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SUMMARY
Generalized Isoperimetric Problem*

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We consider the generalized isoperimetric problem stated as follows.

$$\int_0^1 \mid u(\tau) \mid^2 d\tau \to inf, \tag{1}$$

where $u(\cdot) : [0,1] \to \mathbf{R}^m$ is subject to the additional constraining relations, i.e.,

$$\begin{aligned} \dot{x} &= u(t), \\ \dot{y} &= B(x)u(t), \\ x(0) &= \bar{x}, x(1) = \hat{x}, \\ y(0) &= \bar{y}, y(1) = \hat{y} \end{aligned} \tag{2}$$

with $x \in \mathbf{R}^m$, $y \in \mathbf{R}^n$, $B(x) = \{b_1(x), \dots, b_m(x)\}$.

The points $(\bar{x}, \bar{y}), (\hat{x}, \hat{y}) \in \mathrm{R}^{m+n}$ are assumed to be fixed beforehand. The minimum of (1) is said to be a sub-Riemannian distance between $(\bar{x}, \bar{y})$ and $(\hat{x}, \hat{y})$. We address it also as a sub-Riemannian length.

The vector fields $B(x) = \{b_1(x), b_2(x), \dots, b_m(x)\}$ are assumed to be C^∞-vector-fields such that the Lie algebra generated by

$$\{\frac{\partial}{\partial x_j} + \sum_i b_{ji}(x) \frac{\partial}{\partial y_i}\}_{j=1}^m$$

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has the full rank at any point $x \in \mathrm{R}^m$. We will call such family of vector fields controllable and also refer to $B(x)$ as controllable family of vector fields.

Introduce the matrix

$$G(x(t))p = \left\{ \left\langle \frac{\partial}{\partial x_i} b_j(x) - \frac{\partial}{\partial x_j} b_i(x), p \right\rangle \right\}_{j,i=1}^{m}, \tag{3}$$

where j and i enumerate rows and columns, respectively.

Theorem 1 *Let $B(x)$ be a controllable family of vector fields. Then for any $(\bar{x}, \bar{y}), (\hat{x}.\hat{y}) \in R^{m+n}$ one can find a sub-Riemannian length minimizer $(x(t), y(t))$ which measures the sub-Riemannian distance between $(\bar{x}, \bar{y})$ and $(\hat{x}, \hat{y})$. Moreover, $(x(t), y(t))$ is a solution of the following boundary value problem*

$$\begin{cases} p_0 \cdot \ddot{x} = (G(x(t))p)\dot{x}, \\ \dot{y} = B(x)\dot{x}, \end{cases} \tag{4}$$

$$x(0) = \bar{x}, x(1) = \hat{x},$$

$$y(0) = \bar{y}, y(1) = \hat{y}$$

where $(p_0,p) \in R^{1+n} \setminus 0$ is a real vector.

We propose a necessary condition of extremum when $B(x)$ being Λ-uniform, i.e., defined as follows.

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Definition *The family of vector fields $B(x)$ is said to be Λ-uniform if there exists a linear operator $\Lambda : R^n \to R^n$ such that*

$$\sum_{\nu} x_\nu \cdot \frac{\partial}{\partial x_\nu} \left(\frac{\partial}{\partial x_i} b_j(x) - \frac{\partial}{\partial x_j} b_i(x) \right) = \Lambda \left(\frac{\partial}{\partial x_i} b_j(x) - \frac{\partial}{\partial x_j} b_i(x) \right)$$

$$\forall x \in R^m.$$

Theorem 2 *Let $B(x)$ be a controllable family of vector fields. Suppose further that $B(x)$ is Λ -uniform. Then for any $(\bar{x}, \bar{y}), (\hat{x}.\hat{y}) \in R^{m+n}$ one can find a sub-Riemannian length minimizer $(x(t), y(t))$ which measures the sub-Riemannian distance between $(\bar{x}, \bar{y})$ and $(\hat{x}, \hat{y})$. Moreover, $(x(t), y(t))$ is a solution of the boundary value problem (4), where for $(p_0, p) \in R^{1+n} \setminus 0$ the following condition holds*

$$p_0 \cdot |\dot{x}(t)|^2 = p_0 \cdot (\langle \hat{x}, \dot{x}(1) \rangle - \langle \bar{x}, \dot{x}(0) \rangle) +$$
$$\left\langle (I + \frac{1}{2} \cdot \Lambda)^T p. \bar{y} - \hat{y} - \int_0^1 B(\hat{x} + (\bar{x} - \hat{x})\tau)(\bar{x} - \hat{x}) d\tau \right\rangle +$$
$$\int_0^1 \langle (G(\hat{x} + (\bar{x} - \hat{x})\tau)p)(\bar{x} - \hat{x}), \hat{x} + (\bar{x} - \hat{x})\tau \rangle \, d\tau, \ \forall t \in [0, 1],$$

where $G(x)p$ is defined in (3).

We illustrate the application of Theorems 1, 2 by the analysis of generalized Dido's problem of the N-th order.

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