) = \psi(x+z/2,t)\,\psi^*(x-z/2,t), \tag{7.19}$$

which should be contrasted with the expression (3.27) for $q(x,t;y,t)$. Thus, the finite joint densities we have constructed for quantum diffusions can be viewed as extended Wigner distributions for the position variables at arbitrary times. Observe also that in the expression (2.7) for the finite joint densities, the only quantity that can be negative is the end-point density $q(x,s;y,t)$. The marginal densities $\rho(x,t) = q(x,t;x,t)$ are always non-negative, since by construction the density $q(x,t;y,t)$ given by (3.27) is always non-negative, and the three-point transition densities $r(x, s;y,t;z,u)$ with $s<t<u$ are also positive.

VIII. CONCLUSIONS

In this paper, we have presented a comprehensive reformulation of stochastic mechanics, which instead of using Markov diffusions, relies on a subclass of reciprocal diffusions, the quantum diffusions, whose conservation laws are equivalent to Schrödinger's equation. This new form of stochastic mechanics presents several advantages over earlier Markovian theories of Schrödinger or Nelson. First, it is not necessary to introduce a quantum potential in the Newton law satisfied by the diffusions in order to make their evolution consistent with Schrödinger's equation. The new mechanics is local, and the uncertainty principle arises naturally from the closure rules defining quantum diffusions.

In spite of the apparent agreement between quantum mechanics and the stochastic mechanics of quantum diffusions, the two theories have significant differences. Quantum mechanics is less ambitious than stochastic mechanics in the sense that it is primarily concerned with the evolution of marginal densities for the position or momentum variables of a physical system. No attempt is made at evaluating joint probability densities for the positions at different times, since, in general, the Heisenberg operators $X(t)$ and $X(s)$ representing the positions at different times t and s do not commute, so that these positions are not simultaneously observable. On the other hand, stochastic mechanics assigns joint probability densities $p(x_0,t_0;x_1,t_1;...,x_N,t_N)$ to the positions at different times. These joint densities yield marginals for the position and velocity variables at a single time t, which are consistent with the rules of quantum mechanics, in the sense that the conservation laws for $\rho(x,t)$ and $v(x,t)$ are equivalent to Schrödinger's equation. However, the finite joint densities may themselves be devoid of physical significance, as evidenced by the fact that they can be negative. Through Newton's law, stochastic mechanics provides a nice interpretation of the relation existing between quantum and classical mechanics, but it represents a model rather than a physical theory.

The new stochastic mechanics sketched here is incomplete in several respects. First, many important quantum mechanics phenomena, such as interference, scattering, statistics of indistinguishable particles, or measurement theory, need to be given a stochastic formulation within the new theory. It would also be of interest to obtain a variational derivation of the closure rules (3.1a)–(3.1b) similar to the one proposed in Ref. 33 for Markovian stochastic mechanics, but possibly with a different action functional. Finally, as was noted in the discussion of the harmonic oscillator of Example 4.2, we need to examine how quantum diffusions with a finite lifetime can be combined to describe quantum processes over longer periods of time.

ACKNOWLEDGMENT

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