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A Hybrid Computational Approach to Nonlinear Estimation

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Abstract

A general theory of nonlinear observers is developed. It is broad enough to include all existing approaches both deterministic and stochastic. We show that all observers reduce to the solution in a viscosity sense of a partial differential equality of Hamilton-Jacobi-Bellman type. Based on this, we have developed a hybrid algorithm for a nonlinear state observer that utilizes two levels of computation. On the higher level we solve the HJB equality by approximating it by a discrete time and space nonlinear program. At the lower level we level, we initiate local observers that resemble extended Kalman filters at the local minima of the HJB solution. These are computed on a much faster time scale than the solution of the HJB equality. One also computes a measure of how well each local observer explains the observations. The current estimate is the local observer that best explains the observations to date.

We will present numerical results of the hybrid method with the inclusion of a forgetting factor to speed up the convergence. The algorithm has been tested for the the low-dimensional systems. The performance of the hybrid estimator is contrasted with that of an extended Kalman filter. We have found that given a badly chosen initial condition of a nonlinear system, an extended Kalman filter can get trapped in a region far from the true value while the hybrid approach achieves an accurate estimate.

1 A General Framework for Observers

For simplicity we restrict our attention to systems with observations but without controls,

$$\dot{x} = f(x) \tag{1}$$
$$y = h(x). \tag{2}$$

An observer is a causal mapping

$$\begin{bmatrix} \hat{x}^0 \\ y(s) \end{bmatrix} \longmapsto \hat{x}(t), \qquad 0 \le s \le t$$

from the initial state estimate $\hat{x}^0$ and the past observations, $y(s), 0 \le s \le t$, to the current state estimate, $\hat{x}(t)$ satisfying the following conditions.

(a) The estimate $\hat{x}(t)$ as a function of t is continuous from the left and limits from the right exist.

(b) There exists a function $\beta(r,t)$ of class K-L such that

$$|x(t) - \hat{x}(t)| \le \beta(|x(s) - \hat{x}(s)|, t - s).$$

(Recall a function $\beta(r,t)$ is of class K-L if for fixed t, it is of class K, i.e., continuous, strictly increasing and $\beta(0,t) = 0$, and for fixed r it goes to 0 as t goes to ∞.)

Using arguments similar to those of [3], one can show that condition (b) is equivalent to the two conditions (c,d).

(c) There exists a class K_∞ funtion $\delta(\epsilon)$ such that $|x(t) - \hat{x}(t)| < \epsilon$ whenever $|x(s) - \hat{x}(s)| < \delta(\epsilon)$ and $0 \le s \le t$. (Recall a function $\delta(\epsilon)$ is of class K_∞ if it is of class K and goes to ∞ as ϵ goes to ∞.)

(d) For any $r, \epsilon > 0$ there exists $T(r,\epsilon)$ such that $|x(t) - \hat{x}(t)| < \epsilon$ whenever $|x(s) - \hat{x}(s)| < r$ and $t - s > T$.

Either (b) or (c) implies (e).

(d) If $x(s) = \hat{x}(s)$ then $x(t) = \hat{x}(t)$ for all $s \le t$.

The above defined observer is global in $x, \hat{x}$ and is uniform in $x, \hat{x}, s, t$. One could consider local and/or nonuniform observers but we shall not do so.

Theorem If there exists an observer for (1, 2) then there exists a function $Q(x,t)$ with the following properties

(i) $Q(x,t)$ depends causally on $\hat{x}^0$ and $y(s), 0 \le s \le t$,

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(ii) $0 \le Q(x,t) \le \frac{1}{2}|x - \hat{x}(t)|^2$,

(iii) $Q(x,t) = 0$ iff $x = \hat{x}(t)$,

(iv) Along any trajectory $x(t)$ consistent with the observations $y(t) = h(x(t))$, $Q(x(t),t)$ is monotone decreasing to 0,

(v) $Q(x,t)$ is a viscosity subsolution of the partial differential equation

$$Q_t + Q_x f + \frac{1}{2} Q_x Q_x' - \frac{1}{2}|y - h|^2 = 0. \tag{3}$$

(vi) $Q(x,t)$ is a viscosity supersolution of (3) where $Q(x,t) < \frac{1}{2}|x - \hat{x}(t)|$

