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Design of controllers for MG3 compressor models with general characteristics using graph backstepping☆

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Graph backstepping procedure for designing stabilizing controllers for Moore–Greitzer three-dimensional models describing rotating stall and surge in compressors with general characteristics is presented.

Abstract

We discuss the design of controllers for compressors with general characteristics, including the right-skewed ones. The controllers can be represented graphically by plotting their throttle surfaces. The graphical representation of controllers enhances understanding of controller action and allows to compare various controllers. We discuss the issue of the right skewness of a compressor characteristic. We show how to design controllers using a graph backstepping procedure involving the construction of the throttle surface. We show that for a quite general compressor characteristic, every potential axisymmetric equilibrium on the decreasing part of the compressor characteristic, the peak of the characterisitic, and every rotating stall equilibrium close to the peak can be globally or semi-globally stabilized by an appropriate choice of the throttle surface and the controller gains. We obtain lower bounds on the gains of the controller in terms of the divided differences related to compressor characteristic. These bounds can be expressed using some bounds on the first and second derivatives of the characteristic in the region of operation. In this way we establish a direct relationship between the shape of the compressor characteristic and the required controller gains. We discuss controllers that stabilize a range of the desired equilibria and guarantee a soft bifurcation of the equilibria as a set-point parameter varies. We give an example of controller design for a right-skew compressor. We provide simple general guidelines for choosing the throttle surface. © 1999 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Surge and rotating stall are complex nonlinear phenomena that limit the perfrmance of compressors. Moore and Greitzer (1986) developed a simple model (MG3) that mirrors the observed behaviour. In this model, the compressor is treated as an actuator disk which is coupled to the upstream and downstream flow fields. This results in a system of 2 ODEs and a PDE. The PDE is Galerkin projected onto its first circumferential Fourier mode and the result is a system of 3 ODEs.

$$\dot{A} = \sigma I_2(\Phi, A),$$

$$\dot{\Phi} = \frac{1}{l_c}(I_1(\Phi, A) - \Psi), \tag{1}$$

$$\dot{\Psi} = \frac{1}{4l_cB^2}(\Phi - K_T\sqrt{\Psi}),$$

$$I_1(\Phi, A) := \frac{1}{2\pi}\int_0^{2\pi}\Psi_c(\Phi + A\sin\theta)\,d\theta,$$

$$I_2(\Phi, A) := \frac{1}{\pi}\int_0^{2\pi}\Psi_c(\Phi + A\sin\theta)\sin\theta\,d\theta. \tag{2}$$

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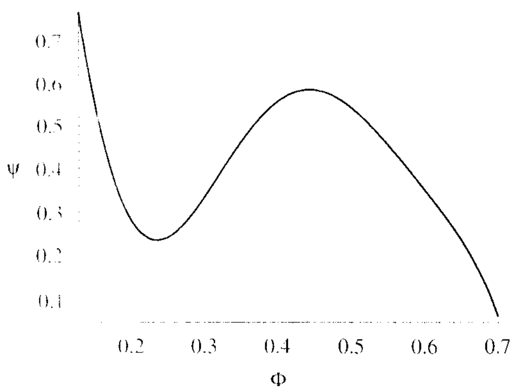


Fig. 1. C_3 compressor characteristic.

The three state variables of this model are Φ, the nondimensionalized annulus averaged mass flow coeficient through the compressor, Ψ, the nondimensionalized annulus averaged pressure rise coeficient across the compressor, and A, the nondimensionalized amplitude of the first Fourier mode of the mass flow through the compressor. The function $\Psi_c(\phi)$ is called the compressor characteristic and is found empirically. Its value is the local pressure rise when the local mass flow is ϕ. For most compressors it has an S shape as seen in Fig. 1. The parameters σ, l_c, B are determined by the geometry of the compressor and the throttle parameter K_T is the fraction of the throttle opening. Since the throttle parameter can be varied, it will be used as the control.

We assume that A, the magnitude of the first Fourier coefficient of the flow coefficient at the compressor face, is always nonnegative.

We assume that the compressor characteristic $\Psi_c(\Phi)$ is a general S-shaped curve (like the one shown on Fig. 1). In particular, we assume that:

1. The characteristic has one peak (Φ_0, Ψ_0) and, to the left of the peak, one well. The characteristic is strictly decreasing to the right of the peak and to the left of the well, it is strictly increasing between the well and the peak.
2. The characteristic has exactly one inflection point $(\Phi_{\text{infl}}, \Psi_{\text{infl}})$ between the well and the peak. One has $\Psi_c''(\Phi) > 0$ for $\Phi < \Phi_{\text{infl}}$ and $\Psi_c''(\Phi) < 0$ for $\Phi > \Phi_{\text{infl}}$, i.e., the characteristic is strictly concave to the right of the inflection point, and its strictly convex to the left of the inflection point.

Fig. 1 shows a characteristic that was obtained by fitting a sixth-degree polynomial to a piecewise cubic characteristic of C_3 compressor considered in (Mansoux et al., 1994). Unless otherwise specified, all figures and examples presented in this paper refer to this characteristic, that we will call C_3' characteristic.

The throttle parameter K_T is considered to be the control variable. The control goal is to maintain a maximum possible pressure rise coefficient at the design point. This requires choosing the throttle parameter such that the closed-loop system has a unique equilibrium at the peak or close to the peak. However, this goal cannot be achieved with a constant throttle parameter, since for the low values of the throttle parameter corresponding to equilibria on the compressor characteristic close to the peak one usually creates some additional *rotating stall* equilibria through a (hard) *saddle-node bifurcation*. Moreover, for a low value of the throttle parameter, as the axisymmetric equilibrium approaches the peak, its domain of attraction shrinks and a small disturbance can cause the state of the system to settle at a rotating stall equilibrium with a smaller value of the pressure rise coefficient or to undergo the so-called *surge* cycle. If the state of the system settles at a rotating stall equilibrium, one must open the throttle (i.e., increase the value of the throttle parameter) until the stall equilibria disappear through a saddle-node bifurcation. However, the state of the system does not return immediately to the desired equilibrium on the compressor characteristic close to the peak. Instead, it settles on the compressor characteristic at a point with much lower pressure rise coefficient than desired. To force the state of the system to go back to the design point with a higher value of the pressure rise coefficient one has to close the throttle again at the cost of creating the stall equilibria. As we can see, returning to a design point close to the peak after stability loss involves going through a *hysteresis* loop. (See McCaughan (1990) for a detailed bifurcation analysis of the MG3 model with a cubic compressor characteristic and a constant throttle parameter.)

By allowing the throttle parameter to be a *function* of the state rather than a *constant*, we can improve performance of the compression system. First of all, by an appropriate choice of the throttle function and controller gains we can stabilize the axisymmetric equilibria to the right of the peak, or the peak itself, and, at the same time, eliminate the rotating stall equilibria.

The compressor characteristic is often not known exactly and it depends on many factors that can change in time. We would like to choose the throttle parameter so that the closed-loop system has only one stable equilibrium on the compressor characteristic close to the peak. However, it may happen that the throttle surface may be off from its desired position (for instance, because of an incorrect estimation of the peak position or a disturbance) and some rotating stall equilibria will appear. In this case, we will be interested in designing the throttle surface so that there is only one stable rotating stall equilibrium close to the peak that is created via a (soft) *supercritical pitchfork bifurcation*. By changing a set-point parameter the equilibrium will continuously go back to the design point, avoiding a hysteresis.

In Liaw and Abed (1996) a controller that locally stabilized a desired equilibrium and eliminated hysteresis has been designed. It used information about the second derivative of the characteristic at the peak. In Eveker et al. (1995) a controller that simultaneously controlled stall and surge and avoided hysteresis has been designed. The controller was experimentally validated for a low-speed 3-stage compressor. Behnken et al. (1995) discusses using the air injection for control of the rotating stall. An application of oscillatory control to compressors has been studied in Baillieul et al. (1995).

Most approaches for designing controllers for compression systems assumed some particular form of the compressor characteristic. For instance, following the original suggestion of Moore and Greitzer (1986), Eveker et al. (1995) and Krstić (1995) (see also Krstić et al., 1995) used a *cubic* parametrization. Assuming this parametrization, it has been shown in Krstić (1995), Krstić et al. (1995) that *global* stabilization of a range of equilibria is possible. However, as it was observed, the cubic parametrization could not account for the existence of stall equilibria for the values of the mass flow coefficient to the right of the peak of the characteristic, which was experimentally observed for some compressors. As it was noticed in Jankovic (1995), such compressors exhibit a deep hysteresis and are difficult to control. Deep hysteresis has been related in Sepulchre and Kokotovic (1996) to the fact that the characteristic to the left of the peak drops off faster than to the right. This property has been since called the *right-skewness* of the characteristic.

To study the right-skewness and its impact on controller design, a two-sine parametrization has been used in Sepulchre and Kokotovic (1996). Recently, another parametrization that allows to describe the right-skewness has been proposed in Krstić and Wang (1996). Local stabilization of a range of equlibria has been obtained in Sepulchre and Kokotovic (1996) and Krstić and Wang (1996). Minimum sensing requirements for feedback control have been also studied in these papers. While Sepulchre and Kokotovic (1996) used a specific parametrization to obtain local stabilization of the desired equilibria, Krstić and Wang (1996) allowed a general characteristic and used information about derivatives of the compressor characteristic near the peak for a local analysis.

Using specific parametrizations of compressor characteristics was certainly useful for a *local* study of the right skewness and its impact on *local* stabilization of a range of equilibria. However, using specific parametrizations had several deficiences. The parametrizations had to be simple enough to allow for explicit evaluation of the integrals in Eqs. (1) and keep the number of terms occurring on the right-hand sides of these equation resulting from the integration low enough to make the analysis tractable. None of the simple parametrizations used so far was general enough to fit experimentally obtained data for most existing compressors. A piecewise cubic used in Mansoux et al. (1994) provided a good fit to data, but it did not allow explicit evaluation of the integrals in (1). On the other hand, using a general high-degree polynomial parametrization would allow for both good fit to experimental data and explicit evaluation of the integrals in Eqs. (1), but (because of a large number of terms involved) would make the analysis tractable. Even a simple cubic characteristic $\Psi_c(\Phi)$ when substituted into Eqs. (1) creates *many terms* on the right-hand sides of these equation. As a result, in analysis and controller design one has to deal with complicated interactions of these terms. In contrast, in the original form, Eqs. (1) have *few terms* that, as we will show, can be relatively easily analysed on a general level. It is our opinion that using specific form of a compressor characteristic makes analysis and controller design *less general* and *more difficult* and actually *obscures* understanding of what features of the characteristic are crucial for analysis and control design.

As a result of exploiting particular parametrizations the existing results on controller design for compressors are either global for specific classes of characteristics, or local for general characteristics. The present paper tries to fill the gap by providing *global* results for *general* characteristics.

Our approach differs from existing approaches for designing controllers for compression systems. It does not use any *particular parametrization* of the compressor characteristic. Instead, it directly uses information about the *slopes* and *curvatures* of the characteristic, i.e., information about the *shape* of the characteristic. The knowledge of the exact values of the slopes and curvatures of the characteristic is *not required*, knowing some upper and lower bounds suffices. In that sense the designed controller is *robust* with respect to changes in the shape of the characteristic, as long as certain inequalities involving the controller gains and some quantities depending on the slopes and curvatures of the characteristic are satisfied. The controller is obtained in a *graphical* way by specifying the position of the corresponding *throttle surface*.

We will show that for a quite general compressor characteristic, if one neglects the actuator saturation and bandwidth limitations, *every potential axisymmetric equilibrium on the decreasing part of the compressor characteristic, the peak of the characterisitic, and every rotating stall equilibrium close to the peak can be globally or semi-globally stabilized by an appropriate choice of the throttle surface and value of controller gains.*

Of course, global or semi-global stabilization of a compression system is not a practical goal. The Moore–Greitzer model is certainly not valid globally and some initial conditions far from the design point will never occur in practice. We prove global or semi-global stabilization merely to achieve two practical goals:

1. We can guarantee an *arbitrarily large domain of attraction* of the desired equilibrium.

2. We construct a family of *invariant sets* for the flow of the closed-loop system (the level sets of Lyapunov function that we construct in the paper).

The analytical description of the controller design that we present in this paper uses some divided differences related to the compressor characteristic and requires some notation that makes the main results difficult to read. However, all results presented in the paper have a *simple graphical interpretation.* To make the results easier to understand we provided many figures. We encourage the reader to think geometrically and try to sketch the missing graphs whenever possible.

The paper is organized as follows. In Section 2 we introduce the notion of a throttle surface. We show how various controllers can be analysed in a uniform way. We also study possible equilibria of the closed-loop system. In Section 3 we recall some basic facts about divided differences. In Section 4 we present a simple explanation of the effect of the right skewness of the compressor characteristic on the location of the stall equilibria. Section 5 contains a graph backstepping procedure for construction of a controller. The main results of the paper are contained in this section. Section 6 discusses controllers that depend on a set-point parameter. Conditions for stabilization of a range of equilibria and enforcing a soft bifurcation are given in this section. In Section 7 we give an example of a controller design using the graph backstepping procedure. We provide simple general guidelines for choosing the throttle surface in Section 8.

2. Controller design by choosing the throttle function

The controllers considered in the recent papers (Krstić, 1995; Krstić et al., 1995; Krstić and Wang, 1996; Sepulchre and Kokotovic, 1996), are of the form

$$K_T := (c_\Psi \Psi + h(\Phi, A))/\sqrt{\Psi}\,, \tag{3}$$

$$K_T := (d_\Phi \dot{\Phi} + h(\Phi, A))/\sqrt{\Psi}\,, \tag{4}$$

or

$$K_T := (c_\Psi \Psi + d_\Phi \dot{\Phi} + h(\Phi, A))/\sqrt{\Psi}\,, \tag{5}$$

where $h(\Phi, A)$ is a function and c_Ψ, d_Φ are nonzero constants. The implementation of the controller defined by (4) is different from that defined by (3) because of the different sensing requirements. However, since $\dot{\Phi} = (1/l_c)(I_1(\Phi, A) - \Psi)$, from the point of view of the dynamics of the closed-loop system, one can treat a controller defined by (4) or (5) (if $c_\Psi - d_\Phi/l_c \neq 0$) as a special case of the controller defined by (3). In the sequel we will work with a controller of form (3) keeping in mind that the analysis of controllers of form (4) or (5) will be similar. This allows us to treat various controllers in a uniform way.

For our purpose it is convenient to represent the system equations (1) in the equivalent form

$$\begin{aligned} \dot{A} &= \sigma I_2(\Phi, A), \\ \dot{\Phi} &= \frac{1}{l_c}(I_1(\Phi, A) - \Psi), \\ \dot{\Psi} &= \frac{-c_\Psi}{4 l_c B^2}(\Psi - \tilde{\Psi}(\Phi, A)), \end{aligned} \tag{6}$$

where

$$\tilde{\Psi}(\Phi, A) := (\Phi - h(\Phi, A))/c_\Psi\,. \tag{7}$$

We call the function $\tilde{\Psi}(\Phi, A)$ the *throttle function* and its graph the *throttle surface.* Now, construction of a controller within the specified class is equivalent to choosing the throttle function $\tilde{\Psi}(\Phi, A)$ and the value of gain c_Ψ. (In general, c_Ψ does not have to be a constant. If c_Ψ is a function of Φ and A, we will use notation $c_\Psi(\Phi, A)$.) The corresponding function $h(\Phi, A)$ can be calculated from (7) and then the throttle area K_T can be obtained from (3).

Observe that the equilibria of the system occur at the intersections of the surfaces on which $\dot{A} = 0$, $\dot{\Phi} = 0$, and $\dot{\Psi} = 0$. We denote these surfaces by S_A, S_Φ, and S_Ψ, respectively. Note that the time derivatives of the variables A, Φ, and Ψ change sign when the trajectories of the system (6) cross the surfaces S_A, S_Φ, and S_Ψ. This justifies calling S_A, S_Φ, and S_Ψ the *turning surfaces* for the corresponding variables. Since the turning surface for Ψ is determined by the throttle opening, we will also refer to S_Ψ as the *throttle surface.* Note that we have

$$\begin{aligned} S_A &= \{(A, \Phi, \Psi)\colon I_2(\Phi, A) = 0\}, \\ S_\Phi &= \{(A, \Phi, \Psi)\colon \Psi = I_1(\Phi, A)\}, \\ S_\Psi &= \{(A, \Phi, \Psi)\colon \Psi = \tilde{\Psi}(\Phi, A)\}. \end{aligned} \tag{8}$$

We will also distinguish the curves

$$\begin{aligned} C_{A,\Phi} &:= S_A \cap S_\Phi, \\ C_{\Phi,\Psi} &:= S_\Phi \cap S_\Psi, \\ C_{A,\Psi} &:= S_A \cap S_\Psi, \end{aligned} \tag{9}$$

and we will call them the *turning curves* for, respectively, A and Φ, Φ and Ψ, and A and Ψ. Note that, in general, some of the turning curves may be empty. Turning surfaces S_A and S_Φ, the curve of potential equilibria $c_{A,\Phi}$, and its projections onto A–Φ and Φ–Ψ planes for C'_3 compressor are shown on Fig. 2.

