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Control of Center Manifolds

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Abstract— **In this paper, we use a feedback to change the orientation and the shape of the center manifold of a system with uncontrollable linearization. This change will directly affect the reduced dynamics on the center manifold, and hence will change the stability properties of the original system.**

I. Introduction and Problem Statement

Center manifold theory plays an important role in the study of the stability of nonlinear systems when some eigenvalues of the linearized system are on the imaginary axis and the others are in the open left half plane. The center manifold is an invariant manifold of the differential equation which is tangent at the equilibrium point to the eigenspace of the neutrally stable eigenvalues. In practice, one does not compute the center manifold and its dynamics exactly, since this requires the resolution of quasilinear partial differential equation which is not easily solvable. In most cases of interest, an approximation of degree two or three of the solution is sufficient. Then, we determine the reduced dynamics on the center manifold, study its stability and then conclude about the stability of the original system [23], [26], [19], [8], [14].
The combination of this theory with the normal form approach of Poincaré [24] was used extensively to study parameterized dynamical systems exhibiting bifurcations.
This approach is also useful in control theory. The combination of the normal form approach for control systems [18] and center manifold theory enabled the analysis and stabilization of systems with uncontrollable linearization [16], [17], [13], [21], [11]. It can also be viewed as a reduction technique for some classes of ordinary and partial differential equations which show dominant subdynamics. The synthesis of a feedback for these differential equations is then performed for the reduced ordinary differential equation.
In this paper, we show that we can use a feedback to change the orientation and the shape of the center manifold of a system with uncontrollable linearization. This change will directly affect the reduced dynamics on the center manifold, and hence will change the stability properties of the original system.
The paper is organized as follows : In section §1, we determine the linear part of the center manifold and show that a linear feedback is sufficient to change the orientation, then, in section §2 we determine the quadratic part of the center manifold, and show how a quadratic feedback can change its shape. We show also that this permits to use Lyapunov functions to study the stabilization problem of systems with uncontrollable linearization.

Consider the following nonlinear system

$$\dot{\zeta} = f(\zeta, v) \tag{1}$$

the variable $\zeta \in \mathbb{R}^n$ is the state, $v \in \mathbb{R}$ is the input variable. The vectorfield $f(\zeta)$ is assumed to be C^k for some sufficiently large k.
Assume $f(0,0) = 0$, and suppose that the linearization of the system at the origin is (A, B),

$$A = \frac{\partial f}{\partial \zeta}(0,0), \quad B = \frac{\partial f}{\partial v}(0,0),$$

with

$$\text{rank}([B \; AB \; A^2B \; \cdots \; A^{n-1}B]) = n - r. \tag{2}$$

Let us denote by $\Sigma_{\mathcal{U}}$ the system (1) under the above assumptions.
The system $\Sigma_{\mathcal{U}}$ is not linearly controllable at the origin, and a change of some control properties may occur around this equilibrium point, this is called a control bifurcation if it is linearly controllable at other equilibria. [21].
From linear control theory, we know that there exist a linear change of coordinates and a linear feedback transforming the system $\Sigma_{\mathcal{U}}$ to

$$\begin{aligned} \dot{\bar{z}} &= A_1 \bar{z} + f_1^{[d\geq 2]}(\bar{z}, \bar{x}, \bar{u}), \\ \dot{\bar{x}} &= A_2 \bar{x} + B_2 \bar{u} + f_2^{[d\geq 2]}(\bar{z}, \bar{x}, \bar{u}), \end{aligned} \tag{3}$$

with $\bar{z} \in \mathbb{R}^r$, $\bar{x} \in \mathbb{R}^{n-r}$, $\bar{u} \in \mathbb{R}$, $A_1 \in \mathbb{R}^{r\times r}$ is in the Jordan form, $A_2 \in \mathbb{R}^{(n-r)\times(n-r)}$, $B_2 \in \mathbb{R}^{(n-r)\times 1}$,

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

and $f_k^{[d\geq 2]}(\bar{z}, \bar{x}, \bar{u})$, for $k = 1, 2$, designates a vector field which is a homogeneous polynomial of degree $d \geq 2$.

Moreover, from [21] we know the quadratic normal form of the system $\Sigma_{\mathcal{U}}$.

