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Reciprocal diffusions in flat space

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Summary. We describe the theory of reciprocal diffusions in flat space. A reciprocal process is a Markov random field on a one dimensional parameter space. Every Markov process is reciprocal but not vice versa. We descibe the first and second order mean differential characteristics of reciprocal diffusions. This includes a new definition of stochastic acceleration. We show that reciprocal diffusions satisfy stochastic differential equations of second order. Associated to a reciprocal diffusion is a sequence of conservation laws, the first two of which are the familiar continuity and Euler equations. There are two cases where these laws can be closed after the first two. They are the mutually exclusive subclasses of Markov and quantum diffusions. The latter corresponds to solutions of the Schrödinger equation and may be part of a stochastic description of quantum mechanics.

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1 Introduction

In 1931, E. Schrödinger [18] described some similarities between the partial differential equation of quantum mechanics that bears his name, and a partial differential equation associated to a Markov process that is pinned at both endpoints. The following year, S. Bernstein [1] formalized some of Schrödinger's ideas by introducing the concept of a reciprocal process. In current terminology, a reciprocal process is a Markov random field on a one dimensional parameter space. Apparently P. Levy was unaware of Bernstein's work when he defined the Markov property for random fields [15], otherwise he might have called them reciprocal fields.

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Let $\xi(t)$ be a n vector valued stochastic process depending on the one dimensional parameter $t \in [t_0, t_f]$. We refer to t as time although it could be a spatial parameter. Such a process is *reciprocal* if conditioned on its values $\xi(t_1), \xi(t_2)$ at the end points of any subinterval $[t_1, t_2] \subseteq [t_0, t_f]$, the process on the interior of the subinterval is conditionally independent of the process on the exterior of the subinterval.

Reciprocal processes have also been called quasi-Markov [3] or Bernstein [22] processes. They are closely related to conditionally Markov processes. The goal of Schrödinger to develop a stochastic theory of quantum mechanics was the inspiration for stochastic mechanics, as developed by Nelson [16] and others. For the most part this effort has focused on Markov processes. The Markov processes form a proper subclass of the reciprocal processes. We believe, for reasons that are discussed in [14] and in Sect. 7 of this paper, that a satisfactory stochastic theory of quantum mechanics cannot be based on Markov processes. If such a theory is possible within the framework of reciprocal processes, it will involve a different and completely disjoint subclass of the reciprocal processes. We have termed this subclass the quantum processes. For similar reasons Zambrini has developed the theory of Euclidean quantum mechanics [23]. We refer the reader to the above citations and [9] for further results on reciprocal processes.

In [12] it is shown that continuous Gaussian reciprocal process can be realized as the solution of a linear stochastic differential equation of second order satisfying boundary conditions. In this paper we extend these results to nonGaussian processes and nonlinear stochastic differential equations of second order. For simplicity, we restrict our attention to processes evolving in flat space. The scale of the noise imposes a Riemannian or subRiemannian metric on the space. In flat space the noise is invariant under translations and rotations. This paper is based on early work in [11]. A path integral description of similar results can be found in [13].

It follows immediately from the definitions that every Markov process is reciprocal and every reciprocal process is conditionally Markov in the following sense. Let $\xi(t)$ be a reciprocal process on $[t_0, t_f]$ and let $s \in [t_0, t_f]$. Let $\xi(t|x, s)$ be the conditioned subprocess that satisfies $\xi(s) = x$ with the conditional probability measure. Then $\xi(t|x, s)$ is Markov on the subinterval $[t_0, s]$ and is also Markov on $[s, t_f]$ It need not be Markov on $[t_0, t_f]$.

The law of a Markov process $\xi(t)$ is determined by its initial distribution and its two time (forward) Markov transition distribution. We shall assume throughout that densities exist. Let $\rho(x_0, t_0; \ldots; x_k, t_k)$ denote the joint density of $\xi(t_i) = x_i,\ i = 0, \ldots, k$ and let $p(x, s; y, t)$ be the probability density of $\xi(t) = y$ given that $\xi(s) = x$ for $t_0 \leq s \leq t \leq t_f$. The Markov property implies that for $t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_k \leq t_f$,

$$\rho(x_0, t_0; \ldots; x_k, t_k) = \rho(x_0, t_0) p(x_0, t_0; x_1, t_1) \ldots p(x_{k-1}, t_{k-1}; x_k, t_k). \tag{1.1}$$

The Markov property is invariant under time reversal. The law of a Markov process is also determined by its final density $\rho(x_f, t_f)$ and its backward Markov transition density, $\bar{p}(x, s; y, t)$. This is the probability density of $\xi(s) = x$ given that $\xi(t) = y$ for $t_0 \leq s \leq t \leq t_f$. Unless stated otherwise, a Markov transition density is assumed to be a forward Markov transition density.

A function $p(x,s;t,y)$ is a Markov transition density if for fixed s,t it is Borel measurable in x and a probability density in y and, in addition, it satisfies the familiar Chapman-Kolmogorov equation

$$p(x,s;t,y) = \int p(x,s;\xi,\tau)p(\xi,\tau;y,t)d\xi \tag{1.2}$$

for $t_0 \leq s \leq \tau \leq t \leq t_f$.

Two Markov processes are in the same *forward Markov class* if they have the same forward Markov transition density. The Markov class of a process is not invariant under time reversal, two Markov processes with the same forward transition density need not have the same backward transition density.

The law of a reciprocal process $\xi(t)$ is determined by its joint density at the end times $\rho(x_0,0;x_f,t_f)$ and its *three time reciprocal transition density* $q(x,s;y,t;z,u)$. This is the conditional density of $\xi(t) = y$ given that $\xi(s) = x$, $\xi(u) = z$ where $t_0 \leq s \leq t \leq u \leq t_f$. The reciprocal property implies that for $t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_k \leq t_f$

$$\begin{aligned}\rho(x_1,t_1;\ldots;x_k,t_k) &= q(x_1,t_1;x_2,t_2;x_k,t_k)\ldots q(x_{k-2},t_{k-2};x_{k-1},t_{k-1};x_k,t_k) \\ &\quad \times\rho(x_1,t_1;x_k,t_k).\end{aligned} \tag{1.3}$$

Notice how the density is propagated inward through the nested sequence of subintervals

$$[t_1,t_k] \supseteq [t_2,t_k] \supseteq \ldots \supseteq [t_{k-2},t_k]$$

obtained by moving one endpoint, in this case the left one, inward at each step. This respects the reciprocal property. There are other ways of propagating the density inward, while respecting the reciprocal property. For example, we could move the right endpoint inward

$$\begin{aligned}\rho(x_1,t_1;\ldots;x_k,t_k) &= \rho(x_1,t_1;x_k,t_k)\times \\ &\quad q(x_1,t_1;x_{k-1},t_{k-1};x_k,t_k)\ldots q(x_1,t_1;x_2,t_2;x_3,t_3),\end{aligned} \tag{1.4}$$

or alternately move the left and right endpoints inward

$$\begin{aligned}\rho(x_1,t_1;\ldots;x_k,t_k) &= \rho(x_1,t_1;x_k,t_k)\times \\ &\quad q(x_1,t_1;x_{k-1},t_{k-1};x_k,t_k)q(x_1,t_1;x_2,t_2;x_{k-1},t_{k-1})\ldots\end{aligned} \tag{1.5}$$

For q to be a reciprocal transition density, all must yield the same value. Reciprocal densities can also be propagated outward but we shall not do so in this paper.

Jamison [9] has shown that for a function $q(x,s;y,t;z,u)$ to be a reciprocal transition density, it must be a density in y and satisfy for $t_0 \leq t_1 \leq t_2 \leq t_3 \leq t_4 \leq t_f$

$$q(x_1,t_1;x_2,t_2;x_4,t_4)q(x_2,t_2;x_3,t_3;x_4,t_4)$$
$$= q(x_1,t_1;x_3,t_3;x_4,t_4)q(x_1,t_1;x_2,t_2;x_3,t_3). \quad (1.6)$$

This ensures that (1.3), (1.4), (1.5) and other inward propagations of the density yield the same joint densities. Notice that (1.6) is the four time transition density of the process, the joint density of $\xi(t_2) = x_2,\ \xi(t_3) = x_3$ given $\xi(t_1) = x_1,\ \xi(t_4) = x_4$ computed in two different ways using Bayes' rule and the reciprocal property. We denote the four time transition density by $q(x_1,t_1;x_2,t_2;x_3,t_3;x_4,t_4)$.

Suppose that $p(x,s;y,t)$ and $\bar{p}(x,s;y,t)$ are the forward and backward two time transition densities of a Markov process. Its three time transition density is given by Bayes' rule,

$$q(x,s;y,t;z,u) = \frac{p(x,s;y,t)p(y,t;z,u)}{p(x,s;z,u)} = \frac{\bar{p}(x,s;y,t)\bar{p}(y,t;z,u)}{\bar{p}(x,s;z,u)}. \quad (1.7)$$

This is a reciprocal transition density since Markov processes are reciprocal. Similar expressions exist for other multi-time transition densities, e.g., the four time transition density is

$$q(x_1,t_1;x_2,t_2;x_3,t_3;x_4,t_4) = \frac{p(x_1,t_1;x_2,t_2)p(x_2,t_2;x_3,t_3)p(x_3,t_3;x_4,t_4)}{p(x_1,t_1;x_4,t_4)}, \quad (1.8)$$

and so (1.6) holds trivially.

Two reciprocal processes (in particular, Markov processes) are said to be in the same *reciprocal class* if they have the same reciprocal transition density. This concept is invariant under time reversal.

Jamison [9] has shown that all reciprocal transition densities arise from Markov transition densities. If $q(x,s;y,t;z,u)$ is a reciprocal transition density then, by the conditionally Markov property of reciprocal processes,

$$p(x,s;y,t) = q(x,s;y,t;x_f,t_f) \quad (1.9)$$

is a Markov transition density for each x_f, t_f and all $s \le t < t_f$. If we start with a reciprocal transition density, define a Markov transition density by (1.9) and another reciprocal transition density by (1.7) then the two reciprocal transition densities agree on $[t_0, t_f)$. If we start with a Markov transition density, define a reciprocal transition density by (1.7) and another Markov transition density by (1.9) then the two Markov transition densities are not necessarily equal but they are in the same reciprocal class.

Any reciprocal process with transition density q can be constructed in the following fashion. Suppose $\xi(t)$ is a Markov process in the reciprocal class of q such that its endpoint density $\rho(x_0,t_0;x_f,t_f)$ is positive for all $x_0,\ x_f$. Partition $\xi(t)$ into subprocesses $\xi(t|x_0,t_0;x_f,t_f)$ by conditioning the endpoints $\xi(t_0) = x_0,\ \xi(t_f) = x_f$. Under the conditional measure, the subprocesses are Markov processes in the reciprocal class of q. Choose an arbitrary endpoint density $\bar{\rho}(x_0,t_0;x_f,t_f)$ and form the mixture of the subprocesses with this weight. The result is the general reciprocal process in the class of q.

By a *Markov diffusion*, we mean the strong solution $\xi(t)$ of an Ito stochastic differential equation

$$\begin{aligned} d^+\xi^i &= b^i(\xi,t)dt + \sigma^i_j(\xi,t)d^+w^j & (1.10)\\ \xi(t_0) &= \xi_0 & (1.11)\end{aligned}$$

where $w(t)$ is a standard Wiener process, ξ_0 is a random vector independent of $w(t)$ and d^+ denotes that this is a forward Ito equation. We distinguish between Stratonovich and Ito integrals by the use of $dw(s)$ and $d^+w(s)$ respectively. The process $\xi(t)$ is adapted to the increasing filtration F_t generated by ξ_0 and $\{w(s) : 0 \leq s \leq t\}$. The forward and backward difference operators are defined as

$$d^{\pm}\xi^i(t,dt) = \pm[\xi(t \pm dt) - \xi(t)] \qquad \text{where } dt > 0 \qquad (1.12)$$

In a slight abuse of notation, we use the same notation for their differential limits as $dt \downarrow 0$ as in (1.10). Throughout this paper we adopt the summation convention on repeated indices.

The scale of the noise induces a Riemannian (or subRiemannian) metric on ξ space. Let $a^{ij} = \sigma^i_k\sigma^j_k$. If it exists, the inverse a_{ij} defines the metric. The space is *flat* if $a^{ij} = \delta^{ij}$. We only consider diffusions in flat space, i.e., ξ lives in n dimensional standard Euclidean space, $\mathbf{R^n}$, and $\sigma^i_j = \delta^i_j$.

The diffusion is *smooth* if $b(\xi,t)$ is three times continuously differentiable with partials that are globally bounded. The latter assumption is imposed to simplify the exposition, it can be relaxed. Because of the assumptions on b and σ, the associated Markov transition density $p(x,s;y,t)$ is smooth for all $s < t$. This is easy to see as $p(x,s;y,t)$ satisfies the forward partial differential equation

$$\frac{\partial}{\partial t}p(x,s;y,t) + \frac{\partial}{\partial y^i}(p(x,s;y,t)b^i(y,t)) - \frac{1}{2}\frac{\partial^2}{\partial y^i\partial y^i}p(x,s;y,t) = 0. \qquad (1.13)$$

and the backward equation

$$\frac{\partial p}{\partial s}(x,s;y,t) + \frac{\partial p}{\partial x^i}(x,s;y,t)b^i(x,t) + \frac{1}{2}\frac{\partial^2 p}{\partial x^i\partial x^i}(x,s;y,t) = 0. \qquad (1.14)$$

Partial derivatives will be denoted as follows

$$\begin{aligned} b^i_{,0} &= \frac{\partial b^i}{\partial t},\\ b^i_{,j} &= \frac{\partial b^i}{\partial x^j},\\ b^i_{,jk} &= \frac{\partial^2 b^i}{\partial x^j\partial x^k}.\end{aligned}$$

The above definitions can be extended to reciprocal processes using the conditionally Markov property. For example, a reciprocal process $\xi(t)$ is a *smooth reciprocal diffusion in flat space* if there is a smooth Markov diffusion in flat space in its reciprocal class. We shall only consider smooth Markov and reciprocal diffusions in flat space and for brevity we occasionally refer to such processes as Markov and reciprocal diffusions.