Note that the throttle actuation alone cannot change the turning surfaces for S_A and S_Φ, or their intersection $C_{A,\Phi}$. However, we assume that we can freely choose the throttle surface. (In this paper we neglect the restrictions that the saturation of the control throttle imposes on the position of the throttle surfaces.) The equilibria of the system (6) are the elements of the set $E := C_{A,\Phi} \cap S_\Psi$. For a typical compressor characteristic, the curve

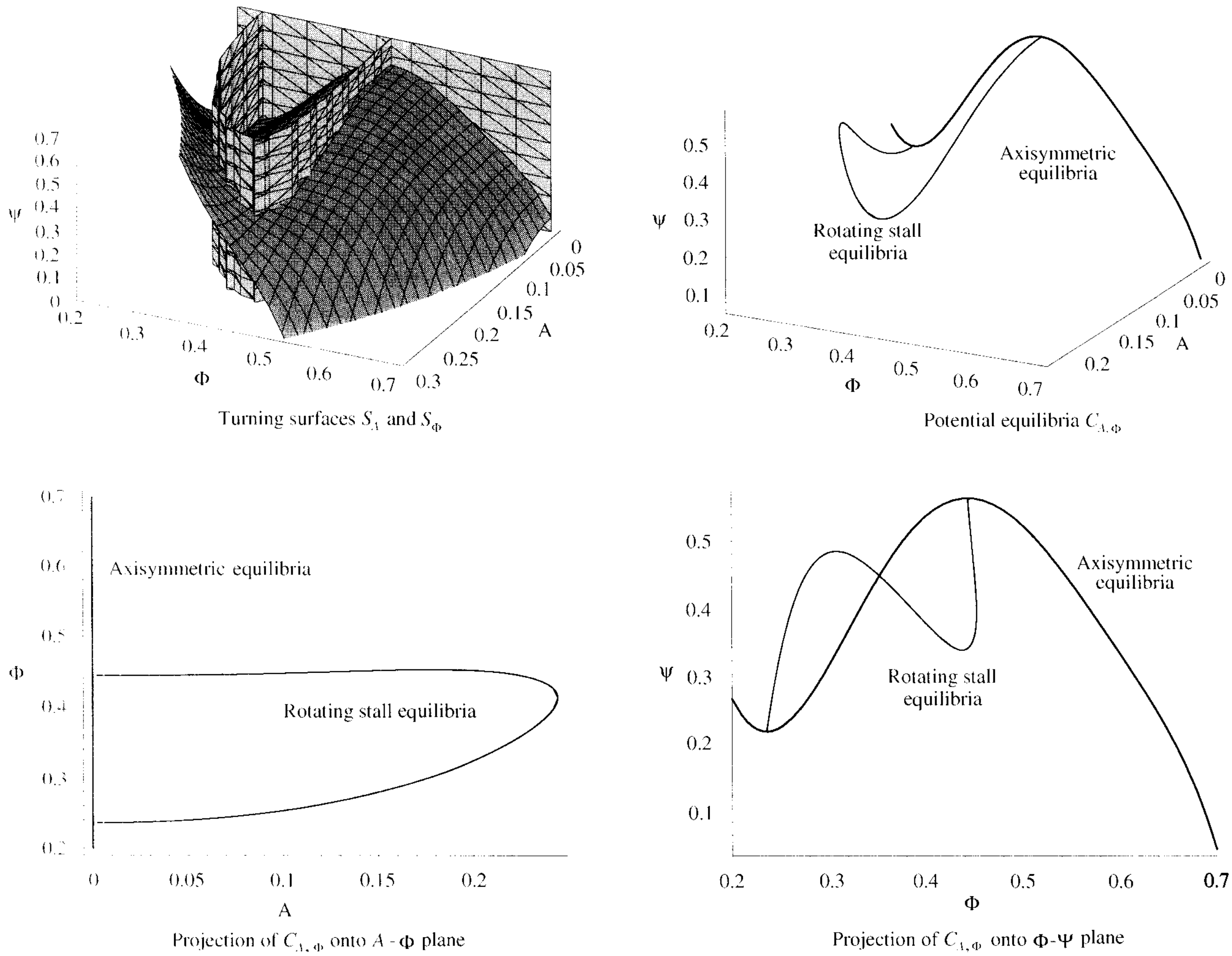


Fig. 2. Turning surfaces S_A and S_Φ for C_3 compressor and potential equilibria.

$C_{A,\Phi}$ has *two branches*: (potential) axisymmetric equilibria $C^{\mathrm{ax}}_{A,\Phi}$ (corresponding to $A = 0$) and (potential) rotating stall equilibria $C^{\mathrm{rs}}_{A,\Phi}$ (corresponding to $A > 0$).

Our goal is to place the throttle surface S_Ψ (and choose the appropriate value of gain c_Ψ) so that there is only one stable equilibrium for the closed-loop system that correspond to a high value of the pressure rise coefficient. We will show that for a quite general compressor characteristic *every potential axisymmetric equilibrium on the decreasing part of the compressor characteristic, the peak of the characterisitic, and every rotating stall equilibrium close to the peak can be globally or semi-globally stabilized by an appropriate choice of the throttle surface and value of gain* c_Ψ.

This result can be used to construct stabilizing controllers in two ways:

1. One can first choose the form of the controller and then prove that the corresponding throttle surface and the gain c_Ψ satisfy the conditions of the main theorem. (In Section 7 we provide an example of such a design.)
2. One can first choose the throttle surface and the gain c_Ψ that satisfy the conditions of the main theorem and then obtain the corresponding controller.

3. Some properties of the divided differences

Since our goal is to study bifurcations of the equilibrium at the peak and design controllers that guarantee some (possibly large) domain of attraction, we are going to use normal forms that allow studying ODEs on large domains. For this, we use *divided differences*.

Let $f(x)$ be a function defined on an open set containing x_0 and suppose that f is differentiable at x_0. We define the *divided difference* of f with respect to x relative to x_0 as

$$\delta_{x,x_0} f(x) := \begin{cases} \dfrac{f(x)-f(x_0)}{x-x_0} & \text{for } x \neq x_0, \\ f'(x_0) & \text{for } x = x_0. \end{cases} \qquad (10)$$

Observe that the divided difference $\delta_{x,x_0} f(x)$ has a graphical interpretation as the slope of the secant line connecting points $(x, f(x))$ and $(x_0, f(x_0))$ on the graph of f.

Let $k \geq 2$. One can define the kth divided difference of f relative to x_0 by the recursive formula

$$\delta^k_{x,x_0} f(x) := \delta_{x,x_0} \delta^{k-1}_{x,x_0} f(x).$$

Note that $\delta_{x,x_0} f(x)$ is a continuous function of x, whenever f is continuous and differentiable at x_0.

Lemma 1. *Assume that f is piecewise differentiable on $[x_0, x]$. Then*

$$\delta_{x,x_0} f(x) = \int_0^1 f'(x_0 + s(x - x_0))\, \mathrm{d}s. \tag{11}$$

Note that one can represent $f(x)$ as

$$f(x) = f(x_0) + \delta_{x,x_0} f(x)(x - x_0). \tag{12}$$

The operation of taking the divided differences is comutative.

Lemma 2. *Assume that f is continuous on its domain and twice differentiable at x_1 and x_2. Then.*

$$\delta_{x,x_1} \delta_{x,x_2} f(x) = \delta_{x,x_2} \delta_{x,x_1} f(x). \tag{13}$$

Let $f(x, y)$ be a function of two variables defined on an open subset of $\mathbb{R}^2$. Assume that f has a partial derivative $\partial f/\partial x$ at (x_0, y) for all real y such that (x_0, y) is in the domain of f. We define the *divided difference* of f with respect to x relative to x_0 as

$$\delta_{x,x_0} f(x, y) := \begin{cases} \dfrac{f(x, y) - f(x_0, y)}{x - x_0} & \text{for } x \neq x_0, \\ \dfrac{\partial f}{\partial x}(x_0, y) & \text{for } x = x_0. \end{cases} \tag{14}$$

Similarly, assume that f has a partial derivative $\partial f/\partial y$ at (x, y_0) for all real x such that (x, y_0) is in the domain of f. We define the *divided difference* of f with respect to y relative to y_0 as

$$\delta_{y,y_0} f(x, y) := \begin{cases} \dfrac{f(x, y) - f(x, y_0)}{y - y_0} & \text{for } y \neq y_0, \\ \dfrac{\partial f}{\partial y}(x, y_0) & \text{for } y = y_0, \end{cases} \tag{15}$$

Lemma 3. *Suppose that $f(x, y)$ has piecewise continuous partial derivatives. Then*

$$\begin{aligned} \delta_{x,x_0} f(x, y) &= \int_0^1 \left.\frac{\partial f(\xi, y)}{\partial \xi}\right|_{\xi = x_0 + s(x - x_0)} \mathrm{d}s, \\ \delta_{y,y_0} f(x, y) &= \int_0^1 \left.\frac{\partial f(x, \xi)}{\partial \xi}\right|_{\xi = y_0 + s(y - y_0)} \mathrm{d}s. \end{aligned} \tag{16}$$

For more properties of divided differences we refer to de Boor (1978).

4. Right skewness

It is convenient to introduce the divided difference of $I_2(\Phi, A)$ with respect to A, relative to $A = 0$:

$$J_2(\Phi, A) := \delta_{A,0} I_2(\Phi, A). \tag{17}$$

Since $I_2(\Phi, 0) = 0$, we have

$$I_2(\Phi, A) = I_2(\Phi, 0) + \delta_{A,0} I_2(\Phi, A) A = J_2(\Phi, A) A. \tag{18}$$

Note that the potential equilibria of system (6) occur when either $A = 0$ (axisymmetric equilibria) or $J_2(\Phi, A) = 0$ for $A > 0$ (rotating stall equilibria) (Fig. 3). It is known that for some compressors the branch of

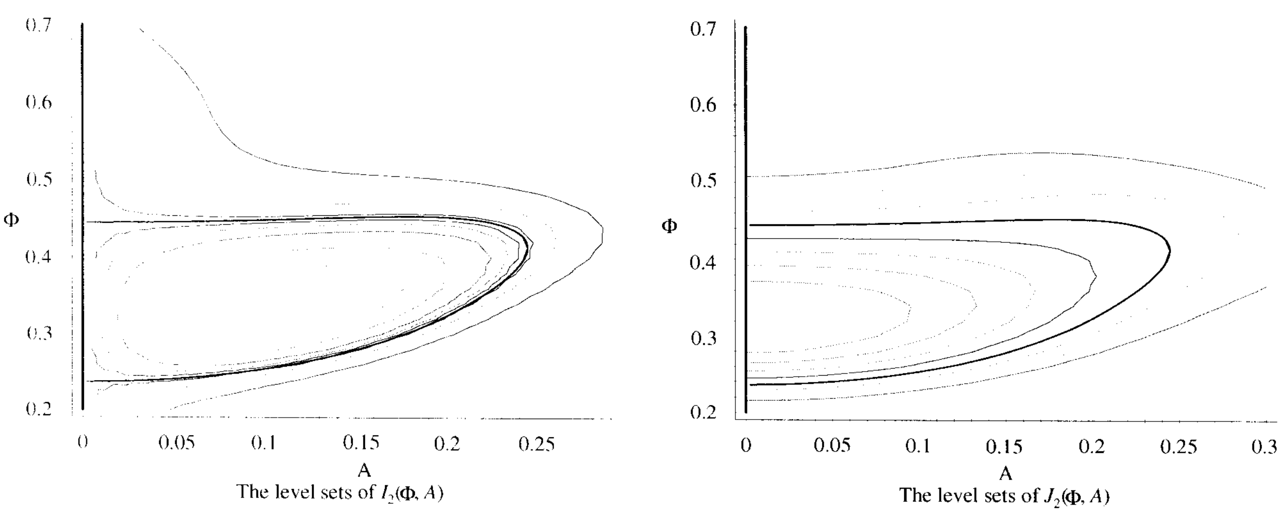


Fig. 3. The level sets of $I_2(\Phi, A)$ and $J_2(\Phi, A)$.

$C_{A,\Phi}$ corresponding to the rotating stall equilibria exists for some values of the mass flow coefficient larger than the value at the peak of the compressor characteristic. Such compressors are said to possess a *right-skewed* characteristic. It has been noticed that the right skewness is caused by the fact that the compressor characteristic decreases faster to the left of the peak than to the right. The recent papers (Sepulchre and Kokotovic, 1996; Krstić and Wang, 1996) explain the right-skewness using some particular forms of the compressor characteristic, where one parameter, the so-called *shape signifier* is responsible for the right skewness. It is possible to choose the parameters in the class of curves considered in the cited references to fit any given characteristic at a point together with first two derivatives and thus analyse the right skewness of the given characteristic *locally*. However, such an approach does not address the fact that the right skewness is actually a *non-local* phenomenon, as it depends on the shape of the compressor characteristic on a large interval. In this section we attempt to analyse the right skewness directly using the shape of the compressor characteristic.

For simplicity, assume that the characteristic has only one inflection point $(\Phi_{\text{infl}}, \Psi_{\text{infl}})$ between the well and the peak.

It is possible to explain the right skewness directly using the definition of $I_2(\Phi, A)$. For this, let us rewrite $I_2(\Phi, A)$ as

$$I_2(\Phi, A) := \frac{2}{\pi}\int_0^{\pi/2} (\Psi_c(\Phi + A\sin\theta) - \Psi_c(\Phi - A\sin\theta))\sin\theta\,\mathrm{d}\theta. \tag{19}$$

Note that the value of $I_2(\Phi, A)$ is an integral of the differences of the values of the function Ψ_c at the points symmetrically located on both sides of the value of Φ, weighted by the sine function. The stall equilibria correspond to $I_2(\Phi, A) = 0$ for $A > 0$. Having $I_2(\Phi, A) = 0$ is possible if the differences $\Psi_c(\Phi + A\sin\theta) - \Psi_c(\Phi - A\sin\theta)$ average to zero as θ varies from 0 to $\pi/2$. If the compressor characteristic drops off faster to the left than to right of the peak, the values of Φ at which the differences average to zero for small values of A will be slightly to the right of the peak. For larger values of A, and for θ near $\pi/2$, $\Phi - A\sin\theta$ will be at the region to the left of the inflection point Φ_{infl} and the values of Ψ_c at that region will not be decreasing as fast (as a function of A) as for small values of A, while the values of $\Psi_c(\Phi + A\sin\theta)$ on the decreasing slope to the right of the peak will be decreasing even faster. To obtain a zero average difference, the value of Φ will have to shift to the left of the peak. As A increases even further, $\Phi - A\sin\theta$ will reach the region of the backflow negative slope of the characteristic, so that the value of Φ corresponding to the stall equilibrium will move even further to the left. Eventually, for a sufficiently large A, the average differences $\Psi_c(\Phi + A\sin\theta) - \Psi_c(\Phi - A\sin\theta)$ will be negative for all values of Φ and hence the stall equilibria will no longer be possible. In particular, this simple analysis also shows that there cannot be any stall equilibria for $\Phi > \Phi_0 + A$, as then the differences $\Psi_c(\Phi + A\sin\theta) - \Psi_c(\Phi - A\sin\theta)$ are all negative. This fact can be used in the control design (see Proposition 6).

An alternative explanation of the right skewness involves examining $J_2(\Phi, A)$. Assume that Ψ_c is piecewise differentiable. Then, from (2) and Lemma 3 we obtain the following expression for $J_2(\Phi, A)$:

$$J_2(\Phi, A) = \frac{1}{\pi}\int_0^1\int_0^{2\pi} \Psi_c'(\Phi + sA\sin\theta)\sin^2\theta\,\mathrm{d}\theta\,\mathrm{d}s. \tag{20}$$

The stall equilibria correspond to $J_2(\Phi, A) = 0$. This is possible only if the average slope of the compressor characteristic between $\Phi - A$ and $\Phi + A$ (weighted by $\sin^2\theta$) is zero. For $\Phi = \Phi_0$ and for small values of A, as long as the positive slopes of the characteristic between $\Phi - A$ and Φ overcome the negative slopes between Φ and $\Phi + A$, the average slope will be positive. To obtain the zero average slope for the same values of A, we must have $\Phi > \Phi_0$. However, for values of A such that $\Phi - A < \Phi_{\text{infl}}$, to maintain the average zero slope between Φ and $\Phi + A$, we must have $\Phi < \Phi_0$. For a sufficiently large A, $\Phi + sA\sin\theta$ varying between $\Phi - A$ and $\Phi + A$ will pick more negative slopes of the characteristic on the backflow slope and to the right of the peak than the positive slopes between the well and the peak of the characteristic and the stall equilibria will no longer be possible with large values of A for any value of Φ.

This analysis shows that the right skewness is actually a *nonlocal* phenomenon, as it depends on the slopes of the compressor characteristic over a large range of values of the mass flow coefficient Φ. Examining the slopes only near the peak is not sufficient to determinine the right or left skewness of the characteristic. To see that, consider a hypothetical compressor whose characteristic initially drops slower to the left than to the right of the peak, but then exhibits a sharp drop to the left (see Fig. 4). The branch of rotating stall equilibria will initially depart to the left from the peak as for a left-skew characteristic, but then will turn back right and enter the region of mass flow coefficients larger than the value at the peak. This characteristic should certainly be classified as a right-skew one, even though near the peak it is locally left skew.

We define the *right-skewness coefficient* $\beta_{\text{rs},1}$ of the characteristic as the smallest slope of the straight line on the A–Φ plane passing through the point $(0, \Phi_0)$ with the property that the projection of the stall equilibria loop onto A–Φ plane lies below that line, i.e.,

$$\beta_{\text{rs},1} := \inf\{a\colon I_2(\Phi_0 + aA, A) \geq 0,\ \forall A \geq 0\}. \tag{21}$$

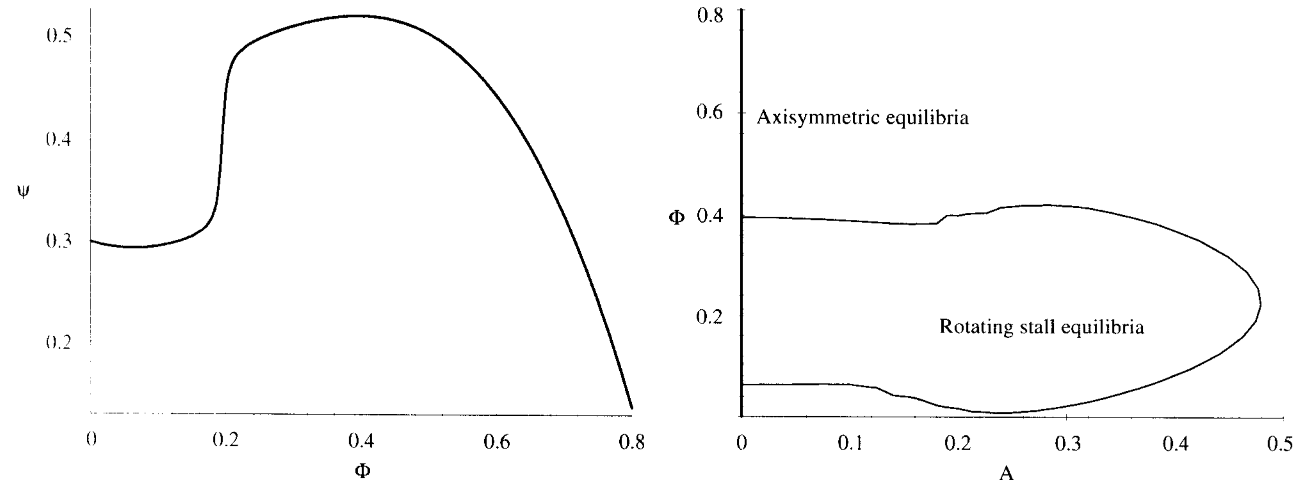


Fig. 4. Characteristic and projection of the stall equilibria onto A–Φ plane for a right skew but locally left skew compressor.