Theorem 1.1: For the system (3), there exists a quadratic change of coordinates and feedback

$$\begin{bmatrix} z \\ x \end{bmatrix} = \begin{bmatrix} \bar{z} \\ \bar{x} \end{bmatrix} + \phi^{[2]}(\bar{z}, \bar{x}) \tag{4}$$

$$\bar{u} = u + \alpha^{[2]}(\bar{z}, \bar{x}, u), \tag{5}$$

which transforms the system into the quadratic normal form

$$\begin{aligned} \begin{bmatrix} \dot{z} \\ \dot{x} \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} z \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u \\ &+ \begin{bmatrix} \tilde{f}_1^{[2;0]} & + & \tilde{f}_1^{[1;1]} & + & \tilde{f}_1^{[0;2]} \\ 0 & + & 0 & + & \tilde{f}_2^{[0;2]} \end{bmatrix} (z, x, u) \\ &+ O(z, x, u)^3 \end{aligned} \tag{6}$$

where $\tilde{f}_i^{[d_1;d_2]} = \tilde{f}_i^{[d_1;d_2]}(z; x, u)$ denotes a polynomial vector field homogeneous of degree d_1 in z and homogeneous of degree d_2 in x, u. The vector field $\tilde{f}_1^{[2;0]}$ is in the quadratic normal form of Poincaré [27],

$$\tilde{f}_1^{[2;0]} = \sum_{i=1}^{r} \sum_{\substack{1 \le j \le k \le r \\ \lambda_j + \lambda_k = \lambda_i}} \beta_i^{jk} \; \mathbf{e}^{1,i} \; z_j \; z_k \tag{7}$$

where $\mathbf{e}^{r,i}$ is the i^{th} unit vector in z space when $r = 1$ and in the x space when $r = 2$. z_i is the i^{th} component of z. The other vector fields are as follows.

$$\tilde{f}_1^{[1;1]} = \sum_{i=1}^{r} \sum_{j=1}^{r} \gamma_i^{j1} \; \mathbf{e}^{1,i} \; z_j \; x_1 \tag{8}$$

$$\tilde{f}_1^{[0;2]} = \sum_{i=1}^{r} \sum_{j=1}^{n-r+1} \delta_i^{jj} \; \mathbf{e}^{1,i} \; x_j^2 \tag{9}$$

$$\tilde{f}_2^{[0;2]} = \sum_{i=1}^{n-r-1} \sum_{j=i+2}^{n-r+1} \epsilon_i^{jj} \; \mathbf{e}^{2,i} \; x_j^2 \tag{10}$$

where for notational convenience we have defined $x_{n-r+1} = u$. The vector field, $\tilde{f}_2^{[0;2]}$, is in quadratic controller form [18]. Let $\sigma_{jk} = 2$ if $j = k$ and $\sigma_{jk} = 1$ otherwise. The quadratic invariants are as follows,

$$\beta_i^{jk} = \tfrac{1}{\sigma_{jk}} \tfrac{\partial^2 f_{1,i}^{[2]}}{\partial z_j \, \partial z_k}(0,0,0)$$
for $1 \le i \le r, 1 \le j \le k \le r$ and $\lambda_i = \lambda_j + \lambda_k$,

$$\gamma_i^{j1} = \textstyle\sum_{k=1}^{n-r+1} (\lambda_i - \lambda_j)^{k-1} \tfrac{\partial^2 f_{1,i}^{[2]}}{\partial z_j \, \partial x_k}(0,0,0)$$
for $1 \le i, j \le r$,

$$\delta_i^{jj} = \tfrac{1}{2} \left[\mathbf{ad}_F^{k+1}(G), \mathbf{ad}_F^{k}(G) \right] (z_i)(0,0,0)$$
where $k = n - r + 1 - j$ for $1 \le i \le n - r - 1$, $i + 2 \le j \le n - r + 1$.

$$\epsilon_i^{jj} = \tfrac{1}{2} \left[\mathbf{ad}_F^{k+1}(G), \mathbf{ad}_F^{k}(G) \right] (x_i)(0,0,0)$$
where $k = n - r + 1 - j$
for $1 \le i \le n - r - 1, i + 2 \le j \le n - r + 1$,

with

$$ad_X(Y) = [X, Y] = \frac{\partial Y}{\partial \xi} X - \frac{\partial X}{\partial \xi} Y,$$

for a given two vector fields $X(\xi)$ and $Y(\xi)$ defined in $\mathbb{R}^n$, and