2 Differential characteristics

Recall that a solution of the first order stochastic differential equation (1.10) satisfies the diffusion (also known as Feller) postulates

$$E(d^+\xi^i|\xi(t)=x) \;=\; b^i(x,t)dt + O(dt^2), \tag{2.1}$$
$$E(d^+\xi^i\ d^+\xi^j|\xi(t)=x) \;=\; a^{ij}dt + O(dt^2), \tag{2.2}$$
$$E(|d^+\xi|^k\ |\xi(t)=x) \;=\; O(dt^2) \qquad \text{if } k>2. \tag{2.3}$$

These postulates assert that the process is forward differentiable in a conditional mean sense. We call b^i the *forward velocity* and $a^{ij}(=\delta^{ij})$ the *forward diffusion coefficient*. They are the *forward Markov characteristics* of the process and are a complete set of invariants of the forward Markov class as they completely determine the forward Markov transition density $p(x,s;y,t)$ by the forward or backward PDE.

The conditional moments of the backward difference have similar expansions

$$E(d^-\xi^i|\xi(t)=x) \;=\; \bar{b}^i(x,t)dt + O(dt^2), \tag{2.4}$$
$$E(d^-\xi^i\ d^-\xi^j|\xi(t)=x) \;=\; \bar{a}^{ij}dt + O(dt^2), \tag{2.5}$$
$$E(|d^-\xi|^k\ |\xi(t)=x) \;=\; O(dt^2) \qquad \text{if } k>2. \tag{2.6}$$

It is well-known [16] that the *backward velocity* $\bar{b}^i$ and the *backward diffusion coefficient* $\bar{a}^{ij}$ satisfy

$$\bar{b}^i \;=\; b^i - \frac{\partial \ln \rho}{\partial x^i} \tag{2.7}$$
$$\bar{a}^{ij} \;=\; a^{ij} \tag{2.8}$$

assuming that $\ln\rho$ is differentiable. The pair $\bar{b}^i$ and $\bar{a}^{ij}$ are called the *backward Markov characteristics* and completely determine the backward Markov class.

Recall that Nelson [16] defines the current velocity $v(x,t)$ and osmotic velocity $u(x,t)$ of a Markov diffusion process $\xi(t)$ as

$$v^i \;=\; (b^i + \bar{b}^i)/2, \tag{2.9}$$
$$u^i \;=\; (b^i - \bar{b}^i)/2. \tag{2.10}$$

Define the centered evaluation and the centered first and second differences as

$$d^0\xi(t,dt) \;=\; [\xi(t+dt)+\xi(t-dt)]/2, \tag{2.11}$$
$$d^1\xi(t,dt) \;=\; [\xi(t+dt)-\xi(t-dt)]/2, \tag{2.12}$$
$$d^2\xi(t,dt) \;=\; \xi(t+dt)-2\xi(t)+\xi(t-dt), \tag{2.13}$$

then

$$E(d^0\xi^i|\xi(t)=x) \;=\; x + O(dt), \tag{2.14}$$
$$E(d^1\xi^i|\xi(t)=x) \;=\; v^i(x,t)dt + O(dt^2), \tag{2.15}$$
$$E(d^2\xi^i|\xi(t)=x) \;=\; 2u^i(x,t)dt + O(dt^2). \tag{2.16}$$

Note u has the dimension of a velocity, but it appears in the leading term of the conditional mean of the second difference where one would expect a quantity with the dimension of an acceleration. Nelson [16] defines the stochastic acceleration of the process, which he denotes by a^i, as follows

$$E(\frac{d^+\overline{b^i}(\xi(t),t) + d^- b^i(\xi(t),t)}{2}|\xi(t) = x) = a^i(x,t)dt + O(dt^2). \tag{2.17}$$

Note that $a^{i,j}$ is the diffusion coefficient and a^i is Nelson's stochastic acceleration. But none of these quantities, $b^i, \overline{b^i}, v^i, u^i, a^i$, are reciprocal invariants, that is, invariants of the reciprocal class of the process.

By changing the conditioning in a way suggested by the reciprocal property, one obtains a more natural definition of stochastic acceleration in terms of quantities which are reciprocal invariants. A smooth reciprocal diffusion is twice mean differentiable in the following sense. Let dt be a small positive scalar and let x, dx be n vectors.

Theorem 2.1 *Let $\xi(t)$ be a smooth reciprocal diffusion in flat space and let b be any forward Markov velocity in the reciprocal class of $\xi(t)$ then*

$$E(d^2\xi^i|d^0\xi = x, d^1\xi = dx) = f^i(x,t)dt^2 + g^i_j(x,t)dx^j\ dt + dt^2\ O(dt, dx), \tag{2.18}$$

$$E(d^2\xi^i\ d^2\xi^j|d^0\xi = x, d^1\xi = dx) = 2\delta^{ij}dt + dt^2\ O(dt, dx), \tag{2.19}$$

$$E(d^2\xi^i\ d^2\xi^j\ d^2\xi^k|d^0\xi = x, d^1\xi = dx) = dt^2\ O(dt, dx), \tag{2.20}$$

$$E(d^2\xi^i\ d^2\xi^j\ d^2\xi^k\ d^2\xi^l|d^0\xi = x, d^1\xi = dx) = 4(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})dt^2 + dt^2\ O(dt, dx), \tag{2.21}$$

$$E(|d^2\xi|^k\ |d^0\xi = x, d^1\xi = dx) = dt^2\ O(dt, dx) \quad if\ k > 4, \tag{2.22}$$

where f and g are reciprocal invariants given by

$$f^i = b^i_{,0} + b^j_{,i}b^j + b^j_{,ij}/2, \tag{2.23}$$

$$g^i_j = b^i_{,j} - b^j_{,i}. \tag{2.24}$$

The first formula (2.18) asserts that in traveling between $x - dx$ and $x + dx$ over the interval $[t - dt, t + dt]$, the process experiences a mean acceleration $f(x,t)+g(x,t)dx/dt$. The second (2.19) asserts that the process also experiences a very large random acceleration whose variance is $O(1/dt)$. In contrast to Nelson, we define the *stochastic acceleration* to be $f(x,t) + g(x,t)dx/dt$ where $x = (\xi(t+dt)+\xi(t-dt))/2$ is the centered position and $dx/dt = (\xi(t+dt)-\xi(t-dt)/(2dt)$

is the centered velocity for small $dt > 0$. It is a consequence of the flat space assumption that in (2.19) the coefficient of dt is $2\delta^{ij}$ and there is no dt^2 term.

Because the conditioning is equivalent to $\xi(t \pm dt) = x \pm dx$, f and g are determined by $q(x-dx, t-dt; y, t; x+dx, t+dt)$ and hence are reciprocal invariants, they depend only on the reciprocal class of the process. Using other methods, Clark [4] has shown that f, g defined by (2.23,2.24) and the scale of the noise $\sigma_k^i \sigma_k^j$ are a complete set of reciprocal invariants. They completely determine the class of a reciprocal diffusion. Since we are restricting our attention to reciprocal diffusions in flat space, f and g completely determine the reciprocal class. In particular this implies that two forward velocities are in the same reciprocal class iff they define the same f and g via (2.23,2.24).

In terms of the backward velocity, $\bar{b}$ (2.7), the reciprocal invariants are given by

$$f^i = \bar{b}^i_{,0} + \bar{b}^j_{,i} \bar{b}^j - \bar{b}^j_{,ij}/2, \tag{2.25}$$

$$g^i_j = \bar{b}^i_{,j} - \bar{b}^j_{,i}. \tag{2.26}$$

It follows immediately from (2.7) that (2.24) and (2.26) are equivalent. The equivalence of (2.23) and (2.25) follows from the Fokker-Planck equation satisfied by ρ,

$$0 = \frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x^j}(\rho(x,t)b^j(x,t)) - \frac{1}{2}\frac{\partial^2}{\partial x^j \partial x^j}\rho(x,t) \tag{2.27}$$

which implies

$$\begin{aligned} 0 &= \frac{\partial^2}{\partial t \partial x^i}\ln\rho(x,t) + b^j(x,t)\frac{\partial^2}{\partial x^j \partial x^i}\ln\rho(x,t) + b^j_{,i}(x,t)\frac{\partial}{\partial x^j}\ln\rho(x,t) \\ &\quad + b^j_{,ij}(x,t) - \frac{\partial}{\partial x^j}\ln\rho(x,t)\frac{\partial^2}{\partial x^j \partial x^i}\ln\rho(x,t) - \frac{1}{2}\frac{\partial^3}{\partial x^j \partial x^j \partial x^i}\ln\rho(x,t) \end{aligned} \tag{2.28}$$

A smooth reciprocal diffusion is also once mean differentiable in both the forward and the backward senses if the logarithm of its density is differentiable, i.e., if there exists $b, a, \bar{b}, \bar{a}$ satisfying (2.1-2.6). Moreover these quantities satisfy (2.7,2.8). When studying the first order behaviour of a reciprocal diffusion, it more natural to employ the centered first difference and the centered conditioning as it is more compatible with that of Theorem 2.1.

Theorem 2.2 *Let $\xi(t)$ be a smooth reciprocal diffusion in flat space whose density satisfies $\rho(x,t) > 0$ for all $t \in (t_0, t_f)$, $x \in \mathbf{R^n}$. There exist vector fields $b(x,t)$ and $\bar{b}(x,t)$ satisfying (2.1-2.8) with $a^{ij} = \bar{a}^{ij} = \delta^{ij}$. For all $t \in (t_0, t_f)$, $x \in \mathbf{R^n}$, the current velocity $v(x,t)$ (2.9) and a symmetric matrix field $P(x,t)$ satisfy*

$$E(d^1\xi^i | d^0\xi = x) = v^i(x,t)dt + O(dt^2), \tag{2.29}$$

$$E(d^1\xi^i \; d^1\xi^j | d^0\xi = x) = \frac{1}{2}\delta^{ij}\; dt + P^{ij}(x,t)dt^2 + O(dt^3), \tag{2.30}$$

$$E(d^1\xi^i\ d^1\xi^j\ d^1\xi^k|d^0\xi = x) = \frac{1}{2}(v^i(x,t)\delta^{jk} + v^j(x,t)\delta^{ik} + v^k(x,t)\delta^{ij})dt^2 + O(dt^3), \quad (2.31)$$

$$E(d^1\xi^i\ d^1\xi^j\ d^1\xi^k\ d^1\xi^l|d^0\xi = x) = \frac{1}{4}(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})dt^2 + O(dt^3), \quad (2.32)$$

$$E(|d^1\xi|^k\ |d^0\xi = x) = O(dt^3) \qquad \text{if } k > 4. \quad (2.33)$$

Furthermore

$$E(d^2\xi^i|d^0\xi = x) = (f^i(x,t) + g^i_j(x,t)v^j(x,t))dt^2 + O(dt^3), \quad (2.34)$$

$$E(d^2\xi^i\ d^2\xi^j|d^0\xi = x) = 2\delta^{ij}dt + O(dt^3), \quad (2.35)$$

$$E(d^2\xi^i\ d^2\xi^j\ d^2\xi^k|d^0\xi = x) = O(dt^3), \quad (2.36)$$

$$E(d^2\xi^i\ d^2\xi^j\ d^2\xi^k\ d^2\xi^l|d^0\xi = x) = 4(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})dt^2 + O(dt^3), \quad (2.37)$$

$$E(|d^2\xi|^k\ |d^0\xi = x) = O(dt^3) \qquad \text{if } k > 4. \quad (2.38)$$

Note the difference in the conditioning in (2.29) and (2.15). Because of (2.29), the vector field v is also called the *centered velocity*. The matrix field P is called the *momentum flux coefficient* of the process. Neither v nor P is a reciprocal invariant.

Zambrini [22] and Cruziero-Zambrini [5] have considered another definition of stochastic acceleration. If $\xi(t)$ is a diffusion and $\phi(x,t)$ a smooth function then define

$$D\phi(x,t) = E(d^+\phi(\xi(t),t)|\xi(t) = x) \quad (2.39)$$

$$D_*\phi(x,t) = E(d^-\phi(\xi(t),t)|\xi(t) = x) \quad (2.40)$$

Their stochastic acceleration is

$$\frac{1}{2}(DD + D_*D_*)\xi \quad (2.41)$$

The Zambrini-Cruziero acceleration appears in an equation which characterizes the extremals of a stochastic variational problem. Thieullen [20] has shown that the Zambrini-Cruziero stochastic acceleration equals $f + gv$ for Gauss-Markov and 1 dimensional Markov diffusions. This holds for all Markov diffusions as we now show. The Zambrini- Cruziero stochastic acceleration is not a reciprocal invariant because of the presence of v.

From equations (2.1), (2.4) and the forward and backward Ito rules we have

$$
\begin{aligned}
D\xi^i &= b^i(\xi,t) && (2.42)\\
D_*\xi^i &= \bar{b}^i(\xi,t)\xi && (2.43)\\
DD\xi^i &= b^i_{,0}(\xi,t)+b^i_{,j}(\xi,t)b^j(\xi,t)+\frac{1}{2}b^i_{,jj}(\xi,t) && (2.44)\\
D_*D_*\xi^i &= \bar{b}^i_{,0}(\xi,t)+\bar{b}^i_{,j}(\xi,t)\bar{b}^j(\xi,t)-\frac{1}{2}\bar{b}^i_{,jj}(\xi,t) && (2.45)
\end{aligned}
$$

From (2.7) and (2.28) it follows that

$$
\begin{aligned}
D_*D_*\xi^i &= b^i_{,0}(\xi,t)+b^i_{,j}(\xi,t)b^j(\xi,t)+b^j_{,i}(x,t)\frac{\partial}{\partial x^j}\ln\rho(x,t)\\
&\quad +b^j_{,ij}(x,t)-\frac{1}{2}b^i_{,jj}(\xi,t) && (2.46)
\end{aligned}
$$

so

$$
\begin{aligned}
\frac{1}{2}(DD+D_*D_*)\xi^i &= b^i_{,0}(\xi,t)+b^i_{,j}(\xi,t)b^j(\xi,t)\\
&\quad +\frac{1}{2}b^j_{,i}(x,t)\frac{\partial}{\partial x^j}\ln\rho(x,t)+\frac{1}{2}b^j_{,ij}(x,t)\\
&= f^i(\xi,t)+g^i_j(\xi,t)v^j(\xi,t) && (2.47)
\end{aligned}
$$

Both of the above theorems are proven by local short time asymptotic expansions of Markov and reciprocal transition densities originally presented in [11]. The details are a bit tedious and will consume the rest of this section. The substance of the proof of Theorem 2.1 can be seen in the following rough asymptotic analysis.