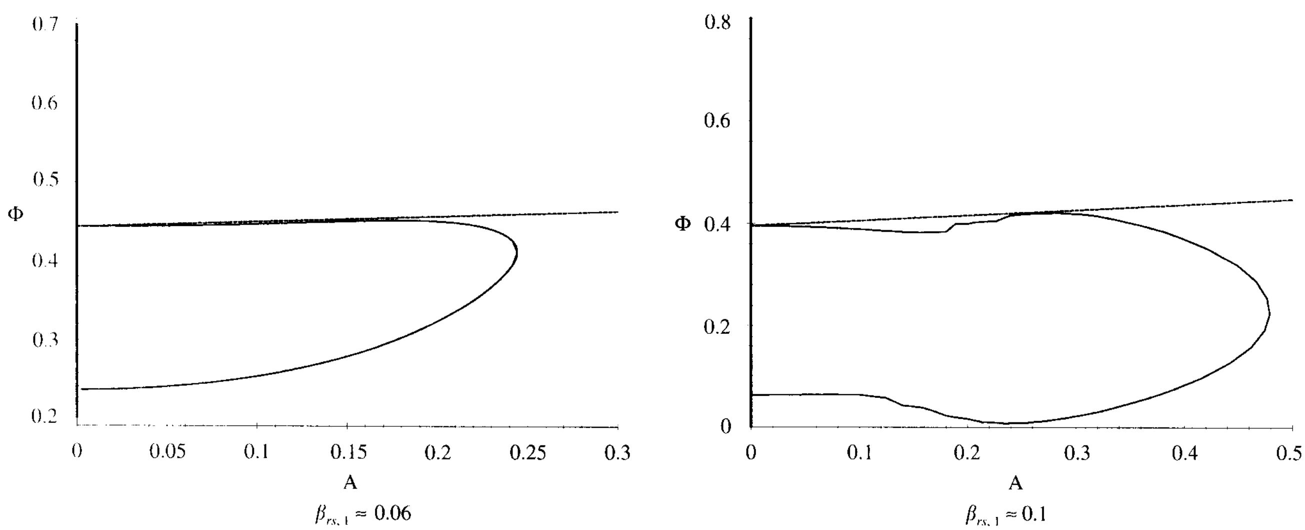


Fig. 5. The right-skewness coefficients for two compressor characteristics.

(The subscript 1 indicates that the right-skewness coefficient is defined with respect to the first Fourier mode A of the rotating stall cell.)

One can prove that for all compressor characteristics $\Psi_c(\Phi)$ with continuous second derivative at the peak one has

$$0 \leq \beta_{rs,1} < 1.$$

We will say that the compressor characteristic is *right skew* iff $\beta_{rs,1} > 0$ and *left skew* iff $\beta_{rs,1} = 0$ (Fig. 5).

5. Stabilization of a desired equilibrium

The goal of compressor design is to obtain maximum possible pressure rise at the design point. This requires choosing the throttle surface such that the closed-loop system has a unique equilibrium at the peak or close to the peak. The compressor characteristic is often not known exactly. Moreover, the characteristic depends on many factors that can change in time. First of all, the characteristic may be different at different wheel speeds. The compressor blades wear off in time, and this may affect the characteristic as well. Moerover, many possible disturbances that may occur in the system, like distortion, noise, as well as the effect of a non-axisymmetric actuation (by air injection), can be roughly represented in the Moore–Greitzer model as changes of the compressor characteristic. Therefore, it is safe to assume that the compressor characteristic and, in particular, the position of the peak are not known exactly and can vary in time. On the other hand, a throttle disturbance can be

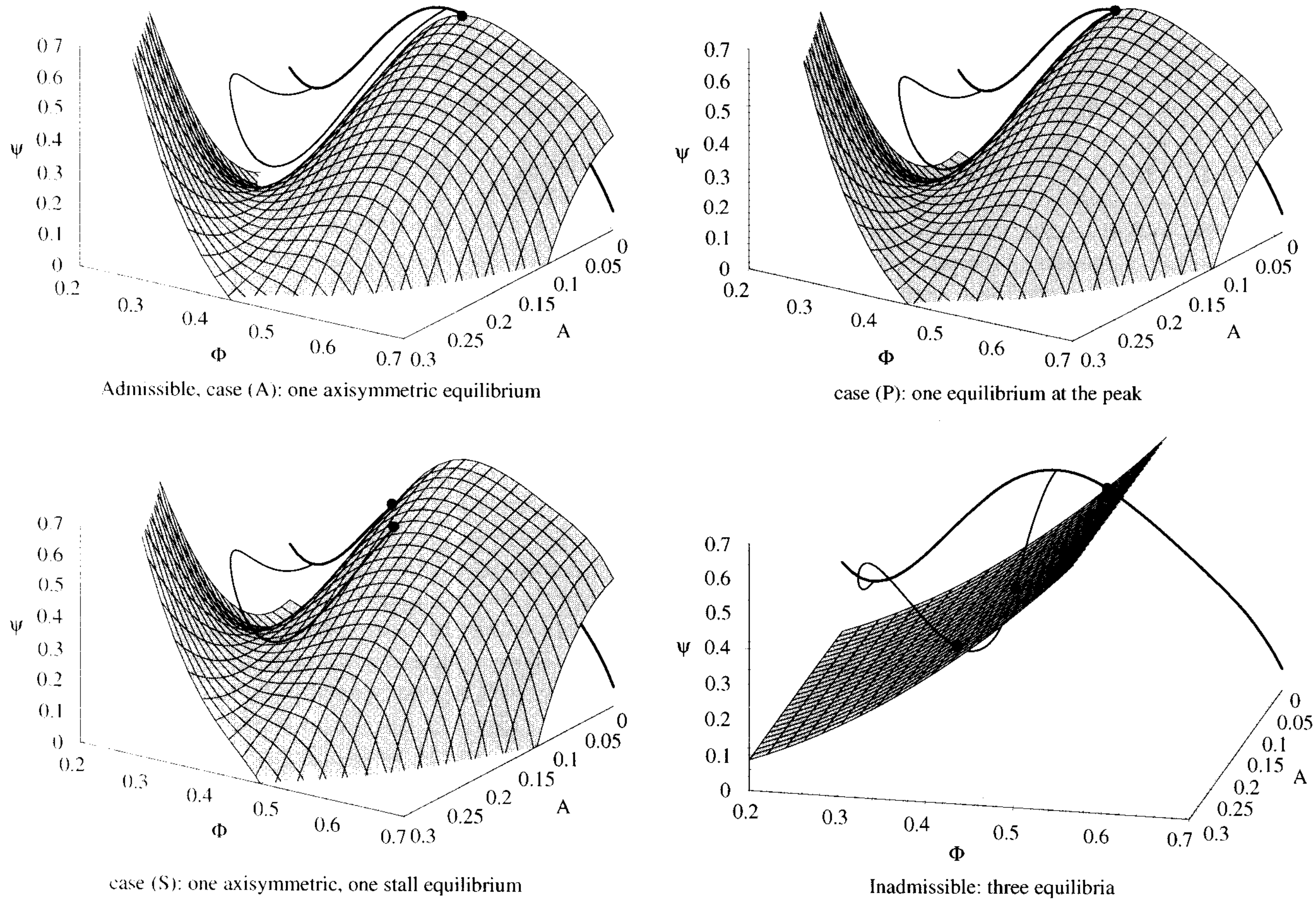


Fig. 6. Admissible and inadmissible positions of the throttle surface.

represented as a shift of the throttle surface. Therefore, the controller design must be robust with respect to possible change of the characteristic and the throttle surface.

Since we want to have only one stable equilibrium of the closed-loop system, we would like to choose the throttle surface so that there is only one intersection of the throttle surface with axisymmetric branch of $C_{A,\Phi}$ close to the peak of the compressor characteristic. However, it may happen that the throttle surface may shift from its desired position (for instance, because of an incorrect estimation of the peak position or the throttle disturbance) and some nonaxisymmetric equilibria will appear. In this case, we will be interested in designing the throttle surface so that there is only one stable rotating stall equilibrium (with a small magnitude of the stall cell). To encompass all possible situations, we define our goal as a stabilization of an equilibrium (A_1, Φ_1, Ψ_1) close to the peak of the characteristic. We say that the throttle surface (function) is *admissible* if we have one of the following three cases:

Case (A): The equilibrium set of the closed-loop system (6), $C_{A,\Phi} \cap S_\Psi$, contains a single axisymmetric equilibrium (A_1, Φ_1, Ψ_1) on the decreasing part of the compressor characteristic to the right of the peak (i.e., $A_1 = 0$ and $\Phi_1 > \Phi_0$).

Case (P): The equilibrium set of the closed-loop system (6), $C_{A,\Phi} \cap S_\Psi$, consists of the peak of the compressor characteristic, i.e., $(A_1, \Phi_1, \Psi_1) = (0, \Phi_0, \Psi_0)$.

Case (S): The equilibrium set of the closed-loop system (6), $C_{A,\Phi} \cap S_\Psi$, consists of two points: (A_1, Φ_1, Ψ_1) with $A_1 > 0$ (rotating stall equilibrium) and (A_2, Φ_2, Ψ_2) with $A_2 = 0$ and $\Phi_2 < \Phi_0$ (axisymmetric equilibrium).

Note that in the case (S) we have $I_2(\Phi, A) > 0$ for all points in a neighborhood of $(0, \Phi_2, \Psi_2)$ with $A > 0$. Thus, any perturbation of $(0, \Phi_2, \Psi_2)$ to a nearby point with $A > 0$ will cause the stall cell to grow. Therefore, the point $(0, \Phi_2, \Psi_2)$ cannot be made stable. However, with an appropriate choice of the throttle function and gain c_Ψ we can make the stall equilibrium (A_1, Φ_1, Ψ_1) stable.

Fig. 6 shows examples of admissible and inadmissible positions of the throttle surface.

In the sequell (A_1, Φ_1, Ψ_1) will denote the equlibrium that we want to stabilize corresponding to one of the cases described above. Note that $A_1 = 0$ in the cases (A) and (P), and $A_1 > 0$ in the case (S). It is possible to consider the cases (A), (P), and (S) one at the time and design a controller separately for each of them. However, we are going to study the cases (A), (P), and (S) simultaneously, even though we realize that this may cause some confusion. The reason for studying all the cases simultaneously is that our ultimate goal is investigation of controllers that depend on a set-point parameter. As the set-point parameter varies, so does the throttle surface, and thus the equilibria of the closed-loop system. Since we are going to stabilize an equilibrium near the (possibly unknown) peak of the characteristic, each of the cases (A), (P), and (S) can occur for some values of the set-point parameter. We want to design a controller with gains independent of the set-point parameter that stabilizes the corresponding equilibrium in each case. For this, it is convenient to have a Lyapunov function that changes continuously with the stabilized equilibria, so that one can obtain bounds for the gains of the controller uniform in the set-point parameter.

We will assume that the projection of the curve $C_{\Phi,\Psi}$ onto A–Φ plane can be parametrized by a piecewise differentiable function $\tilde{\Phi}(A)$. This is equivalent to the assumption that the throttle function $\tilde{\Psi}(\Phi, A)$ is such that the equation $I_1(\Phi, A) = \tilde{\Psi}(\Phi, A)$ has a unique, piecewise differentiable solution $\tilde{\Phi}(A)$, i.e., we have

$$I_1(\tilde{\Phi}(A), A) = \tilde{\Psi}(\tilde{\Phi}(A), A).$$

The *error coordinates* are defined as follows:

$$\begin{aligned} e_A &:= A - A_1 \\ e_\Phi &:= \Phi - \tilde{\Phi}(A) \\ e_\Psi &:= \Psi - \tilde{\Psi}(\Phi, A). \end{aligned} \tag{22}$$

Proposition 4. *The variables e_A, e_Φ, and e_Ψ are global coordinates in $\mathbb{R}^3$.*

Proof. It is straightforward. □

We are going to express the dynamics of (6) in the error coordinates using some divided differences. Let

$$\begin{aligned} a_{11}(A) &:= \delta_{A,A_1} I_2(\tilde{\Phi}(A), A), \\ a_{12}(\Phi, A) &:= \delta_{\Phi,\Phi(A)} I_2(\Phi, A), \\ a_{22}(\Phi, A) &:= \delta_{\Phi,\Phi(A)} I_1(\Phi, A). \end{aligned} \tag{23}$$

We have

$$\begin{aligned} I_2(\Phi, A) &= I_2(\tilde{\Phi}(A), A) + \delta_{\Phi,\Phi(A)} I_2(\Phi, A) e_\Phi \\ &= a_{11}(A) e_A + a_{12}(\Phi, A) e_\Phi \end{aligned} \tag{24}$$

and

$$\begin{aligned} I_1(\Phi, A) &= I_1(\tilde{\Phi}(A), A) + \delta_{\Phi,\Phi(A)} I_1(\Phi, A) e_\Phi \\ &= I_1(\tilde{\Phi}(A), A) + a_{22}(\Phi, A) e_\Phi. \end{aligned} \tag{25}$$

Note that in the cases (A) and (P) we have $A_1 = 0$ and hence $a_{11}(A) = J_2(\tilde{\Phi}(A), A)$. Let $b(\Phi, A) := \delta_{\Phi,\Phi(A)} \tilde{\Psi}(\Phi, A)$.

Note that we have

$$\begin{aligned} \tilde{\Psi}(\Phi, A) &= I_1(\tilde{\Phi}(A), A) + \delta_{\Phi,(\Phi(A))} \tilde{\Psi}(\Phi, A)(\Phi - \tilde{\Phi}(A)) \\ &= I_1(\tilde{\Phi}(A), A) + b(\Phi, A) e_\Phi. \end{aligned} \tag{26}$$

Observe that, given the function $I_1(\Phi, A)$, the functions $\tilde{\Phi}(A)$ and $b(\Phi, A)$ are uniquely determined by the throttle function $\tilde{\Psi}(\Phi, A)$. However, the converse is also true: given the function $I_1(\Phi, A)$, the functions $\tilde{\Phi}(A)$ and $b(\Phi, A)$ uniquely determine the throttle function $\tilde{\Psi}(\Phi, A)$ by Eq. (26). In a backstepping controller design that we are going to present later, the functions $\tilde{\Phi}(A)$ and $b(\Phi, A)$ appear explicitly as the main design objects. Therefore, if the function $I_1(\Phi, A)$ were known exactly, one could actually choose the functions $\tilde{\Phi}(A)$ and $b(\Phi, A)$ first and then define the throttle function $\tilde{\Psi}(\Phi, A)$ by Eq. (26).

Note that the intersections of the graph of $\tilde{\Phi}(A)$ with the projection of the curve of potential equilibria $C_{A,\Phi}$ onto A–Φ plane represent the projections of the equilibria of the closed-loop system onto A–Φ plane. In fact, the graph of $\tilde{\Phi}(A)$ *determines* the equilibria of the closed-loop system. Thus, admissibility of the throttle function $\tilde{\Psi}(\Phi, A)$ can be studied in terms of the graph of $\tilde{\Phi}(A)$. Positions of the graph of $\tilde{\Phi}(A)$ corresponding to admisssible and inadmissible positions of the throttle surface are shown on Fig. 7.

Note that we have

$$\begin{aligned} I_1(\Phi, A) - \Psi &= I_1(\Phi, A) - \tilde{\Psi}(\Phi, A) - e_\Psi \\ &= (a_{22}(\Phi, A) - b(\Phi, A)) e_\Phi - e_\Psi. \end{aligned} \tag{27}$$

The dynamics of the closed-loop system in the error coordinates takes form

$$\begin{aligned} \dot{e}_A &= \sigma a_{11}(A) e_A + \sigma a_{12}(\Phi, A) e_\Phi, \\ \dot{e}_\Phi &= \frac{1}{l_c}(a_{22}(\Phi, A) - b(\Phi, A)) e_\Phi - \frac{1}{l_c} e_\Psi \\ &\quad - \sigma \tilde{\Phi}'(A)\,(a_{11}(A) e_A + a_{12}(\Phi, A) e_\Phi), \\ \dot{e}_\Psi &= \frac{-c_\Psi}{4 l_c B^2} e_\Psi - \frac{\partial \tilde{\Psi}(\Phi, A)}{\partial A} \sigma (a_{11}(A) e_A + a_{12}(\Phi, A) e_\Phi) \\ &\quad - \frac{\partial \tilde{\Psi}(\Phi, A)}{\partial \Phi} \left(\frac{1}{l_c}(a_{22}(\Phi, A) - b(\Phi, A)) e_\Phi - \frac{1}{l_c} e_\Psi \right). \end{aligned} \tag{28}$$

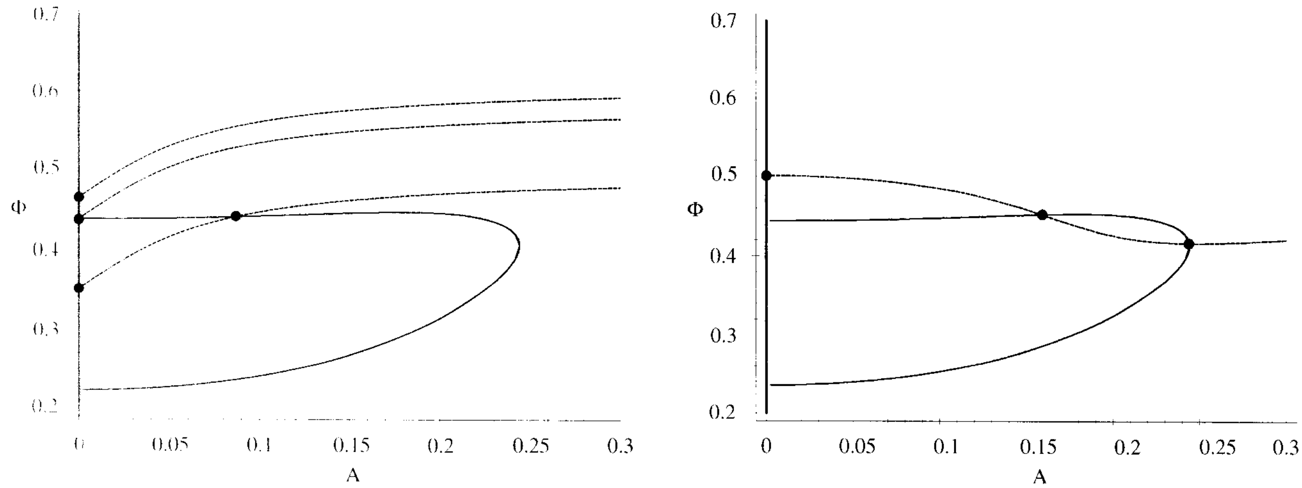
Note that the error dynamics can be made stable if we can make the main damping terms in the corresponding equations:

$\sigma a_{11}(A) e_A$, $(1/l_c(a_{22}(\Phi, A) - b(\Phi, A)) - \sigma \tilde{\Phi}'(A) a_{12}(\Phi, A)) e_\Phi$,

and

$(-c_\Psi / 4 l_c B^2 + 1/l_c \partial \tilde{\Psi}(\Phi, A)/\partial \Phi) e_\Psi$

sufficiently negative to overcome potentially destabilizing action of the crossterms like $\sigma a_{12}(\Phi, A) e_\Phi$, etc. It is convenient to think of construction of the throttle surface



Admissible: the cases (A), (P), and (S)

Inadmissible

Fig. 7. Admissible and inadmissible positions of the graph of $\tilde{\Phi}(A)$.

and the gain c_Ψ as a three-step *backstepping* process. Because this procedure has a simple graphical interpretation we shall call it *graph backstepping*.