Proof. Given the observer, define $Q(x,t)$ as follows

$$Q(x,t) = \inf \left\{ \frac{1}{2} \int_s^t \left| \begin{matrix} w(\tau) \\ v(\tau) \end{matrix} \right|^2 d\tau + \frac{1}{2} |\hat{x}(s) - z(s)|^2 \right\} \tag{4}$$

where the infimum is over all $0 \le s \le t$ and all triples $w(\tau)$, $v(\tau)$, $z(\tau)$ satisfying

$$\dot{z}(\tau) = f(z(\tau)) + w(\tau) \tag{5}$$

$$y(\tau) = h(z(\tau)) + v(\tau) \tag{6}$$

$$z(t) = x \tag{7}$$

where $y(\tau)$ is the output from (1, 2) and $\hat{x}(s)$ is the estimate from the observer.

(i) Clearly $Q(x,t)$ depends causally on $\hat{x}(t)$ which is a causal function of $\hat{x}^0$ and $y(s), 0 \le s \le t$.

(ii) By definition $Q(x,t)$ is nonnegative. By setting $s = t$ in (4) we obtain the other inequality.

(iii) By setting $s = t$ in (4), we obtain $Q(\hat{x}(t),t) = 0$. If $Q(x,t) = 0$ then there exists a sequence s_k, $z_k(\tau)$, $w_k(\tau)$, $v_k(\tau)$ such that (5, 6, 7) hold and such that

$$\frac{1}{2} \int_{s_k}^t \left| \begin{matrix} w_k(\tau) \\ v_k(\tau) \end{matrix} \right|^2 d\tau + \frac{1}{2} |\hat{x}(s_k) - z_k(s_k)|^2 \tag{8}$$

goes to 0 as $k \to \infty$. By passing to a subsequence we can assume s_k converges to some s_0, $0 \le s_0 \le t$. For each k, extend $w_k(\tau)$ to be 0 for $0 \le \tau < s_k$ and extend $z_k(\tau)$ by (5). Since $w_k(\tau)$ converges to 0 in $L_2[0,t]$, $z_k(\tau)$ converges uniformly to the solution $x(\tau)$ of (5,6) with $w = 0$ and $y(\tau) = h(x(\tau))$ for $\tau \in (s_0, t]$. It follows that the infimum of $|x(\tau) - \hat{x}(\tau)|$ for $\tau \in (s_0, t]$ must be zero and so (b) implies that $x = x(t) = \hat{x}(t)$.

(iv) If $t_1 < t_2$ then we wish to show $Q(x(t_1), t_1) \le Q(x(t_2), t_2)$. Let $z(\tau)$, $w(\tau)$, $v(\tau)$ be any admissible trajectory for defining $Q(x(t_1), t_1)$ by (4). Extend to a admissible trajectory for defining $Q(x(t_2), t_2)$ by setting $z(\tau) = x(\tau)$, $w(\tau) = 0$, $v(\tau) = 0$ for $\tau \in [t_1, t_2]$. The value of the bracketed expression in (4) has not been changed by the extension hence $Q(x(t_1), t_1) \le Q(x(t_2), t_2)$. It follows from property (b) and (ii) that $Q(x(t), t)$ goes to 0.

(v) To show that $Q(x,t)$ is a viscosity subsolution of the partial differential equation (3), we must show that if $\Phi(x,t)$ is any smooth test function such that $Q - \Phi$ has a relative maximum at x, t then

$$\Phi_t + \Phi_x f + \frac{1}{2} \Phi_x \Phi_x' - \frac{1}{2}|y - h|^2 \le 0 \tag{9}$$

at x, t. Suppose $Q - \Phi$ has a relative maximum at x, t but (9) fails. Then there exists an $\epsilon > 0$ such that

$$\Phi_t + \Phi_x f + \frac{1}{2} \Phi_x \Phi_x' - \frac{1}{2}|y - h|^2 > \epsilon > 0 \tag{10}$$