$$F = \sum_{j=1}^{r} f_{1,j}(z, x, u) \frac{\partial}{\partial z_j} + \sum_{j=1}^{n-r} f_{2,j}(z, x, u) \frac{\partial}{\partial x_j} \tag{11}$$

$$G = \frac{\partial}{\partial u} \tag{12}$$

Now, let us consider the feedback given by

$$u = K_1 z + K_2 x + z^T Q_{fb} z + O(z, x)^3, \tag{13}$$

with $K_1 = \begin{bmatrix} k_{1,1} & \cdots & k_{1,r} \end{bmatrix}$ and $K_2 = \begin{bmatrix} k_{2,1} & \cdots & k_{2,n-r} \end{bmatrix}$.
Moreover, let us assume that K_2 is such that the eigenvalues of $A_2 + B_2 K_2$ are in the open left half plane, i.e.
Property $\mathcal{P}$: The matrix $\bar{A}_2 = A_2 + B_2 K_2$ is Hurwitz.

Let us denote by $\mathcal{F}$ the feedback (13) with the property $\mathcal{P}$. The closed loop system (6)-(13) possesses r eigenvalues on the imaginary axis, and $n - r$ eigenvalues in the open left half plane. Thus, a center manifold exists. It is represented locally around the origin as

$$W^c = \{(z, x) \in \mathbb{R}^r \times \mathbb{R}^{n-r} | x = \Pi(z), |z| < \delta, \Pi(0) = 0\}. \tag{14}$$

The next sections deal with the computation of the linear and quadratic parts of Π in (14) for the system (6)-(13) and their link with the parameters of the feedback $\mathcal{F}$.

II. Linear Center Manifold

Consider the system (3) and the linear part of the feedback $\mathcal{F}$

$$u = K_1 z + K_2 x + O(z, x)^2. \tag{15}$$

Suppose that the linear part of the center manifold is

$$\Pi^{[1]}(z) = \Pi^{[1]} z \tag{16}$$

with $\Pi^{[1]} \in \mathbb{R}^{(n-r) \times r}$. Since, on the center manifold we have $x = \Pi(z)$, then

$$\dot{x} = \frac{d\,\Pi(z)}{dt} = \frac{\partial\,\Pi(z)}{\partial\, z} \dot{z}.$$

Thus, from (3)

$$\frac{\partial\,\Pi(z)}{\partial\, z} \dot{z} = A_2 \Pi(z) + B_2 \Big(\sum_{i=1}^{r} k_{1,i} z_i + \sum_{i=1}^{n-r} k_{2,i} \Pi_i(z) \Big) + O(z, x)^2. \tag{17}$$

Replacing $\Pi(z)$ by its linear part (16) we deduce that

$$\begin{aligned} \Pi_{i+1}^{[1]}(z) &= \frac{\partial \Pi_i^{[1]}(z)}{\partial\, z} A_1 z, \qquad i \le n - r \\ \sum_{i=1}^{n-r} k_{2,i} \Pi_i^{[1]}(z) + K_1 z &= \frac{\partial \Pi_{n-r}^{[1]}(z)}{\partial\, z} A_1 z \end{aligned}$$

where $\Pi_i^{[1]}(z)$ represents the ith row of $\Pi^{[1]}(z)$. So

$$\begin{array}{ll} \Pi_{i+1}^{[1]} = \Pi_i^{[1]} A_1, & i \leq n-r, \\ \sum_{i=1}^{n-r} k_{2,i}\, \Pi_i^{[1]} + K_1 = \Pi_{n-r}^{[1]} A_1, & \end{array}$$

Hence,

$$\begin{array}{ll} \Pi_{i+1}^{[1]} = \Pi_1^{[1]} A_1^i, & i \leq n-r, \\ \sum_{i=1}^{n-r} k_{2,i}\, \Pi_1^{[1]} A_1^{i-1} + K_1 = \Pi_1^{[1]} A_1^{n-r}. & \end{array} \tag{18}$$

Let $P(\lambda)$ be the characteristic polynomial of $\bar{A}_2$, then

$$P(\lambda) = \lambda^{n-r} - \sum_{i=1}^{n-r} k_{2,i}\lambda^{i-1}, \tag{19}$$

Hence, the equation (18) can be rewritten as

$$\left\{ \begin{array}{ll} \Pi_1^{[1]} = K_1 P(A_1)^{-1}, & \\ \Pi_{i+1}^{[1]} = \Pi_1^{[1]} A_1^{i-1}, & i \leq n-r, \end{array} \right. \tag{20}$$

$P(A_1)$ is invertible since the eigenvalues of A_1, on the imaginary axis, don't coincide with the solutions of $P(\lambda) = 0$, lying in the open left half-plane, since $\bar{A}_2$ is Hurwitz.