Suppose $p(x,s;y,t)$ is the Markov transition density of solutions of the Ito equation (1.10) in flat space. For small $|y-x|$ and $t-s>0$

$$
\begin{aligned}
p(x,s;y,t) &\sim (2\pi(t-s))^{-n/2}\\
&\quad\times (1-b^i_{,j}(x,s)(t-s)/2)\\
&\quad\times \exp\left(-\frac{|y-x-b(x,s)(t-s)|^2}{2(t-s)}\right). && (2.48)
\end{aligned}
$$

From (1.7) it follows that the induced reciprocal transition function is approximately

$$
\begin{aligned}
&q(x-dx,t-dt;y,t;x+dx,t+dt)\\
&\quad\sim (\pi dt)^{-n/2}\,\left(1+\left(b^i_{,j}(x-dx,t-dt)-b^i_{,j}(y,t)\right)dt/2\right)\\
&\quad\times \exp\left(-\frac{|y-x+dx-b(x-dx,t-dt)dt|^2}{2dt}-\frac{|x+dx-y-b(y,t)dt|^2}{2dt}\right.\\
&\quad\left.+\frac{|dx-b(x-dx,t-dt)dt|^2}{dt}\right). && (2.49)
\end{aligned}
$$

We expand and obtain

$$
\begin{aligned}
q \;\sim\; & (\pi dt)^{-n/2} \exp(-\frac{|y-x|^2}{dt}) \\
& \times (1-(y^i - x^i)(b^i_{,0}dt + b^i_{,j}(y^j - x^j + dx^j) - b^j_{,i}(dx^j - b^j dt) + b^j_{,ij} dt/2)
\end{aligned} \tag{2.50}
$$

where the left side is evaluated at $x-dx, t-dt; y, t; x+dx, t+dt$ and the right side at x, t. Assuming the neglected terms are $dt^2\ O(dt, dx)$, one obtains (2.18-2.22).

We shall use related Gaussian transition densities as the leading term in our asymptotic expansions. Suppose we consider the linearization of the Ito differential equation (1.10) around some $x \in \mathbf{R^n}$,

$$
d^+\xi^i = (b^i(x,t) + b^i_{,j}(x,t)(\xi^j - x^j))dt + d^+w^i. \tag{2.51}
$$

The Markov transition density $p_x(y, s; z, t)$ of solutions of this equation is Gaussian with mean $\mu_M(y, s; t)$ and covariance $R_M(s; t)$,

$$
\begin{aligned}
\mu^i_M \;=\; & y^i + \left(b^i + b^i_{,j}(y^j - x^j)\right)(t-s) \\
& + \left(b^i_{,0} + b^i_{,j} b^j + \left(b^i_{,0j} + b^i_{,k} b^k_{,j}\right)(y^j - x^j)\right)(t-s)^2/2 \\
& + O(t-s)^3,
\end{aligned} \tag{2.52}
$$

$$
\begin{aligned}
R^{ij}_M \;=\; & \delta^{ij}(t-s) + \left(b^i_{,j} + b^j_{,i}\right)(t-s)^2/2 + c^{ij}(t-s)^3 \\
& + O(t-s)^4,
\end{aligned} \tag{2.53}
$$

where the evaluations of b and its partials are at $x,\ s$. The form of c^{ij} is immaterial to what follows.

The corresponding reciprocal transition density

$$
q_x(y, s; \xi, \tau; z, t) = \frac{p_x(y, s; \xi, \tau) p_x(\xi, \tau; z, t)}{p_x(y, s; z, t)}
$$

is also Gaussian with mean $\mu_r(y, s; \tau; z, t)$ and covariance $R_r(s; \tau; t)$,

$$
\begin{aligned}
\mu^i_r \;=\; & \psi^i(\tau) - \left(b^i_{,j} + b^j_{,i}\right)(z^j - y^j)\frac{(t-\tau)(\tau-s)}{2(t-s)}, \\
& - \left(b^i_{,0} + b^j_{,i} b^j\right)\frac{(t-\tau)(\tau-s)}{2} \\
& + (t-\tau)(\tau-s)O(t-s, z-y),
\end{aligned} \tag{2.54}
$$

$$
R^{ij}_r \;=\; \delta^{ij}\frac{(t-\tau)(\tau-s)}{(t-s)} + (t-\tau)(\tau-s)O\,(t-s)\,, \tag{2.55}
$$

where the evaluations of b and its partials are at x and any time in $[s, t]$. The curve $\psi(\tau)$ is the straight line (the geodesic in flat space)

$$
\psi^i(\tau) = \frac{t-\tau}{t-s} y^i + \frac{\tau-s}{t-s} z^i. \tag{2.56}
$$

Notice that the second order expansions of the mean and covariance of q_x are a bit simpler than those of p_x, another indication of the utility of the reciprocal point of view.

In particular, the mean and covariance of $q_x(x - dx, t - dt; y, t + \sigma dt; x + dx, t + dt)$ are

$$\begin{aligned} \mu_r^i &= x^i + \sigma dx^i \\ &\quad -\frac{1-\sigma^2}{2}\left(\left(b^i_{,0} + b^j_{,i} b^i\right) dt^2 + \left(b^i_{,j} - b^j_{,i}\right) dx^j \; dt\right) \\ &\quad +(1-\sigma^2) dt^2 \; O(dt, dx), \end{aligned} \tag{2.57}$$

$$R_r^{ij} = \frac{1-\sigma^2}{2}\delta^{ij} dt + (1-\sigma^2) O(dt^3), \tag{2.58}$$

where the evaluations of b and its partials are at x, t and any $\sigma \in [-1, 1]$. This proves Theorem 2.1 for Gaussian reciprocal processes, a result found in [12].

Lemma 2.3 *Let $p(y, s; z, t)$ and $p_x(y, s; z, t)$ denote the Markov transition densities of solutions of the Ito differential equation (1.10) and its linearization (2.51) around x. There exists ϵ sufficiently small so that if y and z are within ϵ of x then*

$$\begin{aligned} p(y, s; z, t) &= p_x(y, s; z, t)\,[1 \\ &\quad -b^i_{,ij}\left[(y^j - x^j) + 2(z^j - x^j)\right]\frac{(t-s)}{6} + b^i_{,jj}(z^i - y^i)\frac{(t-s)}{12} \\ &\quad +b^i_{,jk}\frac{(z^i - y^i)}{6} \\ &\quad \times\left[(y^j - x^j)(y^k - x^k) + (y^j - x^j)(z^k - x^k) + (z^j - x^j)(z^k - x^k)\right] \\ &\quad +O(t-s)^2 + (t-s) O(y-x, z-x)^2 + O(y-x, z-x)^4\big] \end{aligned} \tag{2.59}$$

where the evaluations of b and its partials are at x and any $\tau \in [s, t]$.

Proof. The proof is based as a stochastic variation of the parametrix method of E. E. Levi [7] and Varadhan's estimate [21]. For any τ, $s \le \tau \le t$, define

$$\pi(y, s; \tau; z, t) = \int p(y, s; \xi, \tau) p_x(\xi, \tau; z, t) d\xi. \tag{2.60}$$

Clearly

$$\pi(y, s; s; z, t) = p_x(y, s; z, t), \tag{2.61}$$

$$\pi(y, s; t; z, t) = p(y, s; z, t), \tag{2.62}$$

so

$$p(y, s; z, t) = p_x(y, s; z, t) + \int_s^t \frac{\partial \pi}{\partial \tau}(y, s; \tau; z, t) d\tau. \tag{2.63}$$

Now

$$\frac{\partial \pi}{\partial \tau} = \int \frac{\partial p}{\partial \tau}(y,s;\xi,\tau)p_x(\xi,\tau;z,t) + p(y,s;\xi,\tau)\frac{\partial p_x}{\partial \tau}(\xi,\tau;z,t)d\xi, \qquad (2.64)$$

where p satisfies the forward equation

$$\frac{\partial}{\partial \tau}p(y,s;\xi,\tau) + \frac{\partial}{\partial \xi^i}(p(y,s;\xi,\tau)b^i(\xi,\tau)) - \frac{1}{2}\frac{\partial^2}{\partial \xi^i \partial \xi^i}p(y,s;\xi,\tau) = 0 \qquad (2.65)$$

and p_x satisfies the backward equation

$$\frac{\partial p_x}{\partial \tau}(\xi,\tau;z,t) + \frac{\partial p_x}{\partial \xi^i}(\xi,\tau;z,t)(b^i(x,\tau) + b^i_{,k}(x,\tau)(\xi^k - x^k)) + \frac{1}{2}\frac{\partial^2 p_x}{\partial \xi^i \partial \xi^i}(\xi,\tau;z,t) = 0. \qquad (2.66)$$

Integration by parts yields

$$\begin{aligned}\frac{\partial \pi}{\partial \tau} &= \int p(y,s;\xi,\tau)\frac{\partial p_x}{\partial \xi^i}(\xi,\tau;z,t)\beta^i(\xi,\tau;x)\ d\xi \\ &= \int p(y,s;\xi,\tau)p_x(\xi,\tau;z,t)\frac{\partial \ln p_x}{\partial \xi^i}(\xi,\tau;z,t)\beta^i(\xi,\tau;x)\ d\xi \qquad (2.67)\end{aligned}$$

where

$$\beta^i(\xi,\tau;x) = b^i(\xi,\tau) - b^i(x,\tau) - b^i_{,j}(x,\tau)(\xi^j - x^j) \qquad (2.68)$$

and

$$\frac{\partial \ln p_x}{\partial \xi^i}(\xi,\tau;z,t) = \frac{z^i - \xi^i}{t-\tau} + O(t-\tau)^0. \qquad (2.69)$$

Since b is smooth (C^3 with bounded partials)

$$\beta^i(\xi,\tau;x) = \frac{1}{2}b^i_{,jk}(x,\tau)(\xi^j - x^j)(\xi^k - x^k) + O(\xi - x)^3. \qquad (2.70)$$

Next we employ Varadhan's estimate

$$p(y,s;\xi,\tau) = \frac{1}{(2\pi(\tau-s))^{n/2}}\exp -\frac{1}{2}\left(\frac{|\xi - y|^2}{(\tau - s)} + O(\tau - s)^0\right) \qquad (2.71)$$

which holds uniformly on compact subsets of y,s,ξ,τ space [21]. In particular

$$p(y,s;\xi,\tau) = p_x(y,s;\xi,\tau)O(\tau-s)^0. \qquad (2.72)$$

So plugging (2.69,2.70,2.72) into (2.67) and utilizing (2.54,2.55), we obtain

$$\begin{aligned}\frac{\partial \pi}{\partial \tau} &= p_x(y,s;z,t)\int q_x(y,s;\xi,\tau;z,t)O(\tau-s)^0\frac{\partial \ln p_x}{\partial \xi^i}(\xi,\tau;z,t)\beta^i(\xi,\tau;x)\ d\xi \\ &= p_x(y,s;z,t)\left[O(t-s)^0 + \frac{1}{(t-s)}O(y-x,z-x)^2\right], \qquad (2.73)\end{aligned}$$

and from (2.63)

$$p(y,s;z,t) = p_x(y,s;z,t)\left[1+O(t-s)+O(y-x,z-x)\right]. \tag{2.74}$$

We return to (2.67) and plug in (2.69,2.70,2.74) to obtain

$$\begin{aligned}
\frac{\partial\pi}{\partial\tau} &= p_x(y,s;z,t)\int q_x(y,s;\xi,\tau;z,t)\left[1+O(\tau-s)+O(y-x,z-x)^2\right] \\
&\quad\times\left[\frac{z^i-\xi^i}{t-\tau}+O(t-\tau)^0\right]\left[\frac{1}{2}b^i_{,jk}(x,\tau)(\xi^j-x^j)(\xi^k-x^k)+O(\xi-x)^3\right]\, d\xi \\
&= p_x(y,s;z,t)\Big[-b^i_{,ij}(\psi^j(\tau)-x^j)\frac{(\tau-s)}{(t-s)}+b^i_{,jj}(z^i-y^i)\frac{(t-\tau)(\tau-s)}{2(t-s)^2} \\
&\quad+b^i_{,jk}(z^i-y^i)(\psi^j(\tau)-x^j)(\psi^k(\tau)-x^k)\frac{1}{2(t-s)} \\
&\quad+O(t-s)+O(y-x,z-x)^2+\frac{1}{t-s}O(y-x,z-x)^4\Big]
\end{aligned} \tag{2.75}$$

which after integration with respect to τ yields (2.59). QED

Proof of Theorem 2.1. From (1.7)

$$q(x-dx,t-dt;y,t;x+dx,t+dt) = \frac{p(x-dx,t-dt;y,t)p(y,t;x+dx,t+dt)}{p(x-dx,t-dt;x+dx,t+dt)}. \tag{2.76}$$

By the above lemma

$$\begin{aligned}
q &= q_x\Big[1-b^i_{,ij}(x,t)\left((y^j-x^j)dt/2-dx^j dt/6\right)-b^i_{,jj}(x,t)dx^i dt/6 \\
&\quad+b^i_{,jk}(x,t)\left((y^j-x^j)(y^k-x^k)dx^i/3-(y^i-x^i)(y^j-x^j)dx^k/3\right) \\
&\quad+O(dt^2)+dt\ O(dx,y-x)^2+O(dx,y-x)^4\Big]
\end{aligned} \tag{2.77}$$

where q and q_x are evaluated at $x-dx,t-dt;y,t;x+dx,t+dt$. Now q_x is Gaussian with mean and and covariance given by (2.57,2.58) so

$$\begin{aligned}
\int(y^i-x^i)q dy &= -\frac{1}{2}\left(\left(b^i_{,0}+b^j_{,i}b^i+b^j_{,ij}/2\right)dt^2+\left(b^i_{,j}-b^j_{,i}\right)dx^j\ dt\right) \\
&\quad+dt^2\ O(dt,dx)
\end{aligned} \tag{2.78}$$

$$\int(y^i-x^i)(y^j-x^j)q dy = \frac{1}{2}\delta^{ij}dt+dt^2\ O(dt,dx) \tag{2.79}$$

$$\int(y^i-x^i)(y^j-x^j)(y^k-x^k)q dy = dt^2\ O(dt,dx) \tag{2.80}$$

$$\begin{aligned}
\int(y^i-x^i)(y^j-x^j)(y^k-x^k)(y^l-x^l)q dy &= \frac{1}{4}(\delta^{ij}\delta^{kl}+\delta^{ik}\delta^{jl}+\delta^{il}\delta^{jk})dt \\
&\quad+dt^2\ O(dt,dx)
\end{aligned} \tag{2.81}$$

and the higher moments are $dt^2\ O(dt,dx)$. QED

Proof of Theorem 2.2. Let $\xi(t)$ be a smooth reciprocal diffusion in flat space whose density satisfies $\rho(x,t) > 0$ for all $t \in (t_0, t_f)$, $x \in \mathbf{R^n}$. If we impose the condition that $\xi(t_f) = x_f$ then we obtain a Markov diffusion and we can compute its forward Markov characteristics. Let b be any smooth forward velocity field in the reciprocal class of $\xi(t)$, (2.23,2.24) and let $p(y,s;z,t)$ be the forward Markov transition density of solutions of the Ito equation (1.10). The Markov transition density of $\xi(t)$ conditioned on $\xi(t_f) = x_f$ is

$$q(y,t;z,t+dt;x_f,t_f) = \frac{p(y,t;z,t+dt)p(z,t+dt;x_f,t_f)}{p(y,s;x_f,t_f)}. \tag{2.82}$$