at all z, s sufficiently close to x, t. For any $w(\tau)$ define $z(\tau)$ as the backward solution of (5, 6) and define $v(\tau)$ by (6). Choose δ sufficiently small so that (10) holds on $[t - \delta, t]$. Then

$$Q(x,t) - \Phi(x,t) \ge Q(z(t-\delta), t-\delta) - \Phi(z(t-\delta), t-\delta) \tag{11}$$

or

$$\begin{aligned} Q(x,t) &- Q(z(t-\delta), t-\delta) \\ &\ge \Phi(x,t) - \Phi(z(t-\delta), t-\delta) \\ &= \int_{t-\delta}^t \Phi_t + \Phi_x (f + w) \, d\tau \end{aligned} \tag{12}$$

From the definition, (4), of Q,

$$Q(x,t) \le Q(z(t-\delta), t-\delta) + \frac{1}{2} \int_{t-\delta}^t \left| \begin{matrix} w(\tau) \\ v(\tau) \end{matrix} \right|^2 d\tau \tag{13}$$

Putting these together we obtain

$$\frac{1}{2} \int_{t-\delta}^t \left| \begin{matrix} w(\tau) \\ v(\tau) \end{matrix} \right|^2 d\tau \ge \int_{t-\delta}^t \Phi_t + \Phi_x (f + w) \, d\tau \tag{14}$$

or

$$0 \ge \int_{t-\delta}^t \Phi_t + \Phi_x (f + w) - \frac{1}{2}|w|^2 - \frac{1}{2}|y - h|^2 \, d\tau \tag{15}$$

We let $w = \Phi_x'$ to obtain

$$0 \ge \int_{t-\delta}^t \Phi_t + \Phi_x f + \frac{1}{2} \Phi_x \Phi_x' - \frac{1}{2}|y - h|^2 \, d\tau > \epsilon\delta > 0 \tag{16}$$

which is a contradiction.

(vi) To show that $Q(x,t)$ is a viscosity supersolution of the partial differential equation (3) wherever $Q(x,t) < \frac{1}{2}|x - \hat{x}(t)|$, we must show that if $\Phi(x,t)$ is any smooth test function such that $Q - \Phi$ has a relative minimum at x, t then

$$\Phi_t + \Phi_x f + \frac{1}{2} \Phi_x \Phi_x' - \frac{1}{2}|y - h|^2 \ge 0 \tag{17}$$

w.r.t. By assumption

$$Q(x,t) - \Phi(x,t) \le Q(z,s) - \Phi(z,s) \quad (18)$$

for all z, s sufficiently close to x, t. If (17) is not satisfied then there exists an $\epsilon > 0$ such that

$$\Phi_t + \Phi_x f + \frac{1}{2}\Phi_x \Phi_x' - \frac{1}{2}|y-h|^2 < -\epsilon < 0 \quad (19)$$

for all z, s sufficiently close to x, t. For any $w(\tau)$ choose δ sufficiently small so that (18, 19) hold on $[t-\delta, t]$. Then

$$\begin{aligned} Q(x,t) - Q(z(t-\delta), t-\delta) - \frac{1}{2}\int_{t-\delta}^{t} |w|^2 + |y-h|^2\, d\tau \\ \le \int_{t-\delta}^{t} \Phi_t + \Phi_x(f+w) \ - \frac{1}{2}|w|^2 - \frac{1}{2}|y-h|^2\, d\tau \\ \le \int_{t-\delta}^{t} \Phi_t + \Phi_x(f+w) \ + \frac{1}{2}\Phi_x\Phi_x' - \frac{1}{2}|y-h|^2\, d\tau \\ < -\epsilon\delta \end{aligned} \quad (20)$$

Since $Q(x,t)$ is defined by (4) and $Q(x,t) < \frac{1}{2}|x - \hat{x}(t)|$, we can choose $s < t$, $w(\tau)$, $v(\tau)$, $z(\tau)$ satisfying (5,6,7) such that

$$\begin{aligned} Q(x,t) &\ge \frac{1}{2}\int_s^t \left| \begin{matrix} w(\tau) \\ v(\tau) \end{matrix} \right|^2 d\tau + \frac{1}{2}|\hat{x}(s) - z(s)|^2 - \frac{\epsilon\delta}{2} \\ Q(x,t) &\ge Q(z(t-\delta), t-\delta) + \frac{1}{2}\int_{t-\delta}^{t} \left| \begin{matrix} w(\tau) \\ v(\tau) \end{matrix} \right|^2 d\tau - \frac{\epsilon\delta}{2} \end{aligned} \quad (21)$$

which leads to the contradiction $-\epsilon\delta > -\frac{\epsilon\delta}{2}$. **QED**