Step 1. Choose $\tilde{\Phi}(A)$ so that the main damping coefficient in the e_A dynamics, $a_{11}(A) := \delta_{A,A_1} I_2(\tilde{\Phi}(A), A)$, is negative. (Graphically, this corresponds to choosing the intersection of the throttle surface and the turning surface S_Φ.) In the cases (A) or (P) this can be done simply by making sure that the graph of $\tilde{\Phi}(A)$ stays in the region where $I_2(\Phi, A)$ is negative, i.e., outside the stall equilibria loop (see Figs. 8 and 9). In the case (S), $I_2(\tilde{\Phi}(A), A)$ should be of the opposite sign to $e_A = A - A_1$. This means that the graph of $\tilde{\Phi}(A)$ should stay in the region where $I_2(\Phi, A) > 0$ (i.e., inside the stall equilibria loop) for $A < A_1$, cross the stall equilibria loop at $A = A_1$, and stay in the region where $I_2(\Phi, A) < 0$ (i.e., outside the stall equilibria loop) for $A > A_1$ (see Fig. 10).

Step 2. Choose $b(\Phi, A)$ so that the main damping coefficient in the e_Φ dynamics, $(1/l_c)(a_{22}(\Phi, A) - b(\Phi, A)) - \sigma\tilde{\Phi}'(A)a_{12}(\Phi, A)$, is sufficiently negative to overcome potentially destabilizing action of the crossterms in the first two equations. This corresponds to choosing the divided differences of the sections of the throttle surface with the planes $A = const$, for each fixed A. Graphically, this can be interpreted as choosing the slopes of the appropriate secant lines of the curves obtained as intersections of the throttle surface with the planes $A = const$.

Step 3. Choose c_Ψ so that the main damping coefficient in the e_Φ dynamics,

$$-c_\Psi/4l_c B^2 + \partial\tilde{\Psi}(\Phi, A)/\partial\Phi \; 1/l_c,$$

is sufficiently negative to overcome potentially destabilizing action of the crossterms in all three equations.

For several reasons, Step 1 requires the most detailed explanation. First of all, it is not clear what control one has over position of the graph of $\tilde{\Phi}(A)$. Looking at the graph of the turning surface S_Φ it seems obvious that one can place the throttle surface S_Ψ so that the graph of $\tilde{\Phi}(A)$ (which is the projection of the intersection $C_{\Phi,\Psi}$ of the surfaces S_Φ and S_Ψ onto the A–Φ plane) can be placed in the desired position in each of the cases (A), (P), and (S). (see Fig. 2). If one assumes perfect knowledge of the characterisitic $\Psi_c(\Phi)$ (and thus of the function $I_1(\Phi, A)$), one can simply choose $\tilde{\Phi}(A)$ to be the desired function and then, after performing Step 2 and constructing $b(\Phi, A)$, choose the throttle function to be $\tilde{\Psi}(\Phi, A) := I_1(\tilde{\Phi}(A), A) + b(\Phi, A)e_\Phi$. If one assumes that a measurement of $\dot{\Phi}$ is available, then, as we will see in Section 7, one can indeed freely choose $\tilde{\Phi}(A)$, as the knowledge of $\dot{\Phi}$ substitutes to some extent for the perfect knowledge of the characteristic $\Psi_c(\Phi)$ (and thus of the function $I_1(\Phi, A)$). If the perfect knowledge of the characteristic and the measurement of $\dot{\Phi}$ are not available, the function $\tilde{\Phi}(A)$ can still be chosen so that its graph has the desired properties, but the analysis in this case is a little bit more complicated. In this section we are going to assume that the function $\tilde{\Phi}(A)$ has been chosen so that its graph has the desired properties, i.e., it is admissible.

An important issue in Step 1 that distinguishes the cases (A), (P), and (S) is the character of dependence of the main damping term in the e_A dynamics, $I_2(\tilde{\Phi}(A), A)$, on e_A. The case (A) is the simplest one, as one can guarantee that $I_2(\tilde{\Phi}(A), A) = a_{11}(A)e_A = a_{11}(A)A$, with $a_{11}(A) = J_2(\tilde{\Phi}(A), A)$ negative and bounded away from zero for all $A \geq 0$, as long as the graph of $\tilde{\Phi}(A)$ stays away from the stall equilibria loop, i.e., it stays in the region where $J_2(\tilde{\Phi}(A), A)$ is strictly negative (see Fig. 8). Thus, in the case (A) one can guarantee at least *linear* damping in the e_A dynamics.

In the case (P), if the graph of $\tilde{\Phi}(A)$ stays away from the stall equilibria loop for $A > 0$, we still have $a_{11}(A) = J_2(\tilde{\Phi}(A), A) < 0$ for all $A > 0$. However, $a_{11}(A)$

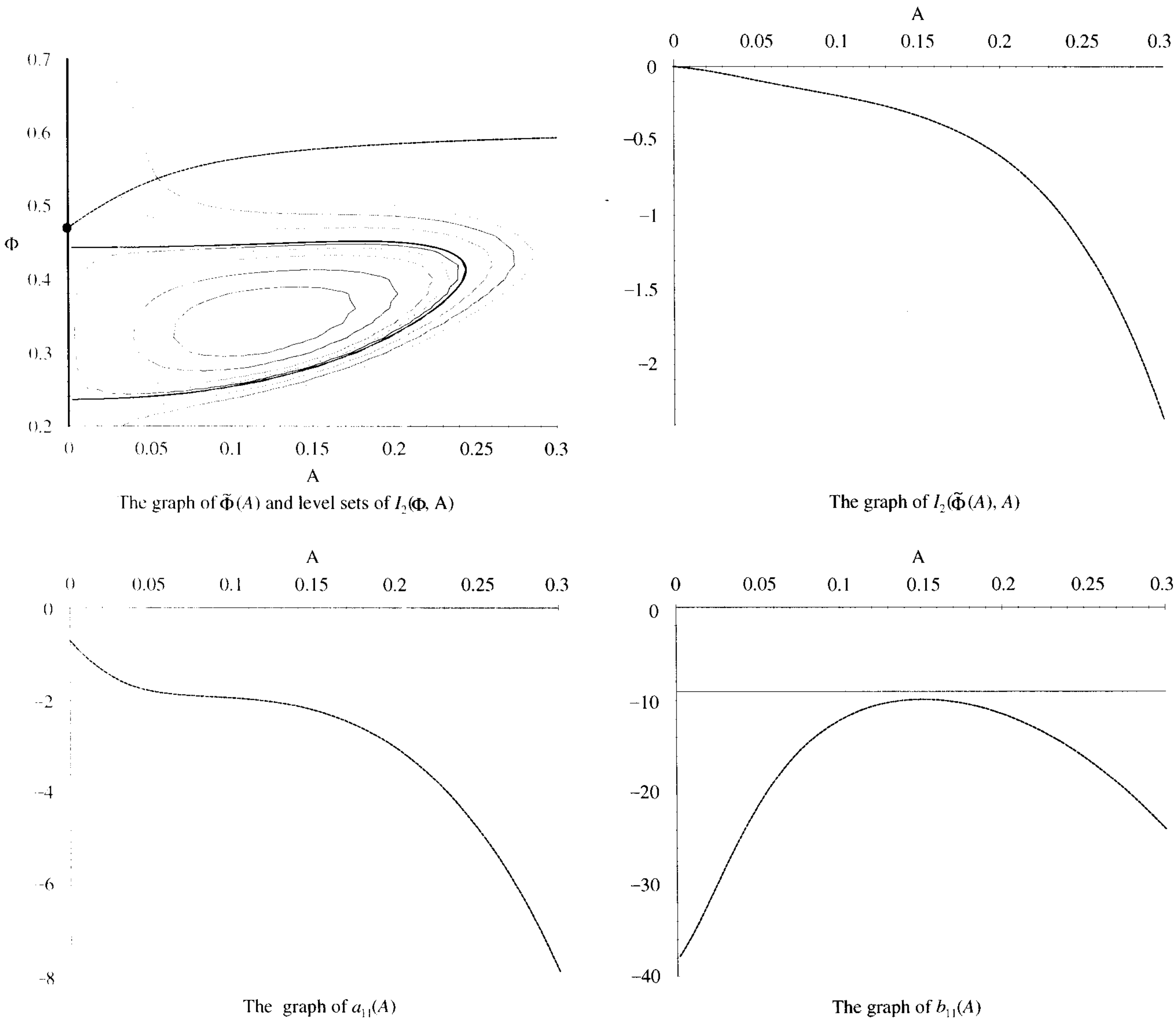


Fig. 8. Illustration of Step 1: desired shape of $\tilde{\Phi}(A)$, $a_{11}(A)$, and $b_{11}(A)$ for the case (A) (axisymmetric equlibrium).

is no longer bounded away from zero, as we have $a_{11}(0) = J_2(\tilde{\Phi}(0), 0) = J_2(\Phi_0, 0) = 0$. Therefore, we cannot guarantee linear damping in the e_A dynamics. Since $a_{11}(0) = 0$, we can factor out A from $a_{11}(A)$, i.e., represent $a_{11}(A)$ as $a_{11}(A) = b_{11}(A)A$, with $b_{11}(A) < 0$ for all $A > 0$. As we shall see, one has $b_{11}(0) = \Psi_c''(\Phi_0)\tilde{\Phi}'(0)$, so that $b_{11}(A)$ can be made negative and bounded away from 0 for all $A \geq 0$, as long as $\Psi_c''(\Phi_0) < 0$ and the graph of $\tilde{\Phi}(A)$ departs linearly with a positive slope from the point $(0, \Phi_0)$ and stays in the region where $J_2(\Phi, A) < 0$. In the case (P) we can only guarantee at most *quadratic* damping in the e_A dynamics (see Fig. 9).

Similarly to the case (A), in the case (S) the main damping term in e_A dynamics, $a_{11}(A)e_A$, behaves *linearly* in e_A near $A = A_1$, provided that $a_{11}(A_1) < 0$. The latter condition is guaranteed if $\nabla J_2(\Phi_1, A_1) \neq 0$ and the graph of $\tilde{\Phi}(A)$ intersects the curve $J_2(\Phi, A) = 0$ (the stall equilibria loop) transversely at (A_1, Φ_1). However, the existence of an unstable axisymmetric equilibrium (the *saddle point*) $(0, \Phi_2, \Psi_2)$ makes the situation essentially different from the case of single stable axisymmetric equlibrium. First of all, strictly speaking, global or semi-global stabilization is impossible. However, for all practical purposes it is sufficient to design the controller that globally or semi-globally stabilizes the open set of points with $A > 0$, and this is what we are going to do. The second (more important) problem that we encounter in the case (S) is that the damping coefficient $a_{11}(A)$ is small for A small, as

$$a_{11}(0) = \frac{I_2(\Phi_2, 0) - I_2(\Phi_1, A_1)}{0 - A_1} = 0.$$

It is possible to represent $a_{11}(A)$ as $a_{11}(A) = b_{11}(A)A$, with $b_{11}(A)$ negative and bounded away from zero for all

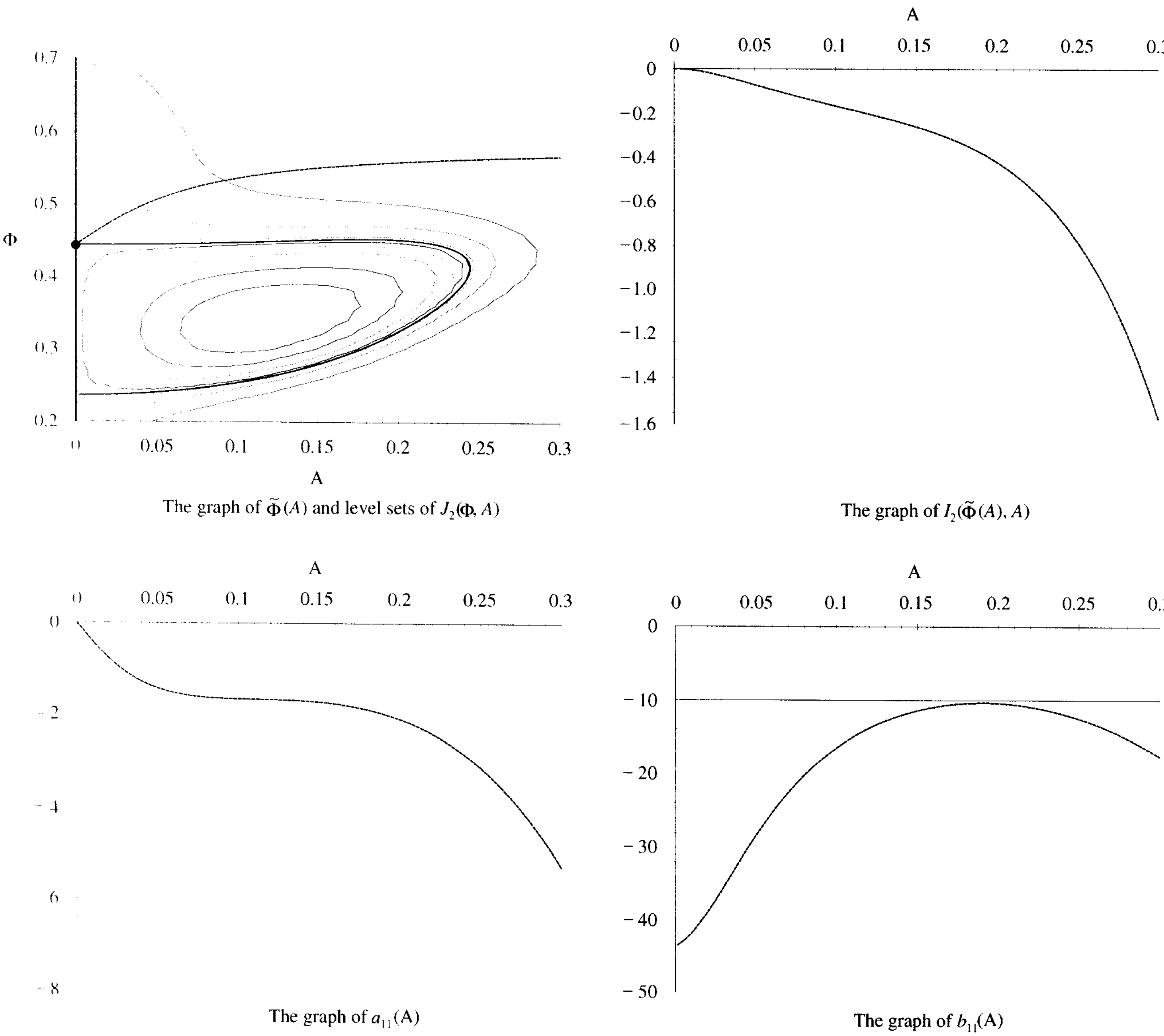


Fig. 9. Illustration of Step 1: desired shape of $\tilde{\Phi}(A)$, $a_{11}(A)$, and $b_{11}(A)$, for the Case (P) (the peak).

$A \geq 0$, provided that the graph of $\tilde{\Phi}(A)$ stays in the region where $I_2(\Phi, A) > 0$ (i.e., inside the stall equilibria loop) for $A < A_1$, crosses the stall equilibria loop at $A = A_1$ *transversely*, and stays in the region where $I_2(\Phi, A) < 0$ (i.e., outside the stall equilibria loop) for $A > A_1$ (see Fig. 10). In any case, damping of the e_A dynamics becomes weak as A approaches zero. We are going to deal with this problem by a special choice of Lyapunov function (similar to that of Krstić (1995) and Krstić et al. (1995) for a cubic characteristic) that becomes unbounded as A approaches 0. This Lyapunov function changes continuously for points with $A > 0$ as the the throttle surface moves. The continuity of the Lyapunov function, and, more importantly, of its time derivative allows us to treat all the cases simultaneously. Moreover, in the next section, we are going to use this Lypaunov function to construct controllers that stabilize the desired equilibria for all values of the set-point parameter, but with gains independent of the set-point parameter.

Note that the effect of the right skewness on control design manifests itself clearly in Step 1. The more right skewed the compressor, the steeper is the slope graph of the stall equilibria loop (plotted as a function $\Phi = f(A)$ near the peak of the compressor characteristic). From the discussion above it follows that the slope of the graph of $\tilde{\Phi}(A)$ has to be steeper than the slope of the stall equilibria loop. Following the graph backstepping procedure one can see in the end that the steep slope of $\tilde{\Phi}(A)$ translates to a high stall feedback gain (i.e., the gain in from of a variable representing stall). Therefore, the more right skewed the compressor, the higher the stall feedback gain has to be.