We expand the second term in the numerator

$$\begin{aligned} q(y,t;z,t+dt;x_f,t_f) \;&=\; p(y,t;z,t+dt)\left[1 + \frac{\partial \ln\, p}{\partial y^i}(y,t;x_f,t_f)(z^i - y^i)\right. \\ &\quad \left. + \frac{\partial \ln\, p}{\partial t}(y,t;x_f,t_f)dt + O(t-s,z-y)^2\right]. \end{aligned} \tag{2.83}$$

Using Lemma 2.3 we obtain

$$\begin{aligned} E(d^+\xi^i|\xi(t) = y, \xi(t_f) = x_f) \;&=\; b^i(y,t) + \frac{\partial \ln\, p}{\partial y^i}(y,t;x_f,t_f) \\ &\quad + O(dt^2), \end{aligned} \tag{2.84}$$

$$E(d^+\xi^i\ d^+\xi^j|\xi(t) = y, \xi(t_f) = x_f) \;=\; \delta^{ij}dt + O(dt^2). \tag{2.85}$$

Note that the conditional forward velocity

$$b^i(x,t|x_f,t_f) = b^i(x,t) + \frac{\partial \ln\, p}{\partial x^i}(x,t;x_f,t_f) \tag{2.86}$$

varies with conditioning but the diffusion coefficient does not. Because of the reciprocal property, adding the additional condition $\xi(t_0) = x_0$ does not change the forward velocity

$$b(x,t|x_0,t_0;x_f,t_f) = b(x,t|x_f,t_f).$$

Suppose the endpoint density of the unconditioned $\xi(t)$ is $\rho(x_0,t_0;x_f,t_f)$ then the unconditioned forward velocity $b(x,t)$ is given by Bayes rule

$$\rho(x,t)b(x,t) = \int\int b(x,t|x_0,t_0;x_f,t_f)q(x_0,t_0;x,t;x_f,t_f)\rho(x_0,t_0;x_f,t_f)dx_0dx_f. \tag{2.87}$$

Next we derive the backward Markov characteristics of a reciprocal diffusion conditioned in the past, $\xi(t_0) = x_0$. The backward Markov transition density is

$$q(x_0,t_0;y,t-dt;z,t) = \frac{p(x_0,t_0;y,t-dt)p(y,t-dt;z,t)}{p(x_0,t_0;z,t)}. \tag{2.88}$$

We expand the first term in the numerator

$$q(x_0,t_0;y,t-dt;z,t) = p(y,t-dt;z,t)\Big[1-\frac{\partial \ln p}{\partial z^i}(x_0,t_0;z,t)(z^i-y^i) \\ -\frac{\partial \ln p}{\partial t}(x_0,t_0;z,t)dt + O(t-s,z-y)^2\Big]. \quad (2.89)$$

Let $p_z(y,t-dt;z,t)$ denote the Gauss Markov transition density of solutions of (2.51) with $x=z$. By Lemma (2.3)

$$\begin{aligned} p(y,t-dt;z,t) &= p_z(y,t-dt;z,t)\left[1+dt\ O(z-y)+O(z-y)^3\right] \\ &= (2\pi\ dt)^{-n/2}\exp-\frac{|z-y|^2}{2dt} \\ &\quad \times\left[1+(z^i-y^i)\left(b^i(z,t)-b^i_{,j}(z,t)(z^j-y^j)\right)\right. \\ &\quad \left.+O(dt)+O(z-y)^3\right], \end{aligned} \quad (2.90)$$

hence

$$\begin{aligned} E(d^-\xi^i|\xi(t)=z,\xi(t_0)=x_0) &= b^i(z,t)-\frac{\partial \ln p}{\partial z^i}(x_0,t_0;z,t) \\ &\quad +O(dt^2), \end{aligned} \quad (2.91)$$

$$E(d^-\xi^i\ d^-\xi^j|\xi(t)=z,\xi(t_0)=x_0) = \delta^{ij}dt+O(dt^2). \quad (2.92)$$

Once again we see that the conditional backward velocity

$$\bar{b}^i(x,t|x_0,t_0) = b^i(x,t)-\frac{\partial \ln p}{\partial x^i}(x_0,t_0;x,t) \quad (2.93)$$

depends on the conditioning but the diffusion coefficient does not. Adding an additional condition in the future does not change the backward velocity

$$\bar{b}(x,t|x_0,t_0;x_f,t_f) = \bar{b}(x,t|x_0,t_0).$$

The unconditioned backward velocity $\bar{b}(x,t)$ satisfies

$$\rho(x,t)\bar{b}(x,t) = \int\int \bar{b}(x,t|x_0,t_0;x_f,t_f)q(x_0,t_0;x,t;x_f,t_f)\rho(x_0,t_0;x_f,t_f)dx_0dx_f. \quad (2.94)$$

To compute the centered mean velocity (2.29), we start with the joint density of $\xi(t-dt),\xi(t+dt)$ conditioned on $\xi(t_0),\xi(t_f)$

$$q(x_0,t_0;x-z,t-dt;x+z,t+dt;x_f,t_f) \quad (2.95)$$

$$= \frac{p(x_0,t_0;x-z,t-dt)p(x-z,t-dt;x+z,t+dt)p(x+z,t+dt;x_f,t_f)}{p(x_0,t_0;x_f,t_f)}.$$

As before we expand the first and third terms in the numerator around x,t and obtain

$$q(x_0, t_0; x-z, t-dt; x+z, t+dt; x_f, t_f) \tag{2.96}$$

$$= p(x - z, t - dt; x + z, t + dt)q(x_0, t_0; x, t; x_f, t_f)$$

$$\times \left[1 + \frac{\partial \ln p}{\partial x^i}(x, t; x_f, t_f)z^i + \frac{\partial \ln p}{\partial t}(x, t; x_f, t_f)dt\right.$$

$$\left. - \frac{\partial \ln p}{\partial x^i}(x_0, t_0; x, t)z^i - \frac{\partial \ln p}{\partial t}(x_0, t_0; x, t)dt + O(t - s, z)^2\right].$$

We apply Lemma (2.3) to obtain

$$\begin{aligned} p(x - z, t - dt; x + z, t + dt) &= p_x(x - z, t - dt; x + z, t + dt) \\ &\quad \times \left[1 + dt \; O(z) + O(z)^3\right] \\ &= (\pi \; dt)^{-n/2} exp - \frac{|z|^2}{dt} \\ &\quad \times \left[1 + 2z^i \left(b^i(x, t) + b^i_{;j}(x, t)z^j\right)\right. \\ &\quad \left. + O(dt) + O(z)^3\right] \end{aligned} \tag{2.97}$$

so the conditional centered velocity

$$v^i(x, t|x_0, t_0; x_f, t_f) = \int \frac{z^i}{2dt} p(x - z, t - dt; x + z, t + dt)dz \tag{2.98}$$

satisfies

$$\begin{aligned} v(x, t|x_0, t_0; x_f, t_f) &= \frac{1}{2}(b^i(x, t|x_0, t_0; x_f, t_f) + \overline{b}^i(x, t|x_0, t_0; x_f, t_f))dt \\ &\quad + O(dt^2). \end{aligned} \tag{2.99}$$

Now

$$\rho(x, t)v^i(x, t) = \int v^i(x, t|x_0, t_0; x_f, t_f)q(x_0, t_0; x, t; x_f, t_f)\rho(x_0, t_0; x_f, t_f)dx_0 dx_f \tag{2.100}$$

and (2.87,2.94) yields (2.29). Assertations (2.30-2.33) are derived in a similar fashion.

The assertions (2.34-2.38) follow immediately from Theorem 2.1 and the nesting of conditional expectations, e.g.

$$E(d^2\xi^i|d^0\xi = x) = E(E(d^2\xi^i|d^0\xi = x, d^1\xi = dx)|d^0\xi = x). \tag{2.101}$$

QED

3 Stochastic ODE's of second order

It is our goal to define a *second order stochastic boundary value problem* of the form

$$d^2\xi^i = f^i(\xi,t)dt^2 + g^i_j(\xi,t)dx^j\ dt + \eta^i(\xi,t)dt^2 \tag{3.1}$$

$$\xi^i(t_0) = x^i_0 \tag{3.2}$$

$$\xi^i(t_f) = x^i_f \tag{3.3}$$

where f, g are defined by (2.23,2.24) for some smooth vector field b, $\eta(x,t)$ is a generalized random field, formally defined by

$$\eta^i(x,t) = \frac{d^2w^i(t)}{dt^2} + b^i_{,j}(x,t)\frac{dw^i(t)}{dt} - \frac{1}{2}b^j_{,ij}(x,t) \tag{3.4}$$

with $w(t)$ a standard n dimensional Wiener process and x_0, x_f are random n vectors.

A first attempt at a definition is the following. A reciprocal process $\xi(t)$ with almost surely continuous sample paths is a solution of the second order stochastic boundary value problem (3.1-3.3) if $\xi(t)$ satisfies the integral equation

$$\begin{aligned}\xi^i(t) &= \frac{t_f - t}{t_f - t_0}x^i_0 + \frac{t - t_0}{t_f - t_0}x^i_f \\ &+ \int_{t_0}^{t_f} \Gamma(t,s)\left[f^i(\xi(s),s)ds + g^i_j(\xi(s),s)d\xi^j(s) + \eta^i(\xi(s),s)ds\right].\end{aligned} \tag{3.5}$$

where $\Gamma(t,s)$ is the Green's function of the differential operator $\frac{d^2}{dt^2}$ with Dirichlet boundary conditions

$$\Gamma(t,s) = \begin{cases} -\left(t_f - t\right)(s - t_0)/\left(t_f - t_0\right) & \text{if } t > s, \\ -\left(t_f - s\right)/(t - t_0)/\left(t_f - t_0\right) & \text{if } t < s \end{cases}. \tag{3.6}$$

From the definition of $\eta(x,t)$ we see that (3.5) is equivalent to

$$\begin{aligned}\xi^i(t) &= \frac{t_f - t}{t_f - t_0}x^i_0 + \frac{t - t_0}{t_f - t_0}x^i_f \\ &+ \int_{t_0}^{t_f} \Gamma(t,s)\left[f^i(\xi(s),s)ds + g^i_j(\xi(s),s)d\xi^j(s)\right] \\ &+ \int_{t_0}^{t_f} \Gamma(t,s)\left[\frac{d^2w^i(s)}{ds^2} + b^i_{,j}(\xi(s),s)\frac{dw^j(s)}{ds} - \frac{1}{2}b^j_{,ij}(\xi(s),s))\right] ds.\end{aligned} \tag{3.7}$$

But this is not a complete definition for we have not defined the various stochastic integrals. Rather than defining them directly, we shall assume that they obey the standard rules of calculus and then transform them into well-defined integrals. For example, integration by parts yields

$$\int_{t_0}^{t_f} \Gamma(t,s)\frac{d^2w^i(s)}{ds^2}ds = w^i(t) - \frac{t_f - t}{t_f - t_0}w^i(t_0) - \frac{t - t_0}{t_f - t_0}w^i(t_f), \tag{3.8}$$

and so (3.5) is equivalent to

$$\begin{aligned} \xi^i(t) &= \frac{t_f - t}{t_f - t_0} x_0^i + \frac{t - t_0}{t_f - t_0} x_f^i \\ &+ \int_{t_0}^{t_f} \Gamma(t,s) \left[f^i(\xi(s),s)ds + g_j^i(\xi(s),s)d\xi^j(s) \right] \\ &+ \int_{t_0}^{t_f} \Gamma(t,s) \left[b_{,j}^i(\xi(s),s)dw^i(s) - \frac{1}{2} b_{,ij}^j(\xi(s),s))ds \right] \\ &+ w^i(t) - \frac{t_f - t}{t_f - t_0} w^i(t_0) - \frac{t - t_0}{t_f - t_0} w^i(t_f). \end{aligned} \tag{3.9}$$

Instead of individually defining the other stochastic integrals

$$\int_{t_0}^{t_f} \Gamma(t,s) g_j^i(\xi(s),s) d\xi^j(s) \tag{3.10}$$

and

$$\int_{t_0}^{t_f} \Gamma(t,s) b_{,j}^i(\xi(s),s) \frac{dw^j(s)}{ds} ds = \int_{t_0}^{t_f} \Gamma(t,s) b_{,j}^i(\xi(s),s) dw^j(s), \tag{3.11}$$

we manipulate (3.9) so that it contains only well-defined integrals.

Suppose $\xi(t)$ is a solution of (3.9), we define

$$\zeta^i(t) = \frac{t_f - t}{t_f - t_0}(x_0^i - w^i(t_0)) + \frac{t - t_0}{t_f - t_0}(x_f^i - w^i(t_f)), \tag{3.12}$$

$$z^i(t) = \xi^i(t) - \zeta^i(t) - w^i(t). \tag{3.13}$$

Then

$$\begin{aligned} z^i(t) &= \int_{t_0}^{t_f} \Gamma(t,s) \left[f^i(\xi(s),s)ds + g_j^i(\xi(s),s)d\xi^j(s) \right] \\ &+ \int_{t_0}^{t_f} \Gamma(t,s) \left[b_{,i}^j(\xi(s),s)dw^j(s) - \frac{1}{2} b_{,ij}^j(\xi(s),s)ds \right], \end{aligned} \tag{3.14}$$

$$\dot{\zeta}^i = x_f^i - x_0^i - w^i(t_f) + w^i(t_0), \tag{3.15}$$

$$dz^i = d\xi^i - \dot{\zeta}^i dt - dw^i. \tag{3.16}$$

From the definitions of f, g we obtain

$$z^i(t) = \int_{t_0}^{t_f} \Gamma(t,s) \left[db^i(\xi(s),s) - b_{,i}^j(\xi(s),s) \left(d\xi^j - b^j(\xi(s),s)ds - dw^j(s) \right) \right]. \tag{3.17}$$

Assuming $z(t)$ is C^1 this becomes

$$\begin{aligned} z^i(t) &= \frac{t_f - t}{t_f - t_0} \int_{t_0}^{t} c^i(z(s),s)ds - \frac{t - t_0}{t_f - t_0} \int_{t}^{t_f} c^i(z(s),s)ds \\ &- \int_{t_0}^{t_f} \Gamma(t,s) c_{,i}^j(z(s),s) \left[\dot{z}^j(s) + \dot{\zeta}^j - c^j(z(s),s) \right] ds, \end{aligned} \tag{3.18}$$

where

$$c(z,t) = b(z + \zeta(t) + w(t), t). \tag{3.19}$$

This motivates the following definition.