Theorem 2.1: Given the feedback $\mathcal{F}$, the center manifold is given by

$$x = \Pi^{[1]} z + O(z^2)$$

with the components of $\Pi^{[1]}$ uniquely determined by (20).

Now, let us show that the orientation of the center manifold can be changed by changing K_1 in (15).

If we view the center manifold represented by $x = \Pi(z)$ as a submanifold in the space of $(z, x) \in \mathbb{R}^n$, we can say that the orientation of the center manifold at the origin is a basis of the orthogonal complement subspace of the tangent space of the center manifold. Indeed, the orientation of the center manifold at the origin is a set of vectors which are orthogonal to the manifold, they are linearly independent and they generate a complement subspace of the manifold.

Theorem 2.2: Given any $(n-r) \times r$ matrix of the form

$$[\mathcal{M}_{(n-r)\times r} \;\; \mathcal{N}_{(n-r)\times(n-r)}]$$

Then, its row vectors define the center manifold orientation at the origin for (3)-(15) if and only if $\mathcal{N}^{-1}$ exists and $\Pi^{[1]} = -\mathcal{N}^{-1}\mathcal{M}$ satisfies (20).

Proof: Suppose that $[\mathcal{M}_{(n-r)\times r} \;\; \mathcal{N}_{(n-r)\times(n-r)}]$ defines the orientation of the center manifold. Then, it is orthogonal to the tangent space of the center manifold. It is known that the tangent space of the center manifold is given by its linear part,

$$x - \Pi^{[1]} z = 0,$$

where $\Pi^{[1]}$ satisfies (20). In the (z, x)-space, a set of orthogonal vectors of the tangent space is the row vectors of $[-\Pi^{[1]} \;\; I]$. Therefore, both $[-\Pi^{[1]} \;\; I]$ and $[\mathcal{M}_{(n-r)\times r} \;\mathcal{N}_{(n-r)\times(n-r)}]$ generate the same space, which is orthogonal to the tangent space of the center manifold. Therefore, the row vectors of $[-\Pi^{[1]} \; I]$ are linear combinations of the row vectors in $[\mathcal{M}_{(n-r)\times r} \;\; \mathcal{N}_{(n-r)\times(n-r)}]$, i.e.

$$[-\Pi^{[1]} \; I] = \mathcal{N}^{-1}[\mathcal{M}_{(n-r)\times r} \;\; \mathcal{N}_{(n-r)\times(n-r)}].$$

So, $\Pi^{[1]} = -\mathcal{N}^{-1}\mathcal{M}$ and it satisfies (20).

On the other hand, suppose $-\mathcal{N}^{-1}\mathcal{M}$ satisfies (20). By Theorem 2.1, the linear space

$$\mathcal{N}^{-1}\mathcal{M}\, z + x = 0$$

represents the linear part of the center manifold. It is the tangent space of the center manifold. Therefore, $[\mathcal{N}^{-1}\mathcal{M} \mid I]$, the row vectors in the coefficient matrix of this equation, forms a basis of the orthogonal space. It is easy to check that the row vectors of $[\mathcal{M}\;\mathcal{N}]$ and $[\mathcal{N}^{-1}\mathcal{M} \;\; I]$ generate the same vector space. Therefore, $[\mathcal{M} \;\; \mathcal{N}]$ defines the orientation of the center manifold. ∎

Now, consider the following change of coordinates

$$\tilde{x}_i = x_i - \Pi_1^{[1]} A_1^{i-1} z, \quad i = 1, \cdots, n-r \tag{21}$$

then,

$$\begin{array}{rcl} \dot{\tilde{x}}_i &=& \tilde{x}_{i+1}, \quad i = 1, \cdots, n-r \\ \dot{\tilde{x}}_{n-r} &=& \sum_{i=1}^{n-r} k_{2,i}\tilde{x}_i \end{array}$$

Hence the coefficient K_1 have been removed from the $x-$part of the dynamics (3)-(15) by a change of coordinates. With $K_1 = 0$, we deduce from (20) that $\Pi^{[1]} = 0$. So, the linear terms of the center manifold have been removed.