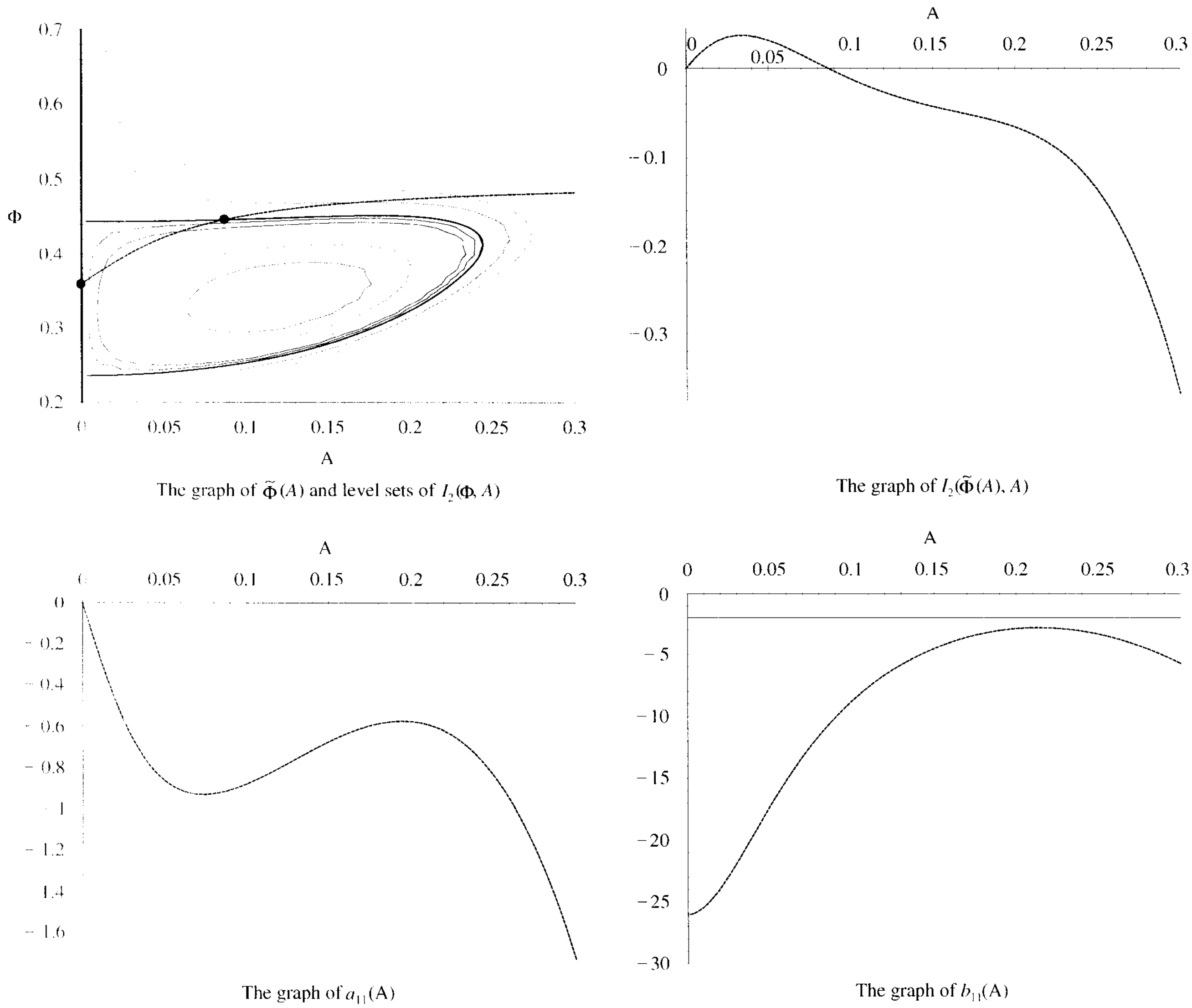


Fig. 10. Illustration of Step 1: desired shape of $\tilde{\Phi}(A)$, $a_{11}(A)$, and $b_{11}(A)$, for the case (S) (rotating stall equilibrium).

In comparison to Step 1, Steps 2 and 3 are relatively straightforward, as the main damping coefficients $b(\Phi, A)$ and c_Ψ appear linearly in the correponding error dynamics. We will obtain sufficient conditions that $b(\Phi, A)$ and c_Ψ must satisfy in the proof of the main result of this section, Theorem 5. However, is possible to gain some insight on the minimum control requirements and to give some indications about the shape of the throttle surface and the value of gain c_Ψ by looking at the main damping terms in the e_Φ and e_Ψ dynamics.

Observe that to have a negative main damping coefficient in the e_Φ dynamics, one should have $b(\Phi, A) > a_{22}(\Phi, A) - l_c\sigma\tilde{\Phi}'(A)a_{12}(\Phi, A)$. As we shall see later, one has $a_{12}(\Phi, 0) = 0$ and $a_{22}(\Phi, 0) = \delta_{\Phi, \Phi(0)}\Psi_c(\Phi)$. Thus, assuming continuity of the divided differences $a_{22}(\Phi, A)$ and $a_{12}(\Phi, A)$ in A, at least for small values of A, the term $a_{22}(\Phi, A)$ dominates the term $l_c\sigma\tilde{\Phi}'(A)a_{12}(\Phi, A)$ so that we can neglect the latter term in a preliminary analysis. We see that a minimum requirement that $b(\Phi, A)$ should satisfy is $b(\Phi, A) > a_{22}(\Phi, A)$. This condition has a simple geometric interpretation: the throttle surface should lie *below* the turning surface S_Φ to the *left* of their intersection (i.e., for $\Phi < \tilde{\Phi}(A)$) and should lie *above* the turning surface S_Φ to the *right* of their intersection.

One can visualize the the divided differences $a_{22}(\Phi, A)$ on the graph of the turning surface S_Φ as the slopes of the secant lines connecting the points $(A, \Phi, I_1(A, \Phi))$ and $(A, \tilde{\Phi}(A), I_1(A, \tilde{\Phi}(A)))$ on the surface S_Φ. Observe that the values of $I_1(A, \Phi)$ are simply averages of the values of the compressor characteristic $\Psi_c(\Phi)$ between $\Phi - A$ and $\Phi + A$. Thus, for a typical compressor characteristic, for a fixed A the graph of $I_1(A, \Phi)$ will lie *below* the compressor characteristic $\Psi_c(\Phi)$ to the *right* of the inflection point Φ_{infl} (where $\Psi_c(\Phi)$ is concave), *above* the compressor characteristic $\Psi_c(\Phi)$ to the *left* of the inflection point

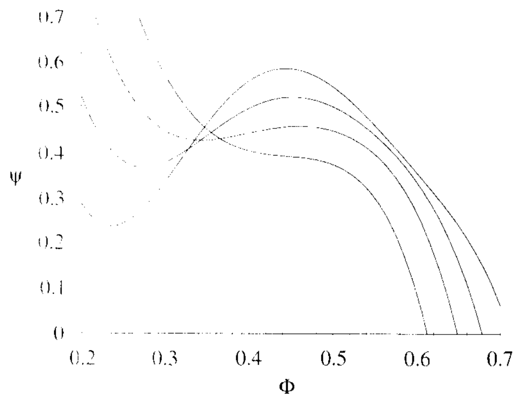


Fig. 11. $I_1(A, \Phi)$ as a function of Φ for $A = 0$, $A = 0.1$, $A = 0.15$, and $A = 0.2$.

Φ_{infl} (where $\Psi_c(\Phi)$ is convex) and will remain at approximately the same level as the compressor characteristic near the inflection point Φ_{infl} (where $\Psi_c(\Phi)$ is close to linear function) (see Fig. 11).

Observe that the divided differences $a_{22}(\Phi, A)$ remain positive, and thus constitute *unfriendly nonlinearities* destabilizing the e_Φ dynamics, only on bounded intervals of values of Φ. Moreover, the lengths of these intervals decrease as A increases and the intervals vanish for sufficiently big A. Thus, for a small A and for Φ to the right of the peak or far away to the left on the back-flow slope of $I_1(A, \Phi)$, as well as for large A and for any values of Φ, the divided differences $a_{22}(\Phi, A)$ are negative, and thus constitute *friendly nonlinearities* stabilizing the e_Φ dynamics. The largest positive values of $a_{22}(\Phi, A)$ are obtained for $A = 0$. Thus, one should have $b(\Phi, 0) > \delta_{\Phi,\Phi(0)}\Psi_c(\Phi)$ for all Φ in the region of interest. Clearly, the largest positive derivative of the compressor characteristic (achieved at the inflection point) is an upper bound on the values of $a_{22}(\Phi, A)$. Thus, $b(\Phi, A) > \sup_\Phi \Psi_c'(\Phi)$ is a sufficient condition for $b(\Phi, A) > a_{22}(\Phi, A)$. However, there is no reason to enforce the condition $b(\Phi, A) > \sup_\Phi \Psi_c'(\Phi)$ globally. It is enough to do that over the bounded region where $a_{22}(\Phi, A) > 0$. $b(\Phi, A)$ could be chosen to be zero, or even negative over the region where $a_{22}(\Phi, A) < 0$. However, one should be careful, for, as we shall see later using a more detailed Lyapunov analysis, $a_{22}(\Phi, A)$ is not the only term that $b(\Phi, A)$ has to dominate. In any case, the negative values of $a_{22}(\Phi, A)$ over some regions will help $b(\Phi, A)$ to dominate the other terms and thus to stabilize the e_Φ dynamics.

In a similar way one can provide some insight on the minimum requirements for the gain c_Ψ. One obvious condition is

$$c_\Psi > 4B^2 \frac{\partial \tilde{\Psi}(\Phi, A)}{\partial \Phi}.$$

Note that

$$\frac{\partial \tilde{\Psi}(\Phi, A)}{\partial \Phi} = b(\Phi, A) + b'(\Phi, A)(\Phi - \tilde{\Phi}(A)).$$

If $b(\Phi, A)$ does not change too rapidly, we can neglect the second term and arrive at the condition $c_\Psi > 4B^2 b(\Phi, A)$ as a minimum requirement for c_Ψ. This condition implies that we have to use some positive gain c_Ψ, whenever $b(\Phi, A) > 0$, which happens (roughly) when $a_{22}(\Phi, A) > 0$, and we can use $c = 0$ whenever $b(\Phi, A) < 0$. (Again, one should be cautious, as $4B^2 b(\Phi, A)$ is not the only term that c_Ψ has to counteract.) In any case, we see that high values of the B parameter require strong actuation that can lead to actuator saturation and thus make the throttle actuation ineffective.

To see the character of dependence of the terms $a_{11}(A)$ and $a_{12}(\Phi, A)$ on A for A small, we are going to represent $a_{11}(A)$ and $a_{12}(\Phi, A)$ as $a_{11}(A) = a_{11}(0) + b_{11}(A)\, A$ and $a_{12}(\Phi, A) = a_{12}(\Phi, 0) + b_{12}(\Phi, A)A$ using the divided differences. Let

$$b_{11}(A) := \delta_{A,0} a_{11}(A), \qquad b_{12}(\Phi, A) := \delta_{A,0} a_{12}(\Phi, A). \tag{29}$$

It follows from Lemma 2 that

$$\begin{aligned} b_{11}(A) &= \delta_{A,0}\delta_{A,A_1} I_2(\tilde{\Phi}(A), A) = \delta_{A,A_1}\delta_{A,0} I_2(\tilde{\Phi}(A), A) \\ &= \delta_{A,A_1} J_2(\tilde{\Phi}(A), A). \end{aligned} \tag{30}$$

In a similar manner one shows that

$$b_{12}(A) = \delta_{\Phi,\tilde{\Phi}(A)} J_2(\Phi, A).$$

Note that for all Φ we have

$$a_{12}(\Phi, 0) = \delta_{\Phi,\tilde{\Phi}(0)}\ I_2(\Phi, 0) = 0.$$

In the sequel the following representation of $I_2(\Phi, A)$ will be useful.

$$\begin{aligned} I_2(\Phi, A) &= I_2(\tilde{\Phi}(A), A) + \delta_{\Phi,\tilde{\Phi}(A)} I_2(\Phi, A) e_\Phi \\ &= a_{11}(A) e_A + a_{12}(\Phi, A) e_\Phi \\ &= a_{11}(0) e_A + b_{11}(A) A e_A + b_{12}(\Phi, A) A e_\Phi. \end{aligned} \tag{31}$$

Note that one has $a_{11}(0) = \Psi_c'(\Phi_1) < 0$ in the case (A) and $a_{11}(0) = 0$ in the cases (P) and (S).

Now, we are ready to prove the main result of the paper. It shows that every point on the decreasing part of the compressor characteristic, the peak of the characterisitic, and every rotating stall equilibrium in the region where the function $b_{11}(A)$ is negative and bounded away from zero can be guaranteed to have an arbitrarily large domain of attraction by an appropriate choice of the throttle surface and value of gain c_Ψ. As before, (A_1, Φ_1, Ψ_1) will denote the equlibrium that we want to stabilize corresponding to one of the cases (A), (P), and (S).

To prove stability of the error dynamics (28) we will need to establish that $b_{11}(A)$ is negative and bounded away from zero in the region of interest. In the proof of the next result we are going to assume that $\tilde{\Phi}(A)$ is such that $b_{11}(A)$ has the desired property. We are going to provide a sufficient condition for $b_{11}(A)$ to be negative and bounded away from zero after the proof of the main theorem.

Theorem 5. *Assume that*

1. *An admissible throttle function $\tilde{\Psi}(\Phi, A)$ is defined for all (Φ, A) with $A \geq 0$.*
2. *The corresponding function $\tilde{\Phi}(A)$ is such that there exists a negative constant η_0 such that, for all $A \geq 0$, $b_{11}(A) < \frac{1}{2}\eta_0$.*
3. *The corresponding function $b(\Phi, A)$ is such that for all (Φ, A) with $A \geq 0$ one has*

$$b(\Phi, A) > a_{22}(\Phi, A) - l_c\sigma\tilde{\Phi}'(A)a_{12}(\Phi, A) + \frac{1}{4}\left(-\frac{a_{12}(\Phi, A)}{a_{11}(A)} - l_c\sigma\tilde{\Phi}'(A)a_{11}(A)\right)^2. \quad (32)$$

Define $V_1(0) = 0$ and, for $e_A > -A_1$,

$$V_1(e_A) := -\int_0^{e_A}\frac{\xi}{a_{11}(A_1 + \xi)}\,d\xi\left(= -\int_{A_1}^{A}\frac{\alpha - A_1}{a_{11}(\alpha)}\,d\alpha\right). \quad (33)$$

Let

$$V(A, \Phi, \Psi) := \frac{1}{\sigma}V_1(e_A) + \frac{l_c}{2}e_\Phi^2 + \frac{4l_cB^2}{2}e_\Psi^2. \quad (34)$$

Then there exists a gain function $c(A, \Phi)$ such that along the solutions of the closed-loop system (6) one has $\dot{V}(A, \Phi, \Psi) < 0$ for all points (A, Φ, Ψ) where $V(A, \Phi, \Psi)$ is defined (except for $(A, \Phi, \Psi) = (A_1, \Phi_1, \Psi_1)$, where $\dot{V}(A, \Phi, \Psi) = 0$). In particular, the sets $V(A, \Phi, \Psi) = \gamma$ are invariant with respect to the flow of the system (6) for all $\gamma > 0$.

Proof. We have

$$\begin{aligned}\dot{V}(A, \Phi, \Psi) &= \left(-e_A^2 - \frac{a_{12}(\Phi, A)}{a_{11}(A)}e_Ae_\Phi\right) + l_ce_\Phi\left(\frac{1}{l_c}(a_{22}(\Phi, A) - b(\Phi, A)\right)e_\Phi - \frac{1}{l_c}e_\Psi \\ &\quad - \sigma(\tilde{\Phi}'(A)(a_{11}(A)e_A + a_{12}(\Phi, A)e_\Phi)) + 4l_cB^2e_\Psi\left(\frac{-c_\Psi}{4l_cB^2}e_\Psi - \frac{\partial\tilde{\Psi}(\Phi, A)}{\partial A}\sigma(a_{11}(A)e_A + a_{12}(\Phi, A)e_\Phi)\right) \\ &\quad - \frac{\partial\tilde{\Psi}(\Phi, A)}{\partial\Phi}\left(\frac{1}{l_c}(a_{22}(\Phi, A) - b(\Phi, A))e_\Phi - \frac{1}{l_c}e_\Psi\right) \\ &= c_{11}e_A^2 + 2c_{12}e_Ae_\Phi + 2c_{13}e_Ae_\Psi + c_{22}e_\Phi^2 + 2c_{23}e_\Phi e_\Psi + c_{33}e_\Psi^2, \end{aligned} \quad (35)$$

where

$$c_{11} := -1,$$

$$c_{12} := \frac{1}{2}\left(-\frac{a_{12}(\Phi, A)}{a_{11}(A)} - l_c\sigma\tilde{\Phi}'(A)a_{11}(A)\right) = \frac{1}{2}\left(-\frac{b_{12}(\Phi, A)A}{a_{11}(0) + b_{11}(A)A} - l_c\sigma\tilde{\Phi}'(A)a_{11}(A)\right),$$

$$c_{22} := a_{22}(\Phi, A) - b(\Phi, A) - l_c\sigma\tilde{\Phi}'(A)a_{12}(\Phi, A),$$

$$c_{13} := -2l_c\sigma B^2\frac{\partial\tilde{\Psi}(\Phi, A)}{\partial A}a_{11}(A), \quad (36)$$

$$c_{23} := -\frac{1}{2} - 2l_cB^2\frac{\partial\tilde{\Psi}(\Phi, A)}{\partial A}a_{12}(\Phi, A) - 2B^2\frac{\partial\tilde{\Psi}(\Phi, A)}{\partial\Phi}(a_{22}(\Phi, A) - b(\Phi, A)),$$

$$c_{33} := -c_\Psi(\Phi, A) + 4B^2\frac{\partial\tilde{\Psi}(\Phi, A)}{\partial\Phi}.$$

Thus, $\dot{V}(A, \Phi, \Psi)$ is a quadratic form in e_A, e_Φ, and e_Ψ that can be made negative definite at any point (Φ, A) by choosing $b(\Phi, A)$ and $c_\Psi(\Phi, A)$ sufficiently large. Sufficient conditions for $\dot{V}(A, \Phi, \Psi)$ to be negative definite at (Φ, A) are

$$\begin{aligned}\Delta_1 &:= c_{11} < 0,\\ \Delta_2 &:= c_{11}c_{22} - c_{12}^2 > 0,\\ \Delta_3 &:= c_{33}\Delta_2 + 2c_{12}c_{13}c_{23} - c_{22}c_{13}^2 - c_{11}c_{23}^2 < 0.\end{aligned} \quad (37)$$

The condition $\Delta_1 < 0$ is obviously satisfied. To enforce the condition $\Delta_2 > 0$ one should choose $b(\Phi, A)$ such that (32) is satisfied. To see that this is possible, note that

$$\frac{a_{12}(\Phi, A)}{a_{11}(A)} = \frac{b_{12}(\Phi, A)A}{a_{11}(0) + b_{11}(A)A}.$$

By assumption, $b_{11}(A)$ is negative and bounded away from zero for all $A \geq 0$. Moreover, $a_{11}(0) \leq 0$, as $a_{11}(0) = \Psi_c'(\Phi_1)$ in the case (A) and $a_{11}(0) = 0$ in the cases (P) and (S). Therefore, we have

$$\left|\frac{a_{12}(\Phi, A)}{a_{11}(A)}\right| \leq \left|\frac{b_{12}(\Phi, A)}{b_{11}(A)}\right| \leq 2\left|\frac{b_{12}(\Phi, A)}{\eta_0}\right|.$$

Hence, the right-hand side of inequality (32) is well-defined at each point (Φ, A). Finally, once $b(\Phi, A)$ satisfies inequality (32), to assure that $\Delta_3 < 0$, at each point (Φ, A), the gain $c_\Psi(\Phi, A)$ should satisfy the inequality

$$c_\Psi(\Phi, A) > 4B^2 \frac{\partial \tilde{\Psi}(\Phi, A)}{\partial \Phi} + \frac{2c_{12}c_{13}c_{23} - c_{22}c_{13}^2 - c_{11}c_{23}^2}{\Delta_2}. \tag{38}$$

The right-hand side of the inequality is defined for every (Φ, A), as $\Delta_2 > 0$ provided that inequality (32) holds for every (Φ, A). Note that $c_\Psi(\Phi, A)$ can be chosen bounded on every compact set.