Definition 3.1 *A stochastic process $\xi(t)$ is a strong solution of the second order stochastic boundary value problem (3.1-3.4) for given $x^0, x^f, w(t)$ if almost surely the processes $\zeta(t), z(t)$ defined by (3.12,3.13) are C^1 and satisfy (3.18).*

Notice that this is essentially a sample path concept, no definition of stochastic integration is required.

Theorem 3.2 *Suppose $\xi(t)$ is a Markov diffusion satisfying the Ito equation (1.10). Let $x_0 = \xi(t_0), x_f = \xi(t_f)$. Then $\xi(t)$ is a strong solution of the second order stochastic boundary value problem (3.1-3.4).*

Proof. By definition

$$\xi^i(t) = \xi^i(t_0) + \int_{t_0}^{t} b^i(\xi(s), s)ds + w^i(t) - w^i(t_0). \tag{3.20}$$

Notice this is an ordinary integral for almost every sample path since b is smooth and the sample paths of $\xi(t)$ are continous a.s. Let $\zeta(t), z(t)$ be defined by (3.12,3.13). Clearly $\zeta(t)$ is C^1 and so is $z(t)$ since

$$z^i(t) = \frac{t - t_0}{t_f - t_0}(x_0^i - w^i(t_0) - x_f^i + w^i(t_f)) + \int_{t_0}^{t} b^i(\xi(s), s)ds. \tag{3.21}$$

If we evaluate (3.20) at $t = t_f$ and plug into (3.21), we obtain

$$z^i(t) = \frac{t_f - t}{t_f - t_0}\int_{t_0}^{t} b^i(\xi(s), s)ds - \frac{t - t_0}{t_f - t_0}\int_{t}^{t_f} b^i(\xi(s), s)ds. \tag{3.22}$$

By differentiating (3.21) we obtain

$$\dot{z}^j(t) + \dot{\zeta}^j - b^j(\xi(t), t) = 0 \tag{3.23}$$

so $z(t)$ satisfies (3.18). QED

Next we state and prove a useful lemma.

Lemma 3.3 *A pair of smooth vector fields $b(x,t)$ and $\bar{b}(x,t)$ are in the same reciprocal class iff*

$$\bar{b}^i(x,t) = b^i(x,t) + h_{,i}(x,t), \tag{3.24}$$

where $h(x,t)$ satisfies the logarithmic backward equation

$$0 = h_{,0} + h_{,j}b^j + \frac{1}{2}h_{,jj} + \frac{1}{2}(h_{,j})^2. \tag{3.25}$$

Remark. This is called the logarithmic backward equation because its solutions are the logarithms of solutions of the backward equation (1.14). This formula (3.24) is the h-transform of Doob.

Proof of Lemma. Suppose $h(x,t)$ satisfies the logarithmic backward equation and $b(x,t)$ and $\bar{b}(x,t)$ satisfy (3.24). Clearly

$$\begin{aligned} \bar{g}^i_j &= \bar{b}^i_{,j} - \bar{b}^j_{,i} \\ &= b^i_{,j} - b^j_{,i} - h_{,ij} + h_{,ji} \\ &= g^i_j . \end{aligned} \tag{3.26}$$

And

$$\begin{aligned} \bar{f}^i &= \bar{b}^i_{,0} + \bar{b}^j_{,i}\bar{b}^j + \frac{1}{2}\bar{b}^j_{,ij} \\ &= b^i_{,0} + b^j_{,i}b^j + \frac{1}{2}b^j_{,ij} + h_{,i0} + h_{,ij}(b^j + h_j) + h_{,j}b^j_{,i} + \frac{1}{2}h_{,ijj} \\ &= f^i + \left(h_{,0} + h_{,j}b^j + \frac{1}{2}(h_{,j})^2 + \frac{1}{2}h_{,jj}\right)_{,i} \\ &= f^i . \end{aligned} \tag{3.27}$$

But by Clark [4], f, g and the scale of the noise $\sigma^i_k \sigma^j_k = \delta^{ij}$ are a complete set of reciprocal invariants so $b(x,t)$ and $\bar{b}(x,t)$ are in the same reciprocal class.

Now suppose $b(x,t)$ and $\bar{b}(x,t)$ are in the same reciprocal class and let

$$k^i(x,t) = \bar{b}^i(x,t) - b^i(x,t). \tag{3.28}$$

Since $g = \bar{g}$, k is a closed one-form on $\mathbf{R^n}$, hence exact so there exist $\bar{h}$ such that

$$\bar{h}_{,i}(x,t) = k^i(x,t). \tag{3.29}$$

Since $f = \bar{f}$,

$$\left(\bar{h}_{,0} + \bar{h}_{,j}b^j + \frac{1}{2}(\bar{h}_{,j})^2 + \frac{1}{2}\bar{h}_{,jj}\right)_{,i} = 0 \tag{3.30}$$

for $i = 1, \ldots, n$. Hence there exists $\alpha(t)$ such that

$$\bar{h}_{,0} + \bar{h}_{,j}b^j + \frac{1}{2}(\bar{h}_{,j})^2 + \frac{1}{2}\bar{h}_{,jj} = \dot{\alpha}. \tag{3.31}$$

Define $h(x,t) = \bar{h}(x,t) - \alpha(t)$ then (3.24,3.25) hold. QED

Theorem 3.4 *Consider the second order stochastic boundary value problem (3.1-3.3) where $w(t)$ is a standard Wiener process independent of the boundary conditions x_0, x_f and assume the joint density of x_0, x_f is positive everywhere. There exists a reciprocal diffusion $\xi(t)$ which satisfies (3.1-3.3) strongly.*

Proof Let $p(x,s;y,t)$ be the Markov transition density of solutions of the Ito equation (1.10), let $q(x,s;y,t;z,u)$ be the corresponding reciprocal transition density (1.7) and let $\rho_{0f}(x_0,x_f)$ denote the density of the endpoints. By a result of Jamison [9], there exist a reciprocal process with these densities q,ρ_{0f} and it is unique up to law. We construct a strong solution of the second order stochastic boundary value problem (3.1-3.3) with this law by conditioning on x_0.

Let $\xi(t_0)=\bar{x}_0$ be a deterministic initial condition and define

$$\begin{aligned} h(x,t) &= \ln\int p(x,t;\bar{x}_f,t_f)\rho_{f|0}(\bar{x}_f|\bar{x}_0)d\bar{x}_f, &(3.32)\\ \bar{b}^i &= b^i+h_{,i} &(3.33)\end{aligned}$$

where $\rho_{f|0}(\bar{x}_f|\bar{x}_0)$ is the conditional density of $\bar{x}_f$ given $x_0=\bar{x}_0$,

$$\rho_{f|0}(\bar{x}_f|\bar{x}_0)=\frac{\rho_{0f}(\bar{x}_0,\bar{x}_f)}{\int\rho_{0f}(\bar{x}_0,\bar{x}_f)d\bar{x}_f}. \tag{3.34}$$

It is not hard to show that h satisfies the logarithmic backward equation so b and $\bar{b}$ are in the same reciprocal class

Let $\bar{w}(t)$ be a standard Wiener process under a probability measure $\bar{P}$ on the space of $\bar{w}$ paths which is independent of $w(t),x_0,x_f$. Consider the Markov diffusion $\bar{\xi}(t)$ that satisfies the Ito equation

$$\begin{aligned} d^+\bar{\xi}^i &= \bar{b}^i(\bar{\xi},t)dt+d^+\bar{w}^i, &(3.35)\\ \bar{\xi}(t_0) &= \bar{x}_0. &(3.36)\end{aligned}$$

The process $\bar{\xi}(t)$ is a reciprocal diffusion in the class of f,g with end point distribution of $\bar{x}_0,\bar{x}_f$ above. Hence the law of $\bar{\xi}(t)$ is the same as that of the desired solution of the second order stochastic boundary value problem (3.1-3.3) conditioned on $x_0=\bar{x}_0$. Moreover, by Theorem 3.2, $\bar{\xi}(t)$ satisfies the integral equation

$$\begin{aligned} \bar{\xi}^i(t) &= \frac{t_f-t}{t_f-t_0}\bar{x}_0^i+\frac{t-t_0}{t_f-t_0}\bar{x}_f^i &(3.37)\\ &+\int_{t_0}^{t_f}\Gamma(t,s)\left[f^i(\bar{\xi}(s),s)ds+g_j^i(\bar{\xi}(s),s)d\bar{\xi}^j(s)\right]\\ &+\int_{t_0}^{t_f}\Gamma(t,s)\left[\frac{d^2\bar{w}^i(s)}{ds^2}+\bar{b}^i_{,j}(\bar{\xi}(s),s)\frac{d\bar{w}^i(s)}{ds}-\frac{1}{2}\bar{b}^j_{,ij}(\bar{\xi}(s),s))\right]ds.\end{aligned}$$

We wish to show is that this holds with b,w replacing $\bar{b},\bar{w}$.

Define a transformation from w paths to $\bar{w}$ paths by

$$\bar{w}^i(t)-\bar{w}^i(t_0)=w^i(t)-w^i(t_0)-\int_{t_0}^{t}h_{,i}(\bar{\xi}(s),s)ds. \tag{3.38}$$

This allows us to define the measure P on $\bar{w}$ paths. The measures $P,\bar{P}$ are mutually absolutely continuous with Radon Nikodym derivative given by Girsanov's Theorem ([17], Theorem 8.22),

$$\frac{d\overline{P}}{dP} = \exp(\zeta_{t_0}^{t_f}), \tag{3.39}$$

where $\zeta_{t_0}^{t}$ satisfies the Ito equation

$$d^+\zeta_{t_0}^t = h_{,i}(\overline{\xi}(t), t)d^+\overline{w}^i + \frac{1}{2}(h_{,i}(\overline{\xi}(t), t))^2 dt. \tag{3.40}$$

We utilize (3.35) and the fact that h satisfies the logarithmic backward equation to obtain

$$\begin{aligned} d^+\zeta_{t_0}^t &= h_{,i}(\overline{\xi}(t), t)d^+\overline{\xi}^i + h_{,0}(\overline{\xi}(t), t)dt + \frac{1}{2}h_{,ii}(\overline{\xi}(t), t)dt \\ &= d^+h(\overline{\xi}(t), t), \end{aligned} \tag{3.41}$$

so

$$\zeta_{t_0}^{t_f} = h(\overline{\xi}(t_f), t_f) - h(\overline{\xi}(t_0), t_0). \tag{3.42}$$

Therefore conditioned on the endpoints, $\overline{\xi}(t_0), \overline{\xi}(t_f)$, the conditional measures of P and $\overline{P}$ are identical. By definition, the statement that $\overline{\xi}(t), \overline{w}(t)$ satisfy (3.37) means that $\overline{z}(t), \overline{\zeta}(t)$ defined by

$$\overline{z}^i(t) = \overline{\xi}^i(t) - \overline{\zeta}^i(t) - \overline{w}^i(t) \tag{3.43}$$

$$\overline{\zeta}^i(t) = \frac{t_f - t}{t_f - t_0}(\overline{x}_0^i - \overline{w}_0^i) + \frac{t - t_0}{t_f - t_0}(\overline{x}_f^i - \overline{w}_f^i) \tag{3.44}$$

satisfy

$$\begin{aligned} \overline{z}^i(t) &= \frac{t_f - t}{t_f - t_0}\int_{t_0}^{t} \overline{b}^i(\overline{\xi}(s), s)ds - \frac{t - t_0}{t_f - t_0}\int_{t}^{t_f} \overline{b}^i(\overline{\xi}(s), s)ds \\ &- \int_{t_0}^{t_f} \Gamma(t, s)\overline{b}^j_{,i}(\overline{\xi}(s), s)\left[\overline{z}^j(s) + \overline{\zeta}^j - \overline{b}^j(\overline{\xi}(s), s)\right] ds. \end{aligned} \tag{3.45}$$

Notice that $\overline{z}(t_0) = \overline{z}(t_f) = 0$ therefore the distribution of the endpoints of solutions to (3.37) is determined not by the distibution of $\overline{w}(t)$ but only by the distribution of $\overline{x}_0, \overline{x}_f$. Hence the measures P and $\overline{P}$ on the space of $\overline{w}$ paths induce the same law on the space of $\overline{\xi}$ paths via (3.37).