2 Hybrid Estimation

Based on this, we have developed a hybrid algorithm for a nonlinear state observer that utilizes two levels of computation. On the higher level we approximately compute a function $Q(x,t)$ similar to the above. At the lower level, we initiate local observers that resemble extended Kalman filters at the local minima of $Q(x,t)$. These are computed on a much faster time scale. We also compute how well they explain the observations. We take as the estimate, the one that best explains the observations to date.

Since the computation of the Q function is expensive, it is done on a relatively coarse spatial and temporal grid. Hence the minimum of Q converges slowly to the true state and is never very accurate due to the coarseness of the grid. The local observers are computationally inexpensive especially since the filter gains are derived from Q rather than solutions of Riccati equations. Moreover when initialized close to the true value, they converge quickly and accurately. However if they are initialized far from the true value, they don't always converge to it. The coarse information in Q allows us to initialize the local observers properly.

Mortensen [4] and Hijab [1] introduced the concept of minimum energy estimation. Given an initial state estimate $\hat{x}^0$, an observation history $\{y(s) : 0 \le s \le t\}$ and an endpoint x one seeks the minimum "energy" triple x^0, $w(s)$, $v(s)$ satisfying

$$\begin{aligned} \dot{x}(s) &= f(x(s)) + w(s) && (22)\\ y(s) &= h(x(s)) + v(s) && (23)\\ x(0) &= x^0 && (24)\\ x(t) &= x. && (25) \end{aligned}$$

The "energy" of the triple x^0, $w(s)$, $v(s)$ is defined as

$$\frac{1}{2}\int_0^t e^{-\alpha(t-s)} \left| \begin{matrix} w(s) \\ v(s) \end{matrix} \right|^2 ds + \frac{e^{-\alpha t}}{2}\left|\hat{x}^0 - x^0\right|^2. \quad (26)$$

Increasing the forgetting factor α decreases the importance of the initial state estimate and earlier observations and increases the importance of the later observations. The value $Q(x,t)$ is a measure of the liklihood that $x(t) = x$ given the initial state estimate and the observations to date. The smaller $Q(x,t)$ is, the more likly $x(t) = x$. Let $Q(x,t)$ denote the infimum of (26) over all triples satisfying (22-25), then the minimum energy estimate is

$$\hat{x}(t) = \text{argmin} Q(x,t). \quad (27)$$

By arguments similar to the above, it is not hard to see that Q is a solution in the viscosity sense of the Hamilton Jacobi Bellman (HJB) PDE

$$\alpha Q + Q_t + Q_x f + \frac{1}{2}Q_x Q_x' - \frac{1}{2}|y-h|^2 = 0. \quad (28)$$

Following Kushner and Dupuis [2], we compute Q not by approximately solving the HJB PDE but rather by solving an approximating nonlinear program. Let r, k be relatively coarse spatial and temporal steps. Choose a subdomain of $\mathbf{R}^n$ where the state is known to be and consider the rectangular lattice of points in the subdomain with spacing r. Following the dynamic program approach (in forward time), we define the approximate $Q(x,t)$ at lattice points x and time steps t by

$$\begin{aligned} Q(x,t+k) &= \inf\{(1-\alpha k)Q(z,t) && (29)\\ &\quad + \left|\frac{x-z}{k} - \frac{f(x,t+k) + f(z,t)}{2}\right|^2 \frac{k}{2} \\ &\quad + \left|y(t) - \frac{h(x,t) + h(z,t)}{2}\right|^2 \frac{k}{2}\Big\} \\ Q(x,0) &= \frac{1}{2}\left|x - \hat{x}^0\right|^2 && (30) \end{aligned}$$

where the infimum is over z in one of the following
a) the whole lattice in the subdomain

b) the $2n$ nearest neighbors $\{z^i\}$ of x in this subdomain
c) the r ball around x in the l_1 norm.
Since Q is only defined on the spatial lattice with stepsize r, in case (c), one makes the following approximations for $|z-x|_1 \le r$..