The above theorem states that one of the conditions for stabilization of the chosen equilibrium is that the function $b_{11}(A)$ is negative and bounded away from zero in the region of interest. Since we have shown that that $b_{11}(A) = \partial_{A,A_1} J_2(\tilde{\Phi}(A), A)$, we see that to achieve $b_{11}(A)$ negative and bounded away from zero for all $A \geq 0$, it is sufficient to make sure that the function $J_2(\Phi, A)$ decreases along the graph of $\tilde{\Phi}(A)$ and that the graph of $\tilde{\Phi}(A)$ intersects the level sets of $J_2(\Phi, A)$ transversely. Since we are interested in stabilization of axisymmetric equilibria to the right of the peak or stall equilibria close to the peak, to simplify the proof of the next result, we are going to restrict our interest to the case when the graph of $\tilde{\Phi}(A)$ originates to the right of the inflection point of the characteristic. Then, a sufficient condition to have $b_{11}(A)$ negative and bounded away from zero for all $A \geq 0$ happens to be $\tilde{\Phi}'(A) \geq 1$. Note that this condition is sufficient for *arbitrary* compresssor characteristic, as long as its second derivative is negative and bounded away from zero to the right of the inflection point.

Proposition 6. *Let $\tilde{\Phi}(A)$ be any continuous and piecewise differentiable function of A that satisfies the following condition:*

1. $\tilde{\Phi}'(A) \geq 1$.
2. $\tilde{\Phi}(0) > \Phi_{\text{infl}}$.

(Here, Φ_{infl} denotes the inflection point of the characteristic.) Assume that for all $\Phi > \tilde{\Phi}(0)$ the compressor characteristic $\Psi_c(\Phi)$ has a piecewise continuous second derivative and there exists a negative constant η_0 such that $\forall \Phi > \tilde{\Phi}(0)\ \Psi_c''(\Phi) < \eta_0$. Then, for all $A \geq 0$,

$$b_{11}(A) < \frac{\eta_0}{2}.$$

Moreover, the graph of $\tilde{\Phi}(A)$ has at most one intersection with the curve $J_2(\Phi, A) = 0$ (i.e., stall equilibria loop).

Proof. Observe that the conditions (1) and (2) imply that the graph of $\tilde{\Phi}(A)$ stays above the line $\Phi = \Phi_{\text{infl}} + A$, so that $\forall s \in [0, 1]\ \tilde{\Phi}(A) + sA\sin\theta > \Phi_{\text{infl}}$. Therefore, $\forall s \in [0, 1]\ \Psi_c''(\tilde{\Phi}(A) + sA\sin\theta) < \eta_0$. Note that it follows from (20) that

$$\frac{\mathrm{d}}{\mathrm{d}A} J_2(\tilde{\Phi}(A), A) = \frac{1}{\pi}\int_0^1\int_0^{2\pi} \Psi_c''(\tilde{\Phi}(A) + sA\sin\theta)(\tilde{\Phi}'(A) + sA\sin\theta)\sin^2\theta\,\mathrm{d}\theta\,\mathrm{d}s.$$

Observe that for all $A \geq 0$ one has $\Psi_c''(\tilde{\Phi}(A) + sA\sin\theta) < \eta_0 < 0$ and $\tilde{\Phi}'(A) + sA\sin\theta \geq 0$. Hence,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}A} J_2(\tilde{\Phi}(A), A) &= \frac{1}{\pi}\int_0^1\int_0^{\pi} \Psi_c''(\tilde{\Phi}(A) + sA\sin\theta)(\tilde{\Phi}'(A) + sA\sin\theta)\sin^2\theta\,\mathrm{d}\theta\,\mathrm{d}s \\ &+ \frac{1}{\pi}\int_0^1\int_{\pi}^{2\pi} \Psi_c''(\tilde{\Phi}(A) + sA\sin\theta)(\tilde{\Phi}'(A) + sA\sin\theta)\sin^2\theta\,\mathrm{d}\theta\,\mathrm{d}s \\ &< \frac{1}{\pi}\int_0^1\int_0^{\pi} \Psi_c''(\tilde{\Phi}(A) + sA\sin\theta)(\tilde{\Phi}'(A) + sA\sin\theta)\sin^2\theta\,\mathrm{d}\theta\,\mathrm{d}s \\ &< \frac{1}{\pi}\int_0^1\int_0^{\pi} \eta_0\tilde{\Phi}'(A)\sin^2\theta\,\mathrm{d}\theta\,\mathrm{d}s < \frac{\eta_0}{2}.\end{aligned} \tag{39}$$

Thus, in view of Lemma 1 and Eq. (30), one has, for all $A \geq 0$, $b_{11}(A) < \eta_0/2$. To show that the graph of $\tilde{\Phi}(A)$ has at most one intersection with the curve $J_2(\Phi, A) = 0$, note that it follows from the fact (established above) that $\mathrm{d}/\mathrm{d}A\, J_2(\tilde{\Phi}(A), A) < 0$ for all $A \geq 0$.

The assumptions about $\tilde{\Phi}(A)$ made in Proposition 6 that guarantee $b_{11}(A)$ to be negative and bounded away from zero for all $A \geq 0$ could be relaxed. In fact, one can show that, under assumption that the gradient of $J_2(\Phi, A)$ is nonzero on the stall equilibria loop, a sufficient condition for $b_{11}(A)$ to be negative and bounded away from zero for all $A \geq 0$ in the case (S) is that the graph of $\tilde{\Phi}(A)$ stays in the region where $J_2(\Phi, A) > 0$ (i.e., inside the stall equilibria loop) for $A < A_1$, transversely crosses the stall equilibria loop at $A = A_1$, and stays in the region where $J_2(\Phi, A) < 0$ (i.e., outside the stall equilibria loop) for $A > A_1$ (see Fig. 10). In the cases (A) and (P) it is enough to require $J_2(\tilde{\Phi}(A), A) < J_2(\tilde{\Phi}(0), 0)$ and $\tilde{\Phi}'(0) > 0$. However, the proof of Proposition 6 that we presented has the advantage that it does not use the assumption that the gradient of $J_2(\Phi, A)$ is nonzero on the stall equilibria loop. In fact, it follows from the proof of Proposition 6 that the gradient of $J_2(\Phi, A)$ indeed is nonzero in the region $\Phi \geq \Phi_{\text{infl}} + A$. In particular, it follows that there is at most one stall equilibrium for every value of A in the region $\Phi \geq \Phi_{\text{infl}} + A$. Besides, there is at most one intersection of the stall equilibria loop with the graphs of $\tilde{\Phi}(A)$ with $\tilde{\Phi}'(A) \geq 1$. This shows that for *all* right-skew compressors, in the region $\Phi \geq \Phi_{\text{infl}} + A$ the positive slopes of the stall equilibria loop represented as a graph of Φ as a function of A never exceeds one.

To have a more precise measure of how large should be the derivative of $\tilde{\Phi}(A)$, it is convenient to introduce the following quantity. Let $\alpha_{\text{rs},1}$ denote the highest positive slope of the upper portion of the stall equilibria loop represented as a graph of Φ as a function of A. One can check that

$$\alpha_{\text{rs},1} = \inf\left\{\alpha: \alpha\frac{\partial J_2(\Phi, A)}{\partial \Phi} + \frac{\partial J_2(\Phi, A)}{\partial A} \geq 0 \quad \text{for all } (\Phi, A) \right.$$
$$\left. \text{such that } J_2(\Phi, A) = 0 \ \& \ \frac{\partial J_2(\Phi, A)}{\partial \Phi} > 0\right\}. \tag{40}$$

One can show that always $\alpha_{\text{rs},1} \geq 0$. The proof of Proposition 6 shows that for all compressors satisfying the standard assumptions one has $\alpha_{\text{rs},1} < 1$. In the formulation of Proposition 6 and in what follows, one can replace the condition $\tilde{\Phi}'(A) \geq 1$ with a weaker condition $\tilde{\Phi}'(A) > \alpha_{\text{rs},1}$.

A simple corollary from Theorem 5 and Proposition 6 is as follows:

Theorem 7. *Every potential axisymmetric equilibrium on the decreasing part of the compressor characteristic, the peak of the characterisitic, and every rotating stall equilibrium (A_1, Φ_1, Ψ_1) with the property that $\Phi_1 > \Phi_{\text{infl}} + A_1$ can be globally stabilized by an appropriate choice of the throttle surface and the controller gains.*

Proof. It is enough to choose the throttle function $\tilde{\Psi}(\Phi, A)$ and the gain function $c(A, \Phi)$ such that

1. $\tilde{\Phi}'(A) \geq 1$.
2. $\tilde{\Phi}(0) > \Phi_{\text{infl}}$.
3. For all (Φ, A) with $A \geq 0$ condition (32) is satisfied.
4. For all (Φ, A) with $A \geq 0$ condition (38) is satisfied.

If Ψ_c is piecewise differentiable, then using Lemmas 1 and 3 one can express $a_{11}(A)$, $a_{12}(A)$, and $a_{22}(A)$ in terms of Ψ_c in the following way:

$$a_{11}(A) = \frac{1}{\pi}\int_0^1\int_0^{2\pi} \Psi_c'(\tilde{\Phi}(A_1 + s(A - A_1))$$
$$+ (A_1 + s(A - A_1))\sin\theta)(\tilde{\Phi}'(A_1 + s(A - A_1))$$
$$+ \sin\theta)\sin\theta \mathrm{d}\theta\,\mathrm{d}s, \tag{41}$$

$$a_{12}(\Phi, A) = \frac{1}{\pi}\int_0^1\int_0^{2\pi} \Psi_c'(\tilde{\Phi}(A) + s(\Phi - \tilde{\Phi}(A))$$
$$+ A\sin\theta)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}s,$$

$$a_{22}(\Phi, A) = \frac{1}{\pi}\int_0^1\int_0^{2\pi} \Psi_c'(\tilde{\Phi}(A) + s(\Phi - \tilde{\Phi}(A)) + A\sin\theta)\,\mathrm{d}\theta\,\mathrm{d}s.$$

If Ψ_c is piecewise twice differentiable, using Lemmas 1 and 3 one can express $b_{11}(A)$ and $b_{12}(A)$ in terms of Ψ_c as follows:

$$b_{11}(A) = \frac{1}{\pi}\int_0^1\int_0^1\int_0^{2\pi} \Psi_c''(\tilde{\Phi}(A_1 + w(A - A_1))$$
$$+ s(A_1 + w(A - A_1)\sin\theta))(\tilde{\Phi}'(A_1 + w(A - A_1))$$
$$+ sA\sin\theta)\sin^2\theta\,\mathrm{d}\theta\,\mathrm{d}s\,\mathrm{d}w, \tag{42}$$

$$b_{12}(\Phi, A) = \frac{1}{\pi}\int_0^1\int_0^1\int_0^{2\pi} \Psi_c''(\tilde{\Phi}(A) + s(\Phi - \tilde{\Phi}(A))$$
$$+ sA\sin\theta)\sin^2\theta\,\mathrm{d}\theta\,\mathrm{d}s\,\mathrm{d}w.$$

These formulas can be used to express the lower bounds (32) and (38) in terms of bounds on the first and second derivatives of the compressor characteristic. Namely, one can easily show the following bounds.

Proposition 8. *Let $\mathcal{K}$ be any given compact set on the A–Φ plane. Then*

$$\begin{aligned} &\sup_{\mathcal{K}}|a_{11}(A)| \leq \sup_{\mathcal{K}}|\Psi_c'(\Phi)(1 + \tilde{\Phi}'(A))| \\ &\sup_{\mathcal{K}}|a_{12}(\Phi, A)| \leq \sup_{\mathcal{K}}|\Psi_c'(\Phi)|, \\ &\sup_{\mathcal{K}}|b_{12}(\Phi, A)| \leq \sup_{\mathcal{K}}|\Psi_c''(\Phi)|, \\ &\sup_{\mathcal{K}}|a_{22}(\Phi, A)| \leq \sup_{\mathcal{K}}|\Psi_c'(\Phi)|. \end{aligned} \tag{43}$$

Moreover, if the function $\tilde{\Phi}(A)$ is such that $\tilde{\Phi}'(A) \geq 1$ and $\tilde{\Phi}(0) > \Phi_{\text{infl}}$ then

$$\begin{aligned} &\inf_{\mathcal{K}}|b_{11}(A)| \geq \frac{1}{2}\inf_{\Phi \geq \tilde{\Phi}(0)}|\Psi_c''(\Phi)| \\ &\sup_{\mathcal{K}}\left|\frac{a_{12}(\Phi, A)}{a_{11}(A)}\right| \leq 2\,\frac{\sup_{\mathcal{K}}|\Psi_c''(\Phi)|}{\inf_{\Phi \geq \tilde{\Phi}(0)}|\Psi_c''(\Phi)|}. \end{aligned} \tag{44}$$

One can use the bounds (43) and (44) to obtain explicit lower bounds on the functions $b(\Phi, A)$ in (32) and $c_\Psi(\Phi, A)$ in (38) on compact regions. However, bounds (43) and (44) are very crude, as they estimate the integrals in (41) and (42) in terms of the products of upper or lower bounds of absolute values of all factors of integrands. Nevertheless, bounds (43) and (44) provide some rough indications on how first and second derivatives of the compressor characteristic influence the shape of the throttle function and the gains of the controller.

Observe that in using bounds (43) and (44) in a controller design we neglect the signs of the nonlinear terms and assume a worst-case situation when all the terms have a destabilizing effect. In fact, some of these nonlinear terms are friendly over some regions and allow to use smaller controller gains (see comments on Step 2 earlier in this section).

Note that it may be impossible to satisfy bounds (32) and (38) with constant functions $b(\Phi, A)$ and $c_\Psi(\Phi, A)$. Therefore, if one wants to use constant controller gains, global stabilization of the desired equlibrium may be impossible. However, semi-global stabilization is still possible, as on every compact set, the right-hand sides of inequalities (32) and (38) are bounded. We have the following result.

Theorem 9. *Let $\mathcal{N}$ be any given compact set on the A–Φ plane set. Suppose that the throttle function $\tilde{\Psi}(\Phi, A)$ and the gain function $c(A, \Phi)$ are such that on $\mathcal{N}$:*

1. $d/dA\,\tilde{\Phi}(A) \geq 1$.
2. $\tilde{\Phi}(0) > \Phi_{\text{infl}}$.
3. *Condition* (32) *is satisfied.*
4. *Condition* (38) *is satisfied.*

Then the throttle surface is admissible on $\mathcal{N}$ and the level sets of function $V(A, \Phi, \Psi)$ (defined by (34)) contained inside $\mathcal{N}$ are invariant with respect to the flow of the system (6).

To complete the proof of semi-global stabilization, one can prove that as the size of $\mathcal{N}$ grows, so does the size of the largest level set of $V(A, \Phi, \Psi)$ inside $\mathcal{N}$. Moreover, one can prove that for every given compact set $\mathcal{M}$ containing (A_1, Φ_1, Ψ_1) there is a corresponding compact set $\mathcal{N}$ containing $\mathcal{M}$ such that the largest level set of $V(A, \Phi, \Psi)$ inside $\mathcal{N}$ also contains $\mathcal{M}$.

6. Controller guaranteeing soft bifurcation and stabilization

We consider a family of controllers of the form

$$K_T^\Gamma := \frac{\Gamma + c_\Psi\Psi + h(\Phi - \Gamma, A)}{\sqrt{\Psi}}, \tag{45}$$

where Γ is a set-point and a bifurcation parameter. We assume that Γ belongs to a given interval $[\Gamma_l, \Gamma_r]$. As Γ varies, so does the throttle surface

$$\tilde{\Psi}^\Gamma(\Phi, A) := \frac{\Phi - \Gamma + h(\Phi - \Gamma, A)}{c_\Psi}, \tag{46}$$

the corresponding function $\tilde{\Phi}^\Gamma(A)$, and hence the equilibria of the closed-loop system. We would like to design the family of the controllers K_T^Γ is such way that, for each $\Gamma \in [\Gamma_l, \Gamma_r]$, the throttle surface $\tilde{\Psi}^\Gamma(\Phi, A)$ is such that we have one of the cases (A) (one stable axisymmetric equilibrium), (P) (equilibrium at the peak), or (S) (one stable stall equilibrium and one unstable axisymmetric equilibrium). In other words, we enforce a *soft* bifurcation of an axisymmetric equilibrium of the case (A) into a pair of the equilibria of the case (S). This should be contrasted with a *hard* bifurcation exhibited by the model with C_3' characteristic and constant throttle parameters.

In particular, we are going to show that if $\Gamma_l > \Phi_{\text{infl}}$ then a controller of the form

$$K_T^\Gamma = \Gamma - d_\Phi\dot{\Phi} - c_\Phi(\Phi - \Gamma) + c_A A/\sqrt{\Psi},$$

where d_Φ, c_Φ, and c_A are contants independent of Γ, satisfies for all $\Gamma \in [\Gamma_l, \Gamma_r]$ the conditions of Theorem 9, and thus semi-globally stabilizes each equilibrium corresponding to $\Gamma \in [\Gamma_l, \Gamma_r]$. As the set-point parameter Γ varies, the equlibrium $(A_1^\Gamma, \Phi_1^\Gamma, \Psi_1^\Gamma)$ stabilized by the controller changes from an axisymmetric equilibrium to the right of the peak to a small rotating stall cell.

A simple corollary from Theorem 7 is the following result.

Theorem 10. *Suppose that for each $\Gamma \in [\Gamma_l, \Gamma_r]$ the throttle function $\tilde{\Psi}^\Gamma(\Phi, A)$ and the gain function $c(A, \Phi)$ are such that*

1. $d/dA\,\tilde{\Phi}^\Gamma(A) \geq 1$.
2. $\tilde{\Phi}^\Gamma(0) > \Phi_{\text{infl}}$.
3. *For all (Φ, A) with $A \geq 0$ the condition* (32) *is satisfied uniformly in Γ.*
4. *For all (Φ, A) with $A \geq 0$ the condition* (38) *is satisfied uniformly in Γ.*

Then, for each $\Gamma \in [\Gamma_l, \Gamma_r]$ the throttle surface is admissible and the corresponding equilibrium $(A_1^\Gamma, \Phi_1^\Gamma, \Psi_1^\Gamma)$ is globally stable. (For $A_1 > 0$, global refers to the open set of points with $A > 0$.)