We define $z(t), \zeta(t)$ by

$$z^i(t) = \overline{\xi}^i(t) - \zeta^i(t) - w^i(t), \tag{3.46}$$

$$\zeta^i(t) = \frac{t_f - t}{t_f - t_0}(\overline{x}_0^i - w_0^i) + \frac{t - t_0}{t_f - t_0}(\overline{x}_f^i - w_f^i). \tag{3.47}$$

then by (3.38)

$$\begin{aligned} z^i(t) &= \overline{z}^i(t) + \left(\overline{\zeta}^i(t) - \zeta^i(t)\right) + \left(\overline{w}^i(t) - w^i(t)\right) \\ &= \overline{z}^i(t) + \frac{t - t_0}{t_f - t_0}\left(w_f^i - \overline{w}_f^i - w_0^i + \overline{w}_0^i\right) - \int_{t_0}^{t} h_{,i}(\overline{\xi}(s), s)ds \\ &= \overline{z}^i(t) - \frac{t_f - t}{t_f - t_0}\int_{t_0}^{t} h_{,i}(\overline{\xi}(s), s)ds + \frac{t - t_0}{t_f - t_0}\int_{t}^{t_f} h_{,i}(\overline{\xi}(s), s)ds. \end{aligned} \tag{3.48}$$

From (3.45) we see that

$$\begin{aligned} z^i(t) &= \frac{t_f - t}{t_f - t_0}\int_{t_0}^{t} b^i(\bar{\xi}(s),s)ds - \frac{t - t_0}{t_f - t_0}\int_{t}^{t_f} b^i(\bar{\xi}(s),s)ds \\ &- \int_{t_0}^{t_f} \Gamma(t,s) b^j_{,i}(\bar{\xi}(s),s)\left[\dot{\bar{z}}^j(s) + \dot{\bar{\zeta}}^j - \bar{b}^j(\bar{\xi}(s),s)\right] ds \\ &- \int_{t_0}^{t_f} \Gamma(t,s) h_{,ij}(\bar{\xi}(s),s)\left[\dot{\bar{z}}^j(s) + \dot{\bar{\zeta}}^j - \bar{b}^j(\bar{\xi}(s),s)\right] ds. \end{aligned} \tag{3.49}$$

Now from (3.33,3.38,3.43,3.44)

$$\dot{\bar{z}}^j(s) + \dot{\bar{\zeta}}^j - \bar{b}^j(\bar{\xi}(s),s) = \dot{z}^j(s) + \dot{\zeta}^j - b^j(\bar{\xi}(s),s) \tag{3.50}$$

and from (3.35)

$$\left(\dot{\bar{z}}^j(s) + \dot{\bar{\zeta}}^j - \bar{b}^j(\bar{\xi}(s),s)\right) ds = d^+\bar{\xi}^j(s) - \bar{b}^j(\bar{\xi}(s),s)ds - d^+\bar{w}^j(s) = 0. \tag{3.51}$$

Therefore

$$\begin{aligned} z^i(t) &= \frac{t_f - t}{t_f - t_0}\int_{t_0}^{t} b^i(\bar{\xi}(s),s)ds - \frac{t - t_0}{t_f - t_0}\int_{t}^{t_f} b^i(\bar{\xi}(s),s)ds \\ &- \int_{t_0}^{t_f} \Gamma(t,s) b^j_{,i}(\bar{\xi}(s),s)\left[\dot{z}^j(s) + \dot{\zeta}^j - b^j(\bar{\xi}(s),s)\right] ds \end{aligned} \tag{3.52}$$

so we have shown that $\bar{\xi}(t)$ satisfies

$$\begin{aligned} \bar{\xi}^i(t) &= \frac{t_f - t}{t_f - t_0}\bar{x}_0^i + \frac{t - t_0}{t_f - t_0}\bar{x}_f^i \\ &+ \int_{t_0}^{t_f} \Gamma(t,s)\left[f^i(\bar{\xi}(s),s)ds + g^i_j(\bar{\xi}(s),s)d\bar{\xi}^j(s)\right] \\ &+ \int_{t_0}^{t_f} \Gamma(t,s)\left[\frac{d^2 w^i(s)}{ds^2} + b^i_{,j}(\bar{\xi}(s),s)\frac{dw^i(s)}{ds} - \frac{1}{2}b^j_{,ij}(\bar{\xi}(s),s))\right] ds \end{aligned} \tag{3.53}$$

as desired. QED

Notice that we have not proven the uniqueness of strong solutions to (3.1-3.3). It is possible to show the existence and uniqueness of solutions for any deterministic boundary values x_0, x_f and determistic continuous function $w(t)$ provided $t_f - t_0$ is sufficiently small. But just how small is sufficient depends on x_0, x_f and $w(t)$ as we now show.

By our definition of a smooth vector field (C^3 with bounded partials), there exists a constant $L > 0$ such that

$$|b(x,t)| \le L(1 + |x|) \tag{3.54}$$

$$|b(x,t) - b(y,t)| \le L|x - y| \tag{3.55}$$

$$\left\|\frac{\partial b^*}{\partial x}(x,t)\right\| \le L \tag{3.56}$$

$$\left\|\frac{\partial b^*}{\partial x}(x,t) - \frac{\partial b^*}{\partial x}(y,t)\right\| \le L|x - y| \tag{3.57}$$

where $\| \,.\, \|$ denotes the induced matrix norm and * denotes transpose. Since $\zeta(t) + w(t)$ is bounded for $t \in [t_0, t_f]$, $c(x, t)$ (3.19) also satisfies (3.54-3.57) for some new L depending on x_0, x_f and $w(t)$.

Define an operator on C^1 functions $z(t)$

$$
\begin{aligned}
T[z]^i(t) \;=\; & \frac{t_f - t}{t_f - t_0}\int_{t_0}^{t} c^i(z(s), s)ds - \frac{t - t_0}{t_f - t_0}\int_{t}^{t_f} c^i(z(s), s)ds \qquad (3.58)\\
& - \int_{t_0}^{t_f} \Gamma(t, s) c^j_{,i}(z(s), s)\left[\dot{z}^j(s) + \dot{\zeta}^j - c^j(z(s), s)\right] ds.
\end{aligned}
$$

Notice that $T[z](t)$ is C^1. The next theorem proves the existence and uniqueness of deterministic solutions of (3.18) for sufficiently small $t_f - t_0$ by showing the convergence of the Picard iterates

$$
\begin{aligned}
z^i_{(0)}(t) &= 0, & (3.59)\\
z^i_{(k+1)}(t) &= T[z^i_{(k)}](t). & (3.60)
\end{aligned}
$$

.

Theorem 3.5 *Suppose $c(x, t)$ is smooth vector field satisfying (3.54-3.57) Consider the integral equation (3.18). There exists $\tau > 0$ such that if $0 < t_f - t_0 \leq \tau$ then the solution of (3.18) exists and is unique.*

Proof. This is a modification of a standard proof of existence and uniqueness of solutions for two point boundary value problems. See Hartman [8], Chapter XII, Theorems 0.1 and 4.1. The proof is based on the fact that, for sufficiently small $t_f - t_0$, the operator T is a contraction on the space of C^1 functions equipped with the norm

$$
\|z\| = \max\left\{ \sup_{t_0 \leq s \leq t_f} |z(s)|, \;\; (t_f - t_0) \sup_{t_0 \leq s \leq t_f} |\dot{z}(s)|. \right\} \qquad (3.61)
$$

Choose constants M_1, M_2 such that

$$
\begin{aligned}
M_1 &\geq \sup_{t_0 \leq t \leq t_f} \{1 + |\zeta(t) + w(t)|\}, & (3.62)\\
M_2 &\geq |\dot{\zeta}|, & (3.63)
\end{aligned}
$$

so

$$
\begin{aligned}
|z_{(1)}(t)| \;\leq\; & \frac{t_f - t}{t_f - t_0}\int_{t_0}^{t} LM_1 ds + \frac{t - t_0}{t_f - t_0}\int_{t}^{t_f} LM_1 ds\\
& + \int_{t_0}^{t_f} |\Gamma(t, s)| L(M_2 + LM_1) ds\\
\leq\; & 2\frac{(t_f - t)(t - t_0)}{t_f - t_0} LM_1 + \frac{1}{2}(t_f - t)(t - t_0) L(M_2 + LM_1)\\
\leq\; & \frac{(t_f - t)(t - t_0)}{t_f - t_0} M_3 \leq (t_f - t_0) M_3, \qquad (3.64)
\end{aligned}
$$

where

$$M_3 = 2LM_1 + \frac{1}{2}(t_f - t_0)L(M_2 + LM_1).$$

Also

$$\begin{aligned} |\dot z_{(1)}(t)| &\leq |c(0,t)| + \frac{1}{t_f - t_0}\int_{t_0}^{t_f} |c(0,s)|ds \\ &+ \int_{t_0}^{t_f} \left|\frac{\partial \Gamma(t,s)}{\partial t}\right| L(M_2 + LM_1)ds \\ &\leq 2LM_1 + \frac{1}{2}(t_f - t_0)L(M_2 + LM_1) = M_3. \end{aligned} \tag{3.65}$$

Hence

$$\|z_{(1)}\| \leq (t_f - t_0)M_3. \tag{3.66}$$

Suppose $y(t), z(t)$ are C^1 functions whose norm (3.61) is less than some constant $N > 0$. Then for $t_0 \leq t \leq t_f$,

$$|\dot y(t) + \dot\zeta - c(y(t),t)| \leq \frac{N}{t_f - t_0} + M_2 + L(M_1 + N), \tag{3.67}$$

and similarly for $z(t)$. Then

$$\begin{aligned} |T[y](t) - T[z](t)| &\leq \frac{t_f - t}{t_f - t_0}\int_{t_0}^{t} |c(y(s),s) - c(z(s),s)|\, ds \\ &+ \frac{t - t_0}{t_f - t_0}\int_{t}^{t_f} |c(y(s),s) - c(z(s),s)|\, ds \\ + \int_{t_0}^{t_f} |\Gamma(t,s)| &\times \left|\frac{\partial c^*}{\partial x}(y(s),s)\right| \times |\dot y(s) - \dot z(s) - (c(y(s),s) - c(z(s),s))|\, ds \\ + \int_{t_0}^{t_f} |\Gamma(t,s)| &\times \left|\frac{\partial c^*}{\partial x}(y(s),s) - \frac{\partial c^*}{\partial x}(z(s),s)\right| \times \left|\dot z(s) + \dot\zeta - c(z(s),s)\right| ds. \end{aligned} \tag{3.68}$$

So

$$\begin{aligned} |T[y](t) - T[z](t)| &\leq \frac{2(t_f - t)(t - t_0)}{t_f - t_0}L\,\|y - z\| \\ &+ \int_{t_0}^{t_f} |\Gamma(t,s)|L(\frac{1}{t_f - t_0} + L)\,\|y - z\|\, ds \\ &+ \int_{t_0}^{t_f} |\Gamma(t,s)|L\,\|y - z\| \left(\frac{N}{t_f - t_0} + M_2 + L(M_1 + N)\right) ds \\ &\leq (t_f - t_0)N_2\,\|y - z\| \end{aligned} \tag{3.69}$$

where

$$N_2 = \frac{L}{8}\left(5 + N + (t_f - t_0)(L + M_2 + LM_1 + LN)\right). \tag{3.70}$$

Similarly

$$\begin{aligned}
\left|\dot{T}[y](t)-\dot{T}[z](t)\right| \;&\le\; |c(y(t),t)-(c(z(t),t)| && (3.71)\\
&+\; \frac{1}{t_f-t_0}\int_{t_0}^{t_f}|c(y(s),s)-c(z(s),s)|\,ds\\
+\int_{t_0}^{t_f}\left|\frac{\partial\Gamma}{\partial t}(t,s)\right| \;&\times\; \left|\frac{\partial c^*}{\partial x}(y(s),s)\right|\;\left|\frac{\partial c^*}{\partial x}(y(s),s)\right|\\
&\times\; |\dot{y}(s)-\dot{z}(s)-(c(y(s),s)-c(z(s),s))|\,ds\\
+\int_{t_0}^{t_f}\left|\frac{\partial\Gamma}{\partial t}(t,s)\right| \;&\times\; \left|\frac{\partial c^*}{\partial x}(y(s),s)-\frac{\partial c^*}{\partial x}(z(s),s)\right|\\
&\times\; \left|\dot{z}(s)+\dot{\zeta}-c(z(s),s)\right|\,ds\\
&\le\; 2L\,\|y-z\|\\
&+\; \int_{t_0}^{t_f}\left|\frac{\partial\Gamma}{\partial t}(t,s)\right|(L+L^2)\,\|y-z\|\,ds\\
&+\; \int_{t_0}^{t_f}\left|\frac{\partial\Gamma}{\partial t}(t,s)\right|L\,\|y-z\|\left(\frac{N}{t_f-t_0}+M_2+L(M_1+N)\right)ds\\
&\le\; N_3\,\|y-z\| && (3.72)
\end{aligned}$$

where

$$N_3 = 2L+\frac{LN}{4}+\frac{t_f-t_0}{2}(L(1+M_2)+L^2(M_1+N)). \qquad (3.73)$$

Let $N_4=\max\{N_2,N_3\}$ then

$$\|T[y]-T[z]\| \le (t_f-t_0)N_4\,\|y-z\|\,. \qquad (3.74)$$

We apply Theorem 0.1 on page 404 of [8] and conclude that, for t_f-t_0 sufficiently small, the Picard iterates (3.59,3.60) converge to a unique solution in the space of C^1 functions on $[t_0,t_f]$ equipped with the norm (3.61). QED

4 Girsanov and Onsager-Machlup theorems

In this section we show how the reciprocal characteristics (2.23,2.24) arise naturally in the context of the Girsanov and Onsager-Machlup Theorems. Suppose we have two Ito equations

$$\begin{aligned}
d^+\xi^i &= b^i(\xi,t)dt+d^+w^i && (4.1)\\
d^+\xi^i &= \overline{b}^i(\xi,t)dt+d^+\overline{w}^i && (4.2)
\end{aligned}$$

where $b,\overline{b}$ are smooth vector fields and $w,\overline{w}$ are standard Brownian motions under probability measures $P,\overline{P}$. For a fixed initial condition $\xi(t_0)=x_0$ and time interval $[t_0,t_f]$, each of these equations induces a probability measure on the space of paths $\xi(t),\ t\in[t_0,t_f]$ which we denote by $P^{t_f}_{x_0,t_0},\overline{P}^{t_f}_{x_0,t_0}$. The Cameron-Martin-Girsanov Theorem ([17], Theorem 8.22 and Corollary 8.23) gives the Radom-Nikodym derivative of one of these measures with respect to the other.

$$\frac{d\bar{P}^{t_f}_{x_0,t_0}}{dP^{t_f}_{x_0,t_0}} = \exp\ \zeta^{t_f}_{t_0} \tag{4.3}$$

where ζ satisfies the Ito equation

$$\begin{aligned} d^+\zeta^t_{t_0} &= c^i d^+ w^i - \frac{1}{2} c^i c^i dt \\ &= c^i d^+\xi^i - \frac{1}{2}(\bar{b}^i \bar{b}^i - b^i b^i) dt \qquad (4.4) \\ \zeta^{t_0}_{t_0} &= 0 \qquad (4.5) \end{aligned}$$

and $c^i = \bar{b}^i - b^i$. In Stratonovich form this becomes

$$d\zeta^t_{t_0} = c^i d\xi^i - \frac{1}{2}(\bar{b}^i \bar{b}^i - b^i b^i + c^i_{,i})dt. \tag{4.6}$$

In light of this we define the *Girsanov one-form* of $\bar{b}$ relative to b to be

$$\theta = c^i dx^i - \frac{1}{2}(\bar{b}^i \bar{b}^i - b^i b^i + c^i_{,i})dt \tag{4.7}$$

Let $P^{x_f,t_f}_{x_0,t_0}, \bar{P}^{x_f,t_f}_{x_0,t_0}$ be the induced measures on the space of paths beginning at $\xi(t_0) = x_0$ and ending at $\xi(t_f) = x_f$. Clearly b and $\bar{b}$ are in the same reciprocal class iff $P^{x_f,t_f}_{x_0,t_0} = \bar{P}^{x_f,t_f}_{x_0,t_0}$ for all $x_0, t_0; x_f, t_f$. But this is true iff the integral of the Girsanov form along any path beginning at $\xi(t_0) = x_0$ and ending at $\xi(t_f) = x_f$ depends only on the endpoints and not on the path. In other words, b and $\bar{b}$ are in the same reciprocal class iff θ is a closed one-form. The exterior derivative of θ is

$$\begin{aligned} d\theta &= c^i_{,j} dx^j \wedge dx^i \\ &\quad + (c^i_{,0} + \bar{b}^j \bar{b}^j_{,i} - b^j b^j_{,i} + \frac{1}{2} c^j_{,ji}) dt \wedge dx^i. \end{aligned} \tag{4.8}$$

Hence we see again that b and $\bar{b}$ are in the same reciprocal class iff

$$\begin{aligned} f &= \bar{f} \qquad (4.9) \\ g &= \bar{g} \qquad (4.10) \end{aligned}$$

where f, g are the reciprocal characteristics of b (2.23,2.24) and $\bar{f}, \bar{g}$ are the corresponding reciprocal characteristics of $\bar{b}$.