$$
\begin{aligned}
Q(z,t) &= \sum_1^n |\lambda_i| Q(sign(\lambda_i)z^i,t) \\
f(z,t) &= \sum_1^n |\lambda_i| f(sign(\lambda_i)z^i,t) \\
h(z,t) &= \sum_1^n |\lambda_i| h(sign(\lambda_i)z^i,t) \\
z &= \sum_1^n \lambda_i z^i \\
1 &= \sum_1^n |\lambda_i|
\end{aligned}
$$

where $\{z^i : i = 1, \dots, n\}$ are the n nearest neighbors in the nonegative orthant based at x.

Notice that the computation of Q must be done in real time because of the presence of $y(t)$ and complexity of the computation is inversly proportional to the spatial step r to the power of the state dimension n. Hence there is a tradeoff between accuracy (small r) and computational ease (large r). Of course similar difficulties arise in all nonlinear estimation algorithms, for example, nonlinear filtering requires solving the Zakai stochastic PDE in real time.

The extended Kalman Filtering is an alternative approach which can be very accurate when it converges. However it may fail to converge if the problem is highly nonlinear. If we assume that the w, v in (22,23) are independent standard white Gaussian noises and the initial state estimate is an independent Gaussian random vector with mean $\hat{x}^0$ and covariance P^0 then the extended Kalman Filter (EKF) takes the form

$$
\begin{aligned}
\dot{\hat{x}} &= f(\hat{x},t) + Ph'_x(\hat{x},t)(y - h(\hat{x},t)) && (31) \\
\dot{P} &= f_x(\hat{x},t)P + Pf_x(\hat{x},t)' + I \\
&\quad -Ph'_x(\hat{x},t)h_x(\hat{x},t)P && (32) \\
\hat{x}(0) &= \hat{x}^0 && (33) \\
P(0) &= P^0 && (34)
\end{aligned}
$$

An example of a highly nonlinear problem where an EKF may fail to converge is

$$
\begin{aligned}
\dot{x} &= x(1-x^2) && (35) \\
y &= x^2 + \epsilon x. && (36)
\end{aligned}
$$

If $\epsilon = 0$ the states x, $-x$ are indistinguishable but for nonzero ϵ the system is observable. The dynamics has stable equilibria at $x = \pm 1$ and an unstable equilibrium at $x = 0$. If $\epsilon > 0$, the system is initialized near -1 and the EKF is initialized near 1, the EKF will fail to converge to the true value.

Suppose $Q(x,t)$ is a smooth solution to HJB PDE (28), $\hat{x}(t)$ is a relative minimum of $Q(x,t)$ and $q(t) = Q(\hat{x}(t),t)$ then

$$
\begin{aligned}
0 &= \frac{\partial Q}{\partial x}(\hat{x}(t),t) && (37) \\
0 &= Q_{xx}(\hat{x}(t),t)\dot{\hat{x}}(t) + Q_{xt}(\hat{x}(t),t). && (38)
\end{aligned}
$$

If we partially differentiate (28) with respect to x and evaluate at $\hat{x}(t)$, t we obtain

$$
\begin{aligned}
0 &= Q_{tx}(\hat{x}(t),t) + Q_{xx}(\hat{x}(t),t)f \\
&\quad + h'_x(\hat{x}(t),t)(y - h(\hat{x}(t),t). && (39)
\end{aligned}
$$

From the last two equations we obtain

$$\dot{\hat{x}} = f(\hat{x},t) + Q_{xx}^{-1}(\hat{x},t)h'_x(\hat{x},t)(y - h(\hat{x},t)) \quad (40)$$

and evaluating (28) at $\hat{x}(t)$, t yields

$$\dot{q} = -\alpha\, q + \frac{1}{2}|y - h(\hat{x},t)|. \quad (41)$$

These are the equations of a local observer based on Q. Notice the similarity of (40) to (31) of an EKF.