In the formulation of Theorem 10 one can replace the condition $d/dA\,\tilde{\Phi}^\Gamma(A) \geq 1$ with a weaker condition $d/dA\,\tilde{\Phi}^\Gamma(A) \geq \alpha_{\text{rs},1}$. Note that from a point of view of controller design, $\alpha_{\text{rs},1}$ is a more important measure of right skewness of the compressor characteristic than $\beta_{\text{rs},1}$, as the condition $d/dA\,\tilde{\Phi}^\Gamma(A) \geq \alpha_{\text{rs},1}$ guarantees a soft bifurcation of the equilibria for values of Γ such that $\tilde{\Phi}^\Gamma(0) > \Phi_{\text{infl}}$.

A version of the last theorem useful for a semi-global stabilization is as follows.

Theorem 11. *Let $\mathcal{N}$ be any given compact set on the A–Φ plane set. Suppose that for each $\Gamma \in [\Gamma_l, \Gamma_r]$ the throttle function $\tilde{\Psi}^\Gamma(\Phi, A)$ and the gain function $c(A, \Phi)$ are such that*

1. $d/dA\,\Phi^\Gamma(A) \geq 1$.
2. $\tilde{\Phi}^\Gamma(0) > \Phi_{\text{infl}}$.
3. *Condition* (32) *is satisfied.*
4. *Condition* (38) *is satisfied.*

Then the throttle surface is admissible on $\mathcal{N}$ and the level sets of function $V^{\Gamma}(A, \Phi, \Psi)$ (defined by (34)) contained inside $\mathcal{N}$ are invariant with respect to the flow of system (6).

7. An example of controller design

Consider controller of the form

$$K_{\mathrm{T}} = \frac{\Gamma - d_{\Phi}\dot{\Phi} - c_{\Phi}(\Phi - \Gamma) + c_A A}{\sqrt{\Psi}}, \tag{47}$$

where d_{Φ}, c_{Φ}, and c_A are some positive contants (or, more generally, some piecewise constant functions) and $\Gamma \in [\Gamma_{\mathrm{l}}, \Gamma_{\mathrm{r}}]$, with $\Gamma_l > \Phi_{\mathrm{infl}}$. We are going to show that this controller for some choice of gains satisfies the conditions of Theorem 11, and thus semi-globally stabilizes the desired range of equilibria and guarantees soft bifurcation at the peak.

Since $\dot{\Phi} = (1/l_c)(I_1(\Phi, A) - \Psi)$, one has

$$\dot{\Psi} = \left(-\frac{d_{\Psi}}{l_c}\right)\frac{1}{4l_c B^2}\left(\Psi - I_1(\Phi, A) - \frac{l_c}{d_{\Phi}}(1 + c_{\Phi})(\Phi - \Gamma) + \frac{l_c}{d_{\Phi}} c_A A\right). \tag{48}$$

Let

$$\tilde{\Psi}^{\Gamma}(\Phi, A) := I_1(\Phi, A) + \frac{l_c}{d_{\Phi}}(1 + c_{\Phi})(\Phi - \Gamma) - \frac{l_c}{d_{\Phi}} c_A A,$$
$$c_{\Psi} := d_{\Phi}/l_c. \tag{49}$$

With this notation we can rewrite the $\dot{\Psi}$ dynamics in the standard form

$$\dot{\Psi} = \frac{-c_{\Psi}}{4l_c B^2}(\Psi - \tilde{\Psi}^{\Gamma}(\Phi, A)). \tag{50}$$

It is easy to see that the corresponding function $\tilde{\Phi}^{\Gamma}(A)$ is

$$\tilde{\Phi}^{\Gamma}(A) = \Gamma + \frac{c_A}{1 + c_{\Phi}} A \tag{51}$$

and hence

$$\mathrm{d}/\mathrm{d}A\, \tilde{\Phi}^{\Gamma}(A) = c_A/(1 + c_{\Phi}). \tag{52}$$

We see that the graphs of $\tilde{\Phi}^{\Gamma}(A)$ are straight lines originating at $\tilde{\Phi}^{\Gamma}(0) = \Gamma$ and having slope $c_A/(1 + c_{\Phi})$. Since $\Gamma \geq \Gamma_l > \Phi_{\mathrm{infl}}$, the second condition of Theorem 11 is satisfied. To satisfy the first condition of Theorem 11, it is enough to choose c_A and c_{Φ} so that $c_A/(1 + c_{\Phi}) \geq 1$.

Eq. (49) can be rewritten as

$$\tilde{\Psi}^{\Gamma}(\Phi, A) = I_1(\Phi, A) + \frac{l_c}{d_{\Phi}}(1 + c_{\Phi})(\Phi - \tilde{\Phi}^{\Gamma}(A))$$
$$= I_1(\Phi, A) + \frac{l_c}{d_{\Phi}}(1 + c_{\Phi})e_{\Phi}. \tag{53}$$

Thus

$$\tilde{\Psi}^{\Gamma}(\Phi, A) = I_1(\tilde{\Phi}^{\Gamma}(A), A) + \delta_{\Phi, \tilde{\Phi}^{\Gamma}(A)} I_1(\Phi, A)e_{\Phi} + \frac{l_c}{d_{\Phi}}(1 + c_{\Phi})e_{\Phi}$$
$$= I_1(\tilde{\Phi}^{\Gamma}(A), A) + a_{22}^{\Gamma}(\Phi, A)e_{\Phi} + \frac{l_c}{d_{\Phi}}(1 + c_{\Phi})e_{\Phi}. \tag{54}$$

The corresponding function $b^{\Gamma}(\Phi, A)$ can be expressed as

$$b^{\Gamma}(\Phi, A) := a_{22}^{\Gamma}(\Phi, A) + \frac{l_c}{d_{\Phi}}(1 + c_{\Phi}). \tag{55}$$

Condition (32) is easily seen to be equivalent with

$$\frac{l_c(1 + c_{\Phi})}{d_{\Phi}} > - l_c\sigma\frac{c_A}{1 + c_{\Phi}} a_{12}(\Phi, A) - \frac{1}{4}\left(-\frac{a_{12}(\Phi, A)}{a_{11}(A)} - l_c\sigma\frac{c_A}{1 + c_{\Phi}} a_{11}(A)\right)^2. \tag{56}$$

Let

$$\bar{c}_{\Phi} := \frac{l_c(1 + c_{\Phi})}{d_{\Psi}}$$
$$\alpha := \frac{\mathrm{d}}{\mathrm{d}A}\tilde{\Phi}^{\Gamma}(A) = \frac{c_A}{1 + c_{\Phi}}. \tag{57}$$

Below we present a procedure for choosing the gains of the controller for a stabilization over a desired region of operation $\mathcal{M}$.

1. Choose $\mathcal{N}$ be a compact set on the A–Φ plane that contains the region of the desired operation $\mathcal{M}$. The region $\mathcal{N}$ will be used to obtain bounds on the controller gains. For all Γ in $[\Gamma_{\mathrm{l}}, \Gamma_{\mathrm{r}}]$, the largest level sets of function $V^{\Gamma}(A, \Phi, \Psi)$ contained inside $\mathcal{N}$ are guaranteed to be in the basin of attraction of the corresponding equlibrium $(A_1^{\Gamma}, \Phi_1^{\Gamma}, \Psi_1^{\Gamma})$ and to be an invariant set of the flow. It is difficult to say a priori how big should be $\mathcal{N}$. One should proceed with initial choice of $\mathcal{N}$ and if the largest level sets of $V^{\Gamma}(A, \Phi, \Psi)$ fail to contain the desired region of operation $\mathcal{M}$, one should repeat the procedure with a larger $\mathcal{N}$. Alternatively, one can try to increase the corresponding controller gains.
2. Choose α to be a positive number greater than $\alpha_{\mathrm{rs},1}$. $\alpha \geq 1$ is always sufficient for stabilization of the e_A dynamics and enforcing a soft bifurcation for all characteristics. Higher values of α result in lower values of A_1^{Γ} for any fixed value of the set-point parameter Γ, and thus in a lower pressure drop.
3. Choose $\bar{c}_{\Phi}$ such that

$$\bar{c}_{\Phi} > 1 - l_c\sigma\alpha a_{12}(\Phi, A) + \frac{1}{4}\left(-\frac{a_{12}(\Phi, A)}{a_{11}(A)} - l_c\sigma\alpha a_{11}(A)\right)^2 \tag{58}$$

on $\mathcal{N}$. This choice guarantees that $\Delta_2 > 1$ on $\mathcal{N}$. One can use bounds (43) and (44) to obtain a conservative lower bound for $\bar{c}_{\Phi}$.

4. Choose c_Ψ such that

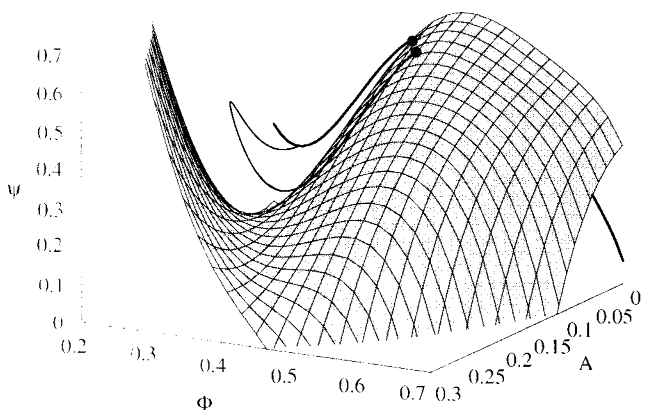
$$c_\Psi > 4B^2\left(\frac{\partial I_1(\Phi, A)}{\partial \Phi} + \bar{c}_\Phi\right) + 2c_{12}c_{13}c_{23} - c_{22}c_{13}^2 - c_{11}c_{23}^2 \tag{59}$$

is satisfied on $\mathcal{A}$ for

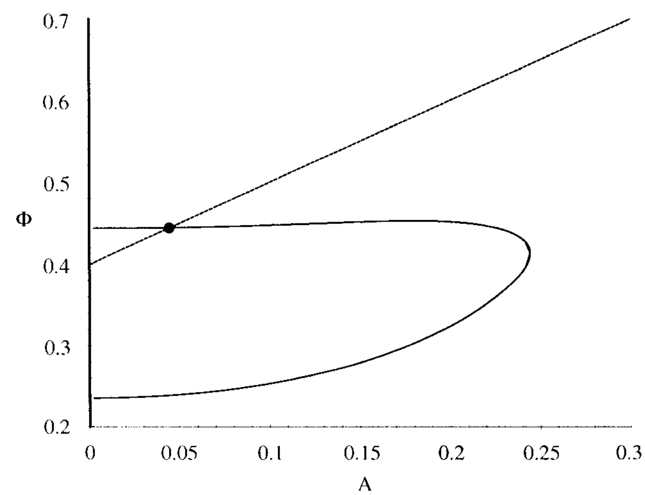
$$\begin{aligned} c_{11} &:= -1, \\ c_{12} &:= \frac{1}{2}\left(-\frac{a_{12}(\Phi, A)}{a_{11}(A)} - l_c\sigma\alpha a_{11}(A)\right), \\ c_{22} &:= -\bar{c}_\Phi - l_c\sigma\alpha a_{12}(\Phi, A), \\ c_{13} &:= -2l_c\sigma B^2\left(\frac{\partial I_1(\Phi, A)}{\partial A} - \alpha\bar{c}_\Phi\right)a_{11}(A), \\ c_{23} &:= -\frac{1}{2} - 2l_c B^2\left(\frac{\partial I_1(\Phi, A)}{\partial A} - \alpha\bar{c}_\Phi\right)a_{12}(\Phi, A) \\ &\quad + 2B^2\bar{c}_\Phi\left(\frac{\partial I_1(\Phi, A)}{\partial \Phi} + \bar{c}_\Phi\right). \end{aligned} \tag{60}$$

One can use the bounds (43) and (44) to obtain a conservative lower bound for c_Ψ, together with the following bounds that are easy to establish:

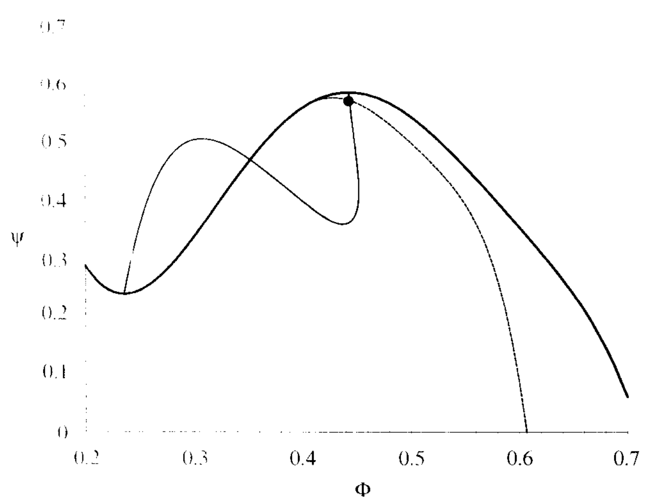
$$\begin{aligned} sup_{\mathcal{A}}\left|\frac{\partial I_1(\Phi, A)}{\partial A}\right| &\leq \sup_{\mathcal{A}}|\Psi_c'(\Phi)| \\ sup_{\mathcal{A}}\left|\frac{\partial I_1(\Phi, A)}{\partial \Phi}\right| &\leq \sup_{\mathcal{A}}|\Psi_c'(\Phi)|. \end{aligned} \tag{61}$$



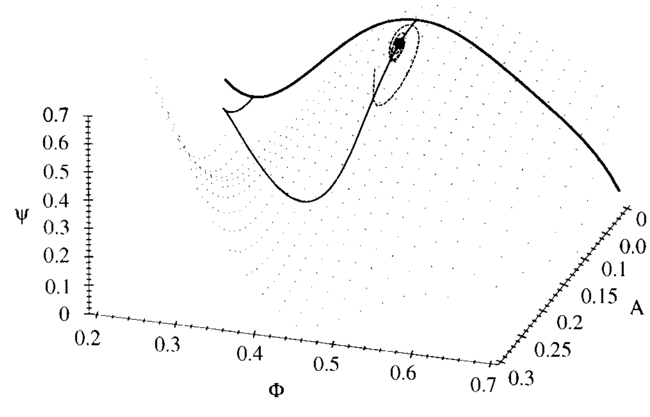
Equilibria of the closed-loop system



Projections of $C_{A,\Phi}$ and $C_{\Phi,\Psi}$ onto A-Φ plane



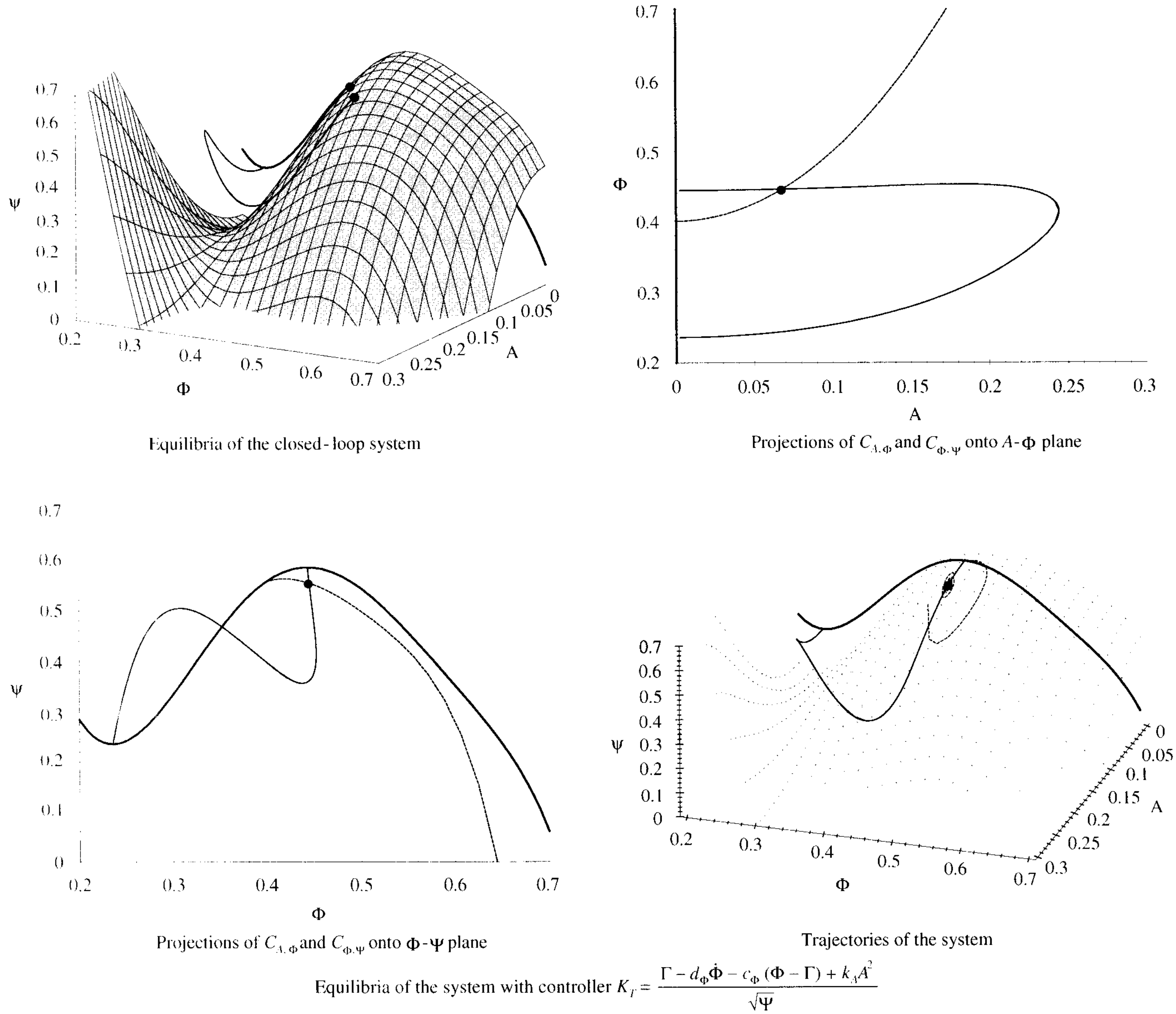
Projections of $C_{A,\Phi}$ and $C_{\Phi,\Psi}$ onto Φ-Ψ plane



Trajectories of the system

Equilibria of the system with controller $K_T = \dfrac{\Gamma - d_\Phi\dot{\Phi} - c_\Phi(\Phi - \Gamma) + c_A A}{\sqrt{\Psi}}$

Fig. 12. Equilibria for the controller with a linear stall feedback.