Now let $\phi(t)$ and $\psi(t)$, $t \in [t_0, t_f]$ be any smooth paths beginning at x_0 and ending at x_f. We define a two dimensional surface bounded by these curves

$$\phi(t,\tau) = (1-\tau)\psi(t) + \tau\phi(t) \tag{4.11}$$

where $\tau \in [0,1]$. Let $\zeta^{t_f}_{t_0}$ be the solution of (4.6,4.5), i.e.

$$\begin{aligned}\zeta_{t_0}^{t_f} &= \int_{t_0}^{t_f} c^i(\phi(t),t))d\phi^i(t) \\ &-\frac{1}{2}(\overline{b}^i(\phi(t),t)\overline{b}^i(\phi(t),t) - b^i(\phi(t),t)b^i(\phi(t),t) \\ &+c^i_{,i}(\phi(t),t))dt \end{aligned} \tag{4.12}$$

and let $\eta_{t_0}^{t_f}$ be the corresponding integral along $\psi(t)$

$$\begin{aligned}\eta_{t_0}^{t_f} &= \int_{t_0}^{t_f} c^i(\psi(t),t))d\psi^i(t) \\ &-\frac{1}{2}(\overline{b}^i(\psi(t),t)\overline{b}^i(\psi(t),t) - b^i(\psi(t),t)b^i(\psi(t),t) \\ &+c^i_{,i}(\psi(t),t))dt. \end{aligned} \tag{4.13}$$

Then

$$\begin{aligned}\zeta_{t_0}^{t_f} - \eta_{t_0}^{t_f} &= \int_0^1 \frac{\partial}{\partial\tau}\int_{t_0}^{t_f} c^i(\phi(t,\tau),t)d\phi^i(t,\tau)d\tau \qquad (4.14)\\ &-\frac{1}{2}(\overline{b}^i(\phi(t,\tau),t)\overline{b}^i(\phi(t,\tau),t) - b^i(\phi(t,\tau),t)b^i(\phi(t,\tau),t) \\ &+c^i_{,i}(\phi(t,\tau),t))dtd\tau \\ &= \int_0^1\int_{t_0}^{t_f} c^i_{,j}(\phi(t,\tau),t)(\phi^j(t)-\psi^j(t))d\phi^i(t,\tau)d\tau \\ &+c^i(\phi(t,\tau),t)d(\phi^i(t)-\psi^i(t))d\tau \\ &-(\overline{b}^i(\phi(t,\tau),t)\overline{b}^i_{,j}(\phi(t,\tau),t) - b^i(\phi(t,\tau),t)b^i_{,j}(\phi(t,\tau),t) \\ &+\frac{1}{2}c^i_{,ij}(\phi(t,\tau),t))(\phi^j(t)-\psi^j(t))dtd\tau. \end{aligned}$$

We integrate the second term by parts with respect to t to obtain

$$\begin{aligned}\zeta_{t_0}^{t_f} - \eta_{t_0}^{t_f} &= \int_0^1\int_{t_0}^{t_f}\left[c^j_{,i}(\phi(t,\tau),t) - c^i_{,j}(\phi(t,\tau),t)\right]\left[\phi^i(t)-\psi^i(t)\right]d\phi^i(t,\tau)d\tau \\ &-\Big[\overline{b}^j(\phi(t,\tau),t)\overline{b}^j_{,i}(\phi(t,\tau),t) - b^j(\phi(t,\tau),t)b^j_{,i}(\phi(t,\tau),t) \\ &+\frac{1}{2}c^j_{,ij}(\phi(t,\tau),t)\Big]\left[\phi^i(t)-\psi^i(t)\right]dtd\tau \\ &= -\int_0^1\int_{t_0}^{t_f}\left[(\overline{f}^i(\phi(t,\tau),t) - f^i(\phi(t,\tau),t))dt\right. \\ &\left.+(\overline{g}^i_j(\phi(t,\tau),t) - g^i_j(\phi(t,\tau),t))d\phi^j(t)\right]\left[\phi^i(t)-\psi^i(t)\right]d\tau. \end{aligned} \tag{4.15}$$

Now $\zeta_{t_0}^{t_f}$ is the logarithm of the ratio of the likelihood of $\phi(t)$ under the measure $\overline{P}$ over the likelihood of $\phi(t)$ under the measure P. Similarly $\eta_{t_0}^{t_f}$ is the logarithm of the ratio of the likelihood of $\psi(t)$ under the measure $\overline{P}$ over the likelihood of

$\phi(t)$ under the measure P. Therefore the difference is the logarithm of the ratio of the relative likelihood of $\phi(t)$ to $\psi(t)$ under the measure $\overline{P}$ over the relative likelihood of $\phi(t)$ to $\psi(t)$ under the measure P. Notice that the right side has the dimensions of work, force times distance. If $b = 0$ then $f = 0, g = 0$ so the change in logarithmic relative likelihood is the work done in perturbing the trajectory from $\psi(t)$ to $\phi(t)$. Other formulas of Girsanov type can be found in [4] and [13].

Suppose $\xi(t)$ is a Markov diffusion satisfying the Ito equation (4.1) with deterministic initial condition $\xi(t_0) = x_0$ and let $\psi(t)$ be any smooth trajectory starting at the same point $\psi(t_0) = x_0$. The Onsager-Machlup formula [19] gives an asymptotic estimate for the probability that $\xi(t)$ lies in a tube of radius ϵ around $\psi(t)$,

$$\ln\ P\left(|\xi(t) - \psi(t)| < \epsilon, \quad t \in [t_0, t_f]\right) - \ln\ P\left(|w(t)| < \epsilon, \quad t \in [t_0, t_f]\right)$$

$$\sim -\frac{1}{2}\int_{t_0}^{t_f} |\dot{\psi} - b(\psi(t), t)|^2 + b_i^i(\psi(t), t)dt. \tag{4.16}$$

The right side is called *the Onsager-Matchlup functional*. The Euler-Lagrange equations for the trajectories which stationarize this functional are

$$\ddot{\psi}^i = f^i(\psi, t) + g_j^i(\psi, t)\dot{\psi}^j \tag{4.17}$$

where f, g are the familiar reciprocal characteristics, (2.23,2.24).

5 Conservation laws

The density of a Markov diffusion satisfies the Fokker-Planck equation, a second order parabolic PDE. As was shown in [10], the regular parts of the conditional moments of the velocity of a reciprocal diffusion satisfy a sequence of conservation laws similar to those of continuum mechanics. More precisely, by the term conditioned on the position, we mean conditioned on $d^0\xi(t, dt) = x$ as in Theorem 2.2. By velocity we mean $d^1\xi(t, dt)$ divided by dt and by the regular part of the moments of the velocity, we mean the part that is $O(1)$ in dt.

The zero order moment of the velocity conditioned on the position is the probability density $\overline{\rho}(x, t, dt)$. This is the density of $d^0\xi(t, dt) = x$ and it is not hard to show that

$$\overline{\rho}(x, t, dt) = \rho(x, t)(1 + O(dt)) \tag{5.1}$$

so the regular part is just $\rho(x, t)$.

The regular part of the first moment of the velocity conditioned on the position is the centered velocity $v(x, t)$ as defined in (2.29). The regular part of the second moment of the velocity conditioned on the position is the momentum flux coefficient $P(x, t)$ as defined in (2.30).

The first conservation law is called the continuity equation and is well-known in stochastic mechanics [16],

$$\frac{\partial}{\partial t}\rho(x,t) = -\frac{\partial}{\partial x^i}\left(\rho(x,t)v^i(x,t)\right). \tag{5.2}$$

It expresses the fact that probability mass is neither created nor destroyed under the mean flow. In other words the time rate of change of the probability of a volume element is only due to the flux of probability through the boundary of the element. The probability flux is $\rho(x,t)v(x,t)$ which can also be interpreted as the momentum density.

The second conservation law was first derived in [10], see also [13]. It is similar to Euler's equation of continuum mechanics,

$$\frac{\partial}{\partial t}\left(\rho(x,t)v^i(x,t)\right) = \rho\left(f^i(x,t) + g^i_j(x,t)v^j(x,t)\right) - \frac{\partial}{\partial x^j}\left(\rho(x,t)P^{ij}(x,t)\right). \tag{5.3}$$

It expresses the fact that the time rate of change of momentum in a volume element is due to the mean forces acting inside the volume element plus the flux of momentum through the boundary of the volume element. The mean forces include a force $f(x,t)$ that depends only on position and one $g(x,t)v(x,t)$ that depends on position and linearly on mean velocity. The momentum flux is $\rho(x,t)P(x,t)$. The kinetic energy density is one half of the trace of the momentum flux.

We shall give a weak derivation of these two conservation laws for reciprocal diffusions using the theorems of Sect. 2 and the assumption that the density $\rho(x,t)$ goes to zero faster than $|x|^{-k}$ for any k as $|x|$ goes to ∞. We conjecture that they are the first two of an infinite sequence of conservation laws satisfied by the regular part of the higher moments. Without going into detail, if $P^{i_1\ldots i_r}(x,t)$ denotes the regular part of the r^{th} moment of $d^1\xi(t)$ conditioned on $d^0\xi(t)$ then the conjecture is

$$\begin{aligned}\frac{\partial}{\partial t}\left(\rho P^{i_1\ldots i_r}(x,t)\right) &= \rho\left(f^{i_j}P^{i_1\ldots\hat{i_j}\ldots i_r} + g^{i_j}_k P^{i_1\ldots\hat{i_j}\ldots i_r k}\right)\\ &\quad - \frac{\partial}{\partial x^k}\left(\rho P^{i_1\ldots i_r k}\right)\end{aligned} \tag{5.4}$$

where the hat as in $\hat{i_j}$ denotes a deleted index. The summation is over the repeated indices j,k. In [10] the third conservation law ($k=2$) was verified for Gaussian reciprocal diffusions for which it takes the form

$$\begin{aligned}\frac{\partial}{\partial t}\left(\rho P^{ij}\right) &= \rho\left(f^i v^j + v^i f^j + g^i_k P^{kj} + P^{ik}g^j_k\right)\\ &\quad - \frac{\partial}{\partial x^k}\left(\rho\left(v^i v^j v^k + \pi^{ij}v^k + \pi^{jk}v^i + \pi^{ki}v^j\right)\right),\end{aligned} \tag{5.5}$$

where

$$\pi^{ij}(x,t) = P^{ij}(x,t) - v^i(x,t)v^j(x,t). \tag{5.6}$$

For nonGaussian processes there is an extra term in the flux of (5.5).

This conservation law expresses the balance of kinetic energy and work for every scalar process of the form $\zeta(t) = \lambda_i\xi^i(t)$. In particular, if we take half of the trace of (5.5), the terms involving g cancel by skewsymmetry and we obtain

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho P^{ii}\right) = \rho f^i v^i - \frac{\partial}{\partial x^k}\left(\rho\left(\frac{1}{2}v^i v^i v^k + \frac{1}{2}\pi^{ii}v^k + \pi^{ik}v^i\right)\right) \quad (5.7)$$

We see that the left side is the time rate of change of the density of kinetic energy. This equals the right side which is the rate at which work is done against the forces acting on particles within a volume element plus the flux of kinetic energy carried by particles entering or leaving the volume element. The flux has contributions from both the mean and random parts of the motion. The mean and random kinetic energy carried by the mean motion are the first and second terms of the flux. The third term of the flux is the mixed mean-random kinetic energy carried by the random motion. This term resembles a viscosity, but it is not. These equations are associated to a random process. In each realization of the process there is only a single particle so there is no viscosity or friction between particles. This term is due to the flux of energy from the random motion of particles between regions of different mean velocities. As a result of this random exchange of particles, regions of slow mean velocity are energized by nearby regions of fast mean velocity and regions of fast mean velocity are deenergized by nearby regions of slow mean velocity. This takes the appearance of viscosity.

These conservation laws hold for reciprocal diffusions whose sample paths are nowhere differentiable almost surely. In particular, it makes no sense to talk of a joint density of position and velocity. Instead we consider the joint density of the recipocal diffusion at two nearby times $\xi(t+dt), \xi(t-dt)$ which we recoordinatize as $d^0\xi(t,dt), d^1\xi(t,dt)$. The density of $d^0\xi(t,dt), d^1\xi(t,dt)$ can be factored into the density $\bar{\rho}(x,t,dt)$ of $d^0\xi(t,dt)$ times the conditional density of $d^1\xi(t,dt)$ given $d^0\xi(t,dt)$. As dt goes to 0, the latter density becomes singular but the regular parts of its conditional moments seem to obey the standard conservation laws.

We now weakly derive the first conservation law (5.2). As test functions, we use C^∞ functions $\phi(x,t)$ with compact support in (t_0,t_f). A test functions and its first and second partials should grow no faster than a polynomial as x goes to ∞, i.e. for some integer k

$$\frac{|\phi(x,t)|}{|x|^k} \to 0 \quad as \quad |x| \to \infty, \quad (5.8)$$

$$\frac{|\phi_{,i}(x,t)|}{|x|^k} \to 0 \quad as \quad |x| \to \infty, \quad (5.9)$$

$$\frac{|\phi_{,ij}(x,t)|}{|x|^k} \to 0 \quad as \quad |x| \to \infty. \quad (5.10)$$

Partition the interval $[t_0,t_f]$ into subintevals $[t_r,t_{r+1}]$ where $t_r = t_0 + rdt, t_f = t_0+(N+1)dt$. For any test function ϕ, define $\phi(t) = \phi(\xi(t),t)$. Because the support of ϕ is in (t_0,t_f), for dt sufficiently small

$$0 = \sum_{r=1}^{N} d^1\phi(t_r) \quad (5.11)$$

and hence

$$\begin{aligned} 0 &= E\left(\sum_{r=1}^{N} d^1\phi(t_r)\right) \\ 0 &= E\left(\sum_{r=1}^{N} E\left(d^1\phi(t_r)|d^0\xi(t_r)\right)\right). \end{aligned} \tag{5.12}$$

But

$$\begin{aligned} d^1\phi(t) &= \frac{1}{2}\left(\phi(t+dt) - \phi(t-dt)\right) \\ &= \frac{1}{2}\left(\phi(t+dt) - \phi(d^0\xi(t),t) + \phi(d^0\xi(t),t) - \phi(t-dt)\right). \end{aligned} \tag{5.13}$$

We expand in a Taylor series around $d^0\xi(t), t$ and obtain

$$d\phi = \phi_{,i} d^1\xi^i + \phi_{,0} dt + O(d^1\xi, dt)^3 \tag{5.14}$$

so by Theorem (2.2)

$$E\left(d^1\phi|d^0\xi\right) = \phi_{,i} v^i dt + \phi_{,0} dt + O(dt)^2. \tag{5.15}$$

We plug this into (5.12) and let dt go to zero to obtain

$$0 = \int \int_{t_0}^{t_f} \rho(x,t)\left(\phi_{,i}(x,t)v^i(x,t) + \phi_{,0}(x,t)\right) dt\,dx \tag{5.16}$$

which we recognize as the weak form of (5.2).