The hybid approach is as follows.
1) Compute $Q(x,t)$ by a nonlinear programming approximation (29) on a coarse spatial and temporal grid,
2) At each relative minimum of Q(x,t), initialize a local observer $\hat{x}(t), q(t)$
3) Let the various local observers $\hat{x}(t), q(t)$ evolve according to (40, 41) on a fast time scale,
4) Eliminate redundant local observers when they come close together,
5) Choose as the current estimate, the $\hat{x}(t)$ of the local observer with smallest $q(t)$.

While this algorithm can result in a large number of local observers, the computational burden associated with each one is quite small, less than an EKF. Each local observer invloves integrating $n+1$ diffferential equations instead of $(n^2+3n)/2$ for an EKF. The total computational burden associated with computing $Q(x,t)$ on a coarse spatial and temporal grid and computing many local observers on a fine temporal scale is considerably lighter than computing $Q(x,t)$ on a fine spatial and temporal grid. Moreover the accuracy of the solution of the HJB PDE (28) is limited by the fineness of the spatial grid while machine precision is the limit on the spatial accuracy of a local observer.

We apply the hybrid method to the one-dimensional example (35,36) with where $\epsilon = 0.5$. The forgetting factor is $\alpha = 0.05$. The spatial domain is $[-3,3]$, the spatial step is $r = 0.4$ and the temporal step is $k = 1$.

The time step of the local observers and the extended Kalman filter is 0.1. The initial state is −0.0009 and the EKF and hybrid filter (HF) are initiated at +1. The EKF gets trapped near +1 while the HF converges to the true state. The estimation errors are shown in fig. 1.

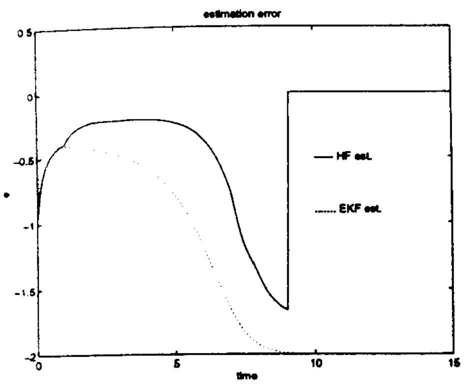

Figure 1:

The second test case is a two dimensional nonlinear system described by

$$
\begin{aligned}
\dot{x}_1 &= -4x_1 - x_2 + 5(x_1^2 + x_2^2)x_1 - (x_1^2 + x_2^2)^2 x_1 \\
\dot{x}_2 &= x_1 - 4x_2 + 5(x_1^2 + x_2^2)x_2 - (x_1^2 + x_2^2)^2 x_2 \\
y &= x_1.
\end{aligned}
$$

This is a system with a stable origin and two limit cycles. The limit cycle of radius 1 is unstable and the limit cycle of radius 2 is stable. Again the forgetting factor is $\alpha = 0.05$. The spatial domain is $[-3,3] \times [-3,3]$, the spatial step is $r = 0.4$ and the temporal step is $k = 1$. The time step of the local observers and the extended Kalman filter is again 0.1. The initial state is $x_1^0 = x_2^0 = 1.4$, so the state is attracted to the stable limit cycle of radius 2. The EKF and hybrid filter (HF) are both initiated at $\hat{x}_1^0 = \hat{x}_2^0 = 0.5$. The EKF gets trapped near the origin while the HF converges to the true state. The estimation errors are shown in fig. 2.

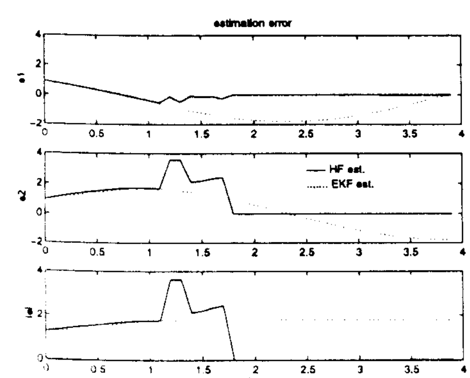

Figure 2:

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