Equilibria of the closed-loop system

Projections of $C_{A,\Phi}$ and $C_{\Phi,\Psi}$ onto A-Φ plane

Projections of $C_{A,\Phi}$ and $C_{\Phi,\Psi}$ onto Φ-Ψ plane

Trajectories of the system

Equilibria of the system with controller $K_\Gamma = \dfrac{\Gamma - d_\Phi\dot{\Phi} - c_\Phi(\Phi - \Gamma) + k_A A^2}{\sqrt{\Psi}}$

Fig. 13. Equilibria for the controller with a quadratic stall feedback.

5. The gains of the controller

$$K_\Gamma = \frac{\Gamma - d_\Phi\dot{\Phi} - c_\Phi(\Phi - \Gamma) + c_A A}{\sqrt{\Psi}}$$

are

$$d_\Phi := l_c c_\Psi, \qquad c_\Phi := \bar{c}_\Phi c_\Psi - 1, \qquad c_A := \alpha\bar{c}_\Phi c_\Psi. \tag{62}$$

Fig. 12 shows the trajectories of the C_3' compressor with parameters $l_c = 4.75$, $\sigma = 0.41$, $B = 0.1$ under control law

$$K_\Gamma = \frac{\Gamma - d_\Phi\dot{\Phi} - c_\Phi(\Phi - \Gamma) + c_A A}{\sqrt{\Psi}}.$$

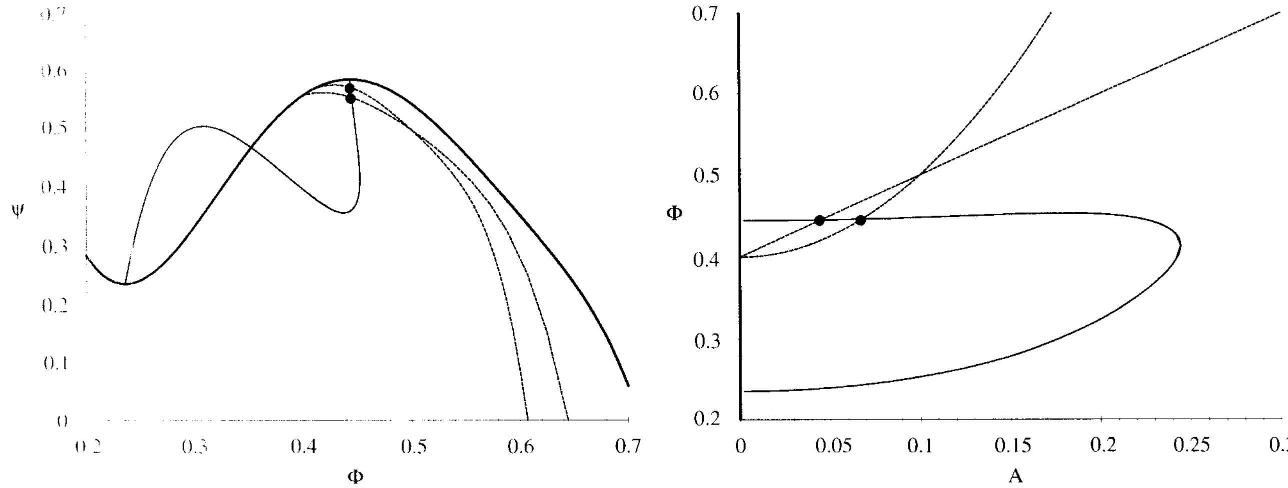
The trajectories start at points (A, Φ_0, Ψ_0), for A equal 0.03, 0.06, 0.1 for $\Gamma = 0.4$. Values of parameters were $\alpha = 1$, $\bar{c}_\Phi = 1$, $c_\Psi = 5$, $d_\Phi = 23.75$, $c_\Phi = 4$, $c_A = 5$.

For comparison, Fig. 13 shows the trajectories of the C_3' compressor under control law

$$K_\Gamma = \frac{\Gamma - d_\Phi\dot{\Phi} - c_\Phi(\Phi - \Gamma) + k_A A^2}{\sqrt{\Psi}}.$$

The trajectories start at points (A, Φ_0, Ψ_0), for A equal 0.03, 0.06, 0.1 for $\Gamma = 0.4$. Values of parameters were $d_\Phi = 23.75$, $c_\Phi = 4$, $k_A = 50$. Note that for this controller we have $\tilde{\Phi}^\Gamma(A) = (\Gamma + k_A/(1 + c_\Phi))A^2$.

For the same value $\Gamma = 0.4$, the quadratic dependence of $\tilde{\Phi}^\Gamma(A)$ on A (despite the high gain $k_A = 50$) causes the stall equilibrium to occur for larger values of A than in the case of the previously considered controller linear in A. This also makes the value of the pressure rise coefficient at the equilibrium to be lower for the controller



The equilibria for controllers $K_l = \frac{\Gamma - d_\Phi \Phi - c_\Phi(\Phi - \Gamma) + c_A A}{\sqrt{\Psi}}$ and $K_T = \frac{\Gamma - d_\Phi \Phi - c_\Phi(\Phi - \Gamma) + k_A A^2}{\sqrt{\Psi}}$ for $\Gamma = .4$

Fig. 14. Comparison of equilibria for the controller with a linear and quadratic stall feedback.

quadratic in A than for the controller linear in A (see Fig. 14). (A similar observation based on local analysis was made in Krener, 1995.)

8. General guidelines for choosing the throttle surface

The physics of compression systems is very complicated and not completely understood. While the three-dimensional Moore–Greitzer model of compressor seems to capture essential qualitative features of compressor dynamics, it is not acurate enough to allow any conclusions on a very detailed level. Therefore, we believe that the procedure for controller design (or validation of exisiting controller candidates) according to graph backstepping procedure presented in the previous sections should be treated as a source of some *general guidelines* for controller design rather than an actual algorithm for designing the controller.

The results of the previous sections suggest the following simple general guidelines for choosing the throttle surface and the gain function:

1. Near the peak the projection of the intersection of the throttle surface S_Ψ (given by equation $\Psi = \tilde{\Psi}(\Phi, A)$) and the turning surface S_Φ (given by equation $\Psi = I_1(\Phi, A)$) graphed as a function $\Phi = \tilde{\Phi}(A)$ should have a positive slope larger than the largest positive slope of the stall equilibria loop (graphed as a function $\Phi = f(A)$ close to the peak). (See Figs. 7 and 14.) The more right skew the compressor the larger should be the slope of $\tilde{\Phi}(A)$. Slope of $\tilde{\Phi}(A)$ equal one or greater suffices. Larger difference of slopes of $\tilde{\Phi}(A)$ and $f(A)$ results in a lower pressure drop when system enters rotating stall. However, the saturation and bandwith limitations of the actuator restrict from above attainable slopes of $\tilde{\Phi}(A)$.

2. The throttle surface S_Ψ should lie below the turning suface S_Φ to the left of their intersection and above the turning surface S_Φ to the right of their intersection. The distance between the throtle surface and the turning surface S_Φ should be an increasing function of Φ. Larger differences of slopes of the throttle surface S_Ψ and turning surface S_Φ make the system more robust with respect to changes of the compressor characteristic and disturbances. However, a transient pressure drop in surge is more severe if the throttle surface S_Ψ drops too fast to the left of the intersection of S_Ψ and S_Φ. Moreover, the saturation and bandwith limitations of the actuator restrict achievable steepness of the throttle surface (Fig. 15).

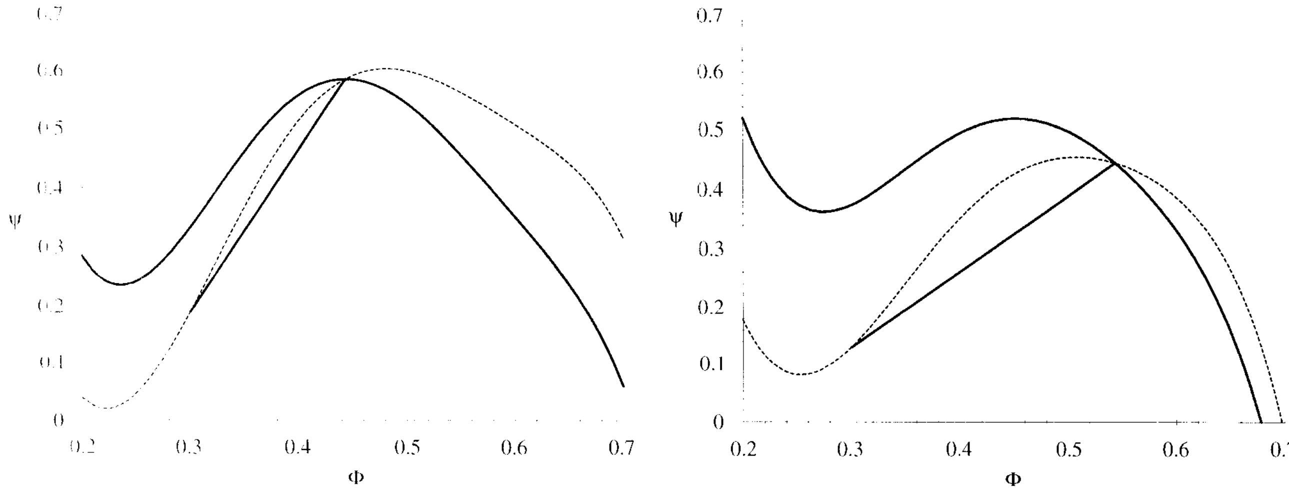
3. The gain function $c_\Psi(\Phi, A)$ should be big enough. Roughly, $c_\Psi(\Phi, A) > 4B^2 b(\Phi, A)$, where B is the Greitzer parameter, $b(\Phi, A)$ is the slope of a secant line connecting points $\tilde{\Psi}(\Phi, A)$ and $\tilde{\Psi}(\tilde{\Phi}(A), A)$ on the throttle surface S_Ψ.

If the differences of slopes mentioned in Steps 1 and 2 and the gain function $c_\Psi(\Phi, A)$ are big enough then one has *global stability*.

Fig. 16 shows turning surfaces for four controllers stabilizing the peak of the characteristic.

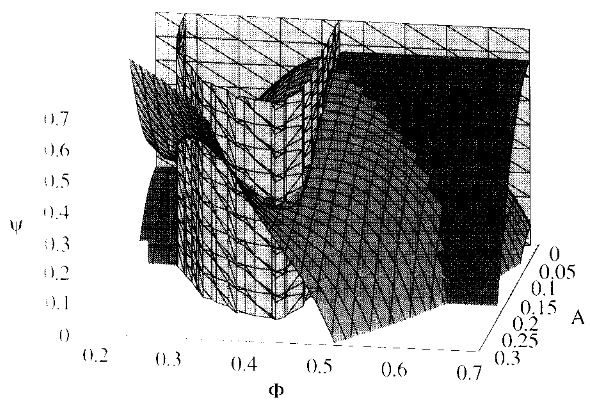
As we can see, all these controllers obey the general guidelines stated above. They have similar throttle surfaces and thus they yield closed-loop systems with similar dynamic properties. While these controllers have been designed in a different way, they all can be treated as graph backstepping controllers using the method presented in Section 5.

To simplify the analysis we neglected the saturation bounds of the actuator. However, these bounds are relatively easy to incorporate. Suppose that the throttle



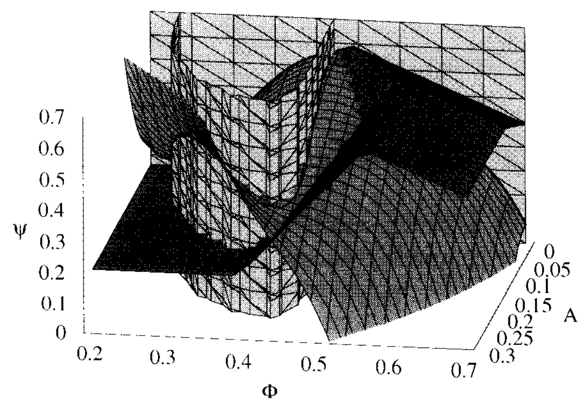
Crossections of S_Φ and S_Ψ with plane $A = 0$ and with plane $A = 0.1$

Fig. 15. Illustration of graph backstepping procedure.



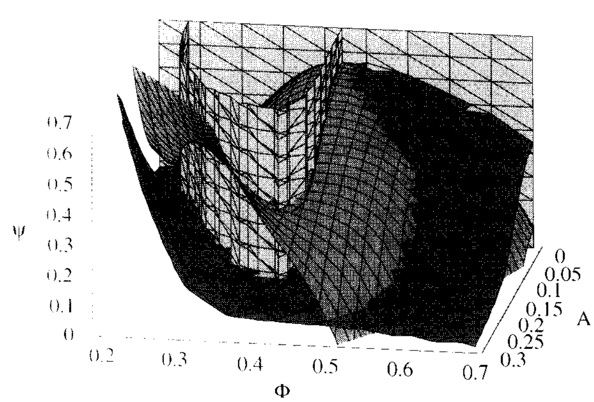
$$K_I = \frac{\Gamma + c_\Psi \Psi - c_\Phi sat(\Phi) + k_A A^2}{\sqrt{\Psi}}$$

Backstepping with Liaw-Abed term



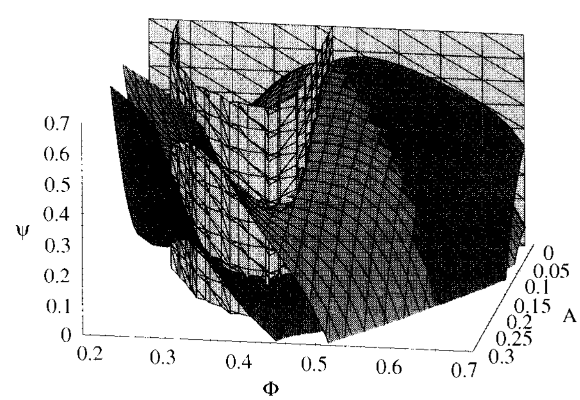
$$K_I = \frac{\Gamma + c_\Psi \Psi - h(\Phi, A)}{\sqrt{\Psi}}$$

Graph backstepping ($h(\Phi, A)$ piecewise linear)



$$K_I = \Gamma - d_\Phi \dot{\Phi} + k_A A^2$$

UTRC with Liaw-Abed term



$$K_I = \frac{\Gamma - d_\Phi \dot{\Phi} - c_\Phi \Phi + c_A A}{\sqrt{\Psi}}$$

Graph backstepping

Fig. 16. Turning surfaces for four controllers.

opening K_T is bounded from below by K_{T1} and from above by K_{T2}. Then one can easily check that the realizable throttle surface has to satisfy the inequalities $K_{T1}\Phi^2 \leq \tilde{\Psi}(\Phi, A) \leq K_{T2}\Phi^2$.

The effect of a limited bandwith is more tricky to deal with, but some insight can still be gained via a graphical analysis of the flow.

9. Conclusion

We have shown that instead of using particular parametrizations of the characteristic, one can work on a general level using slopes and curvatures of the characteristic, i.e., information about the shape of the characteristic.

For compressors with quite general characteristic we considered controllers of the form $K_T = (c_\Psi \Psi + h(\Phi, A))/\sqrt{\Psi}$ and $K_T = (d_\Phi \dot{\Phi} + h(\Phi, A))/\sqrt{\Psi}$. More generally, the results of this paper apply to any controller that allows to put the last equation of (1) in the equivalent form

$$\dot{\Psi} = \frac{-c_\Psi(\Phi, A)}{4l_c B^2}(\Psi - \tilde{\Psi}(\Phi, A)).$$

We have proposed a graph backstepping procedure for controller design involving construction of a throttle surface $\tilde{\Psi}(\Phi, A)$ and the gain function $c_\Psi(\Phi, A)$.

Graph backstepping can be used to design controllers or to verify controller candidates for MG3. It provides simple general guidelines for contruction of controllers.

We have shown that for a general compressor characteristic every potential axisymmetric equilibrium on the decreasing part of the compressor characteristic, the peak of the characterisitic, and every rotating stall equilibrium close to the peak can be globally or semi-globally stabilized by an appropriate choice of the throttle surface and the gain function.

We obtained lower bounds on the gains of the controller in terms of divided differences related to compressor characteristic. These bounds can be expressed using bounds on the first and second derivative of the characteristic in the region of operation.

We presented simple general guidelines for choosing the throttle surface.

Note that a graphical representation of controllers, their graphical comparison, simple analysis of the flow to prove stability, and studying performance limitations is possible because MG3 is a three-dimensional model.

While graphical/topological analysis is more difficult for higher dimensional models, the intuition one gains from working with MG3 is still valuable. In fact, the intuition gained form the backstepping control design of the present paper has happen valuable for solving the pr oblem of a global stabilization of the full Moore–Greitzer infinite-dimensional model in (Banaszuk et al., 1998). The controller proposed in Banaszuk et al. (1998) has a form similar to the one proposed in the present paper, with the magnitude of the first harmonic of the stall cell replaced with the norm of stall cell in the Sobolev space H^1. Alternatively, the minimum of the stall cell could be used for feedback (see Banaszuk et al., 1997). The choice of controller gains in examples provided in Banaszuk et al. (1998) and Banaszuk et al. (1997) has been guided by general design guidelines provided in Section 8.

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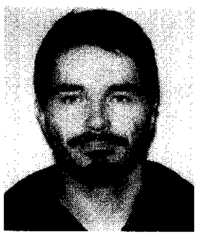

Andrzej Banaszuk has received Ph.D. in Electrical Engineering from Warsaw University of Technology in 1989, and Ph.D. in Mathematics from Georgia Institute of Technology in 1995. From 1989 to 1997 he has held various research and teaching positions at Warsaw University of Technology, Georgia Institute of Technology, University of Colorado at Boulder, and University of California at Davis. During that time he performed research in various areas of control theory including implicit systems, approximate feedback linearization of nonlinear systems, trajectory planning for nonlinear systems, nonlinear observers, feedback stabilization of periodic orbits, and control of surge and rotating stall in jet engines. He is an author or co- author of about 25 journal papers and numerous conference papers.

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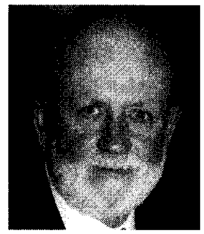
In 1998 Andrzej Banaszuk became an Associate Editor of IEEE Transactions on Control Systems Technology.



Arthur J. Krener was born in Brooklyn, NY, on October 8, 1942. He received the BS degree from Holy Cross College in 1964 and the M.A. and Ph.D. degrees from the University of California, Berkeley, in 1967 and 1971, all in Mathematics. Since 1971 he has been at the University of California, Davis, where he is currently Professor of Mathematics. He has held visiting positions at Harvard University, the University of Rome, Imperial College of Science and Technology, NASA Ames Research Center, the University of California, Berkeley, the University of Paris IX, the University of Maryland, University of Newcastle and the University of Padua.

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