To derive the second conservation law we start with the fact that for dt sufficiently small

$$0 = \frac{1}{dt}\sum_{r=1}^{N} d^2\phi(t_r), \tag{5.17}$$

and hence

$$\begin{aligned} 0 &= E\left(\frac{1}{dt}\sum_{r=1}^{N} d^2\phi(t_r)\right), \\ 0 &= E\left(\frac{1}{dt}\sum_{r=1}^{N} E\left(d^2\phi(t_r)|d^0\xi(t_r)\right)\right). \end{aligned} \tag{5.18}$$

But

$$\begin{aligned} d^2\phi(t) &= \left(\phi(t+dt) - 2\phi(t) + \phi(t-dt)\right) \\ &= \phi(t+dt) - \phi(d^0\xi(t),t) + \phi(t-dt) - \phi(d^0\xi(t),t) \\ &\quad -2(\phi(\xi(t),t) - \phi(d^0\xi(t),t)). \end{aligned} \tag{5.19}$$

Again we expand in a Taylor series around $d^0\xi(t), t$ and obtain

$$
\begin{aligned}
d^2\phi &= \phi_{,i} d^2\xi + \phi_{,ij}\left(d^1\xi^i d^1\xi^j - \frac{1}{4} d^2\xi^i d^2\xi^j\right) \\
&\quad +2\phi_{,i0} d^1\xi^i dt + \phi_{,00} dt^2 \\
&\quad +\frac{1}{24}\phi_{,ijk} d^2\xi^i d^2\xi^j d^2\xi^k \\
&\quad +\frac{1}{12}\phi_{,ijkl}\left(d^1\xi^i d^1\xi^j d^1\xi^k d^1\xi^l - \frac{1}{16} d^2\xi^i d^2\xi^j d^2\xi^k d^2\xi^l\right) \\
&\quad +O(d^2\xi)^5 + O(d^1\xi)^5 + dtO(d^1\xi)^3 + dt^2 O(d^1\xi).
\end{aligned} \tag{5.20}
$$

By Theorem (2.2)

$$
\begin{aligned}
E\left(d^2\phi | d^0\xi\right) &= \phi_{,i}\left(f^i + g^i_j v^j\right) dt^2 + \phi_{,ij} P^{ij} dt^2 \\
&\quad +2\phi_{,i0} v^i dt^2 + \phi_{,00} dt^2 \\
&\quad +O(dt)^3.
\end{aligned} \tag{5.21}
$$

We plug this into (5.18) and let dt go to zero to obtain

$$
0 = \int \int_{t_0}^{t_f} \rho\left(\phi_{,i}(f^i + g^i_j v^j) + \phi_{,ij} P^{ij} + 2\phi_{,i0} v^i + \phi_{,00}\right) dt\, dx. \tag{5.22}
$$

By the weak form of the first conservation law (5.16)

$$
0 = \int \int_{t_0}^{t_f} \rho\left(\phi_{,i0} v^i + \phi_{,00}\right) dt\, dx \tag{5.23}
$$

hence (5.22) becomes after integration by parts

$$
0 = \int \int_{t_0}^{t_f} \phi_{,i}\left(\rho(f^i + g^i_j v^j) - \frac{\partial}{\partial x^j}(\rho P^{ij}) - \frac{\partial}{\partial t}(\rho v^i)\right) dt\, dx \tag{5.24}
$$

which we recognize as the weak form of (5.3).

6 Markov diffusions

Suppose $\xi(t)$ is a Markov diffusion satisfying the Ito equation (1.10). Its density $\rho(x,t)$ satisfies the familiar Fokker-Planck PDE,

$$
\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x^i}(\rho(x,t) b^i(x,t)) - \frac{1}{2}\frac{\partial^2}{\partial x^i \partial x^i}\rho(x,t) = 0. \tag{6.1}
$$

From (2.7) and (2.9) the centered velocity of a Markov process is

$$
v^i(x,t) = b^i(x,t) - \frac{1}{2}\frac{\partial}{\partial x^i}\ln\rho(x,t) \tag{6.2}
$$

and substituting this into the Fokker-Planck equation yields the continuity equation (5.2). From this and the definition of g (2.24) it follows that

$$
g^i_j = v^i_{,j} - v^j_{,i} \tag{6.3}
$$

Recall that π, the regular part of the conditional variance of the velocity (2.30), is given by

$$\pi^{ij}(x,t) = P^{ij}(x,t) - v^i(x,t)v^j(x,t). \tag{6.4}$$

The second conservation law in Lagrangian form is

$$\begin{aligned}\frac{\partial}{\partial t}v^i(x,t) + v^j(x,t)\frac{\partial}{\partial x^j}v^i(x,t) &= f^i(x,t) + g^i_j(x,t)v^j(x,t) \\ &\quad - \frac{1}{\rho(x,t)}\frac{\partial}{\partial x^j}\left(\rho(x,t)\pi^{ij}(x,t)\right). \end{aligned}\tag{6.5}$$

This equation can be solved for the last term using the Fokker-Planck equation (6.1) and the expressions for f (2.23), g (2.24) and v (6.2) to obtain

$$\frac{\partial}{\partial x^j}\left(\rho\left(\pi^{ij} - \frac{1}{4}\frac{\partial^2}{\partial x^i \partial x^j}\ln\rho\right)\right) = 0. \tag{6.6}$$

In [13] it is shown that a stronger condition holds, namely,

$$\pi^{ij} = \frac{1}{4}\frac{\partial^2}{\partial x^i \partial x^j}\ln\rho. \tag{6.7}$$

Since they are satisfied by all Markov diffusions, equations (6.3) and (6.7) are called the *Markov closure rules* for the sequence of conservation laws.

7 Quantum diffusions

Schrödinger's original motivation for studying the reciprocal property was an attempt to give a description of quantum mechanics in stochastic terms. As developed by Fényes [6], Nelson [16] and many others, this program has come to be called *stochastic mechanics*. The basic idea is to associate with a wave function satisfying Schrödinger's equation, a related diffusion process, usually a Markov diffusion. The density of the diffusion should equal the square modulus of the wave function and certain other relations should hold. We refer the reader to [16] for a more complete description of stochastic mechanics.

In this section we shall argue that if there is a stochastic description of quantum mechanics and if this description employs reciprocal diffusions then it does not involve the subclass of Markov diffusions. Rather it must be in terms of a disjoint subclass which we have termed *quantum diffusions*, [13]

Mechanics is intrinsically second order so a definition of stochastic acceleration is needed. There are several possible definitions of stochastic acceleration. We have discussed three of them, the stochastic acceleration a of Nelsen (2.17), the stochastic acceleration of Zambrini-Cruziero (2.41) and our stochastic acceleration in terms of f, g as given in Theorem 2.1. For smooth processes all reduce to the classical acceleration but for diffusions they differ as was pointed out by Thieullen [20]. The latter two are quite similar. The Zambrini-Cruziero stochastic acceleration equals $f(x,t) + g(x,t)v(x,t)$ and is a function of position alone

$x = \xi(t)$. Ours equals $f(x,t) + g(x,t)dx/dt$ and is function of centered position $x = (\xi(t+dt)+\xi(t-dt)/2$ and centered velocity $dx/dt = (\xi(t+dt)-\xi(t-dt)/(2dt)$. In one space dimension they are identical as $g = 0$.

To see the difference between Nelson's stochastic acceleration and the other two, consider an Ornstein-Uhlenbeck velocity process defined as the stationary solution of the scalar Ito equation

$$d^+\xi = -\xi dt + d^+ w \tag{7.1}$$

where the stationary density $\rho(x)$ is Gaussian, zero mean and variance $1/2$. The forward drift is $b(x,t) = -x$, the backward drift as computed from (2.7) is $\bar{b}(x,t) = x$. Nelson's stochastic acceleration is $a(x,t) = -x$ while $f(x,t) = x$ and $g(x,t) = 0$. From Nelson's point of view the particle appears to be moving in attracting force field while from other points of view it is moving in a repelling field. If the equation where not stochastic but instead were deterministic

$$\frac{d}{dt}\xi = -\xi \tag{7.2}$$

then another differentiation yields the second order differential equation

$$\frac{d^2}{dt^2}\xi = \xi \tag{7.3}$$

which is motion in a repelling field.

Moreover Nelson's acceleration is not a reciprocal invariant as it depends on the density through (2.7). For a nonstationary solution of the Ornstein-Uhlenbeck equation (7.1), Nelson's stochastic acceleration changes with time because $\rho = \rho(x,t)$. In space dimensions higher than one, the Zambrini-Cruziero stochastic acceleration is not a reciprocal invariant because it contains v. On the other hand f, g are reciprocal invariants, independent of the density. It is for these reasons that we believe that the appropriate definition of stochastic acceleration is $f(x,t) + g(x,t)dx/dt$ where $x = (\xi(t+dt) + \xi(t-dt)/2$ and $dx/dt = (\xi(t+dt) - \xi(t-dt)/(2dt)$.

In flat space with $\hbar = 1$, the Schrödinger equation takes the form

$$\imath\frac{\partial\psi}{\partial t} = \left[\frac{1}{2}\left(\imath\frac{\partial}{\partial x^j} + A_j\right)\left(\imath\frac{\partial}{\partial x^j} + A_j\right) + \phi\right]\psi \tag{7.4}$$

where ϕ, A_i are a scalar and covector potentials and $\imath$ is the square root of -1. If we assume that

$$\psi = \exp(R + \imath S) \tag{7.5}$$

and take imaginary and real parts of Schrödinger equation (7.4) we obtain

$$\frac{\partial R}{\partial t} = -\frac{1}{2}\frac{\partial^2 S}{\partial x^j \partial x^j} - \frac{\partial R}{\partial x^j}\frac{\partial S}{\partial x^j} + \frac{1}{2}\frac{\partial A_j}{\partial x^j} + A_j\frac{\partial R}{\partial x^j} \tag{7.6}$$

and

$$\frac{\partial S}{\partial t} = \frac{1}{2}\left(\frac{\partial^2 R}{\partial x^j \partial x^j} + \frac{\partial R}{\partial x^j}\frac{\partial R}{\partial x^j} - \frac{\partial S}{\partial x^j}\frac{\partial S}{\partial x^j} - A_j A_j\right) - A_j \frac{\partial S}{\partial x^j} - \phi. \tag{7.7}$$

To this solution $\psi(x,t)$ of the wave equation, we would like to associate a reciprocal diffusion $\xi(t)$ so we make the following assumptions that are standard in stochastic mechanics [16], namely, that the density ρ and centered velocity v of $\xi(t)$ satisfy

$$\rho = |\psi|^2 = \exp 2R, \tag{7.8}$$

$$v^i = \frac{\partial S}{\partial x^i} - A_i. \tag{7.9}$$

It is well-known [16] that under these assumptions the continuity equation (5.2) is equivalent to the imaginary part of Schrödinger's equation (7.6). If we also assume that the stochastic acceleration (in our sense) of the reciprocal diffusion should be the same as that experienced by a classical particle moving in the field induced by the same potentials ϕ, A_i, i.e.,

$$f^i = -\frac{\partial \phi}{\partial x^i} - \frac{\partial A_i}{\partial t} \tag{7.10}$$

$$g^i_j = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} \tag{7.11}$$

then it follows by comparing the Euler equation (5.3) with the real part of Schrödinger's equation that $\pi(x,t)$ of $\xi(t)$ must satisfy

$$\frac{\partial}{\partial x^j}\left(\rho\left(\pi^{ij} + \frac{1}{4}\frac{\partial^2}{\partial x^j \partial x^j}\ln\rho\right)\right) = 0. \tag{7.12}$$

Notice the difference between this and the similar equation for Markov processes (6.7). From this we conclude that if the reciprocal diffusion $\xi(t)$ corresponding to the wave function $\psi(x,t)$ satisfies the above assumptions (7.8-7.11), then $\xi(t)$ is not Markov.

In [13], we defined the class of *quantum diffusions* to be the reciprocal process satisfying the *quantum closure relations*

$$g^i_j = v^i_{,j} - v^j_{,i}, \tag{7.13}$$

$$\pi^{ij} = -\frac{1}{4}\frac{\partial^2}{\partial x^j \partial x^j}\ln\rho. \tag{7.14}$$

The second of these is a strengthened form of (7.12). They are equivalent for Gaussian reciprocal processes because for such processes π does not depend on x, $\pi(x,t) = \pi(t)$, see [12]. As is shown in [14], (7.14) implies the Heisenberg Uncertainty Principle.

Nelson [16] assumes that the stochastic process $\xi(t)$ corresponding to the wave function $\psi(x,t)$ satisfies (7.8,7.9) and that his stochastic acceleration is the same as that experienced by a classical particle moving in the field induced by the same potentials ϕ, A_i, i.e.

$$a^i = -\frac{\partial \phi}{\partial x^i} - \frac{\partial A_i}{\partial t} + \left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} \right) v^j \tag{7.15}$$

instead of (7.10,7.11). If this process is reciprocal then the second conservation law (5.3) implies that (6.6) holds hence the process could be Markov.

Therefore depending on which expressions one takes for the stochastic acceleration, one obtains different reciprocal processes corresponding to a wave function. We refer the interested reader to [14] for a fuller discussion of this point.

8 Conclusions

The theory of Markov diffusions includes the mean differential discription in terms of the diffusion postulates (2.1, 2.2,2.3), the Ito integration of first order stochastic differential equations and the connection with parabolic partial differential equations, the Fokker-Planck and Kolmogorov forward and backward equations. In this paper we have laid out parallel components of the theory of reciprocal diffusions in flat space, the second order mean differential discription, Theorem 2.1, the second order integral discription, Theorems 3.2, 3.4 and the connection with conservation laws, Sect. 5. Much work remains to be done, in particular, in extending the theory to curved space where the geometry will play a significant role.

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