Proposition 2.1: Given any feedback (15) satisfying Property $\mathcal{P}$, and the change of coordinates (21), then the center manifold is given by

$$\tilde{x} = O(z^2)$$

In the following, and for reasons of simplicity, we use (z, x) instead of $(z, \tilde{x})$.

III. Quadratic Center Manifold

In this section we determine the quadratic part of the center manifold and show it can be affected by Q_{fb} of the feedback $\mathcal{F}$. We present also a procedure to have conditions on the entries of Q_{fb} such that we have asymptotic stability of the origin of the closed loop system (6)-(13).

Theorem 3.1: Consider the normal form (6). Under the nonlinear feedback $\mathcal{F}$, the quadratic part of the center manifold satisfies

$$\Pi_i^{[2]}(z) = z^T Q_i z, \tag{22}$$

with Q_i directly linked to Q_{fb} (through equation (23) below).

Proof: Consider a system in the normal form (6). Since the linear part of the center manifold has been removed through the change of coordinates (21), the center manifold has the following form

$$x_i = \Pi_i(z) = z^T Q_i z + O(z^3),$$

where Q_i, $i = 1, \cdots, n-r$, are real $r \times r$ symmetric matrices. From the center manifold equation (17) we deduce that

$$z^T(Q_i A_1 + A_1^T Q_i)z = \begin{cases} z^T Q_{i+1} z, & \text{for } 1 \le i \le n-r-1, \\ \sum_{i=1}^{n-r} k_{2,i} z^T Q_i z + z^T Q_{fb} z, & \end{cases}$$

This equation is equivalent to

$$\begin{array}{rcl} Q_{i+1} & = & \mathcal{S}_{A_1}(Q_i), \\ Q_{fb} & = & \mathcal{S}_{A_1}(Q_{n-r}) - \sum_{i=1}^{n-r} k_{2,i} Q_i = P(\mathcal{S}_{A_1})(Q_1) \end{array} \tag{23}$$

with $\mathcal{S}_{A_1}$ the operator defined $\mathcal{S}_{A_1}(Q_i) = A_1^T Q_i + Q_i A_1$. The spectrum of this operator is in $\{\lambda_i + \lambda_j \ : \text{for } \ i, j = 1, \cdots, r\}$ with λ_ℓ for $\ell = 1, \cdots, r$ are the eigenvalues of A_1. So the eigenvalues of this operator are on the imaginary axis. Since $\bar{A}_2$ is Hurwitz, then the solutions of $P(\lambda) = 0$ are in the open left half plane and so don't coincide with the eigenvalues of $\mathcal{S}_{A_1}$; hence, the operator $P(\mathcal{S}_{A_1})$, whose the eigenvalues are in $\{P(\lambda_i + \lambda_j) \ : \text{for } \ i, j = 1, \cdots, r\}$, is invertible and there is a direct correspondence between Q_1 and Q_{fb}.

■

Using the expressions of the center manifold (22) and the normal form (6), we deduce the normal form of the dynamics of $\Sigma_{\mathcal{U}}$ on the center manifold :

$$\dot{z} = A_1 z + \tilde{f}_1^{[2;0]}(z) + \sum_{i=1}^{r} \sum_{j=1}^{r} \gamma_i^{j1} \ z_j \ z^T Q_1 z \mathbf{e}^{1,i} + \tilde{f}_1^{[3;0]}(z) + O(z^4), \tag{24}$$

with

$$\tilde{f}_1^{[3;0]} = \sum_{i=1}^{r} \sum_{\substack{1 \le j \le k \le r \le \ell \\ \lambda_j + \lambda_k + \lambda_\ell = \lambda_i}} \beta_i^{jk\ell} \ \mathbf{e}^{1,i} \ z_j \ z_k \ z_\ell, \tag{25}$$

and

$$\beta_i^{jkl} = \frac{1}{\sigma_{jkl}} \ \frac{\partial^3 f_{1,i}^{[3]}}{\partial z_j \, \partial z_k \, \partial z_l}(0,0,0), \quad \text{for} \quad 1 \le i, j, k \le r, \tag{26}$$

and $\lambda_i = \lambda_j + \lambda_k + \lambda_l$, with $\sigma_{jkl} = 6$ if $j = k = l$ and $\sigma_{jkl} = \sigma_{jk}\sigma_{jl}\sigma_{kl}$ otherwise.

Remark. In the $x-$coordinates system, the normal form of the dynamics of $\Sigma_{\mathcal{U}}$ on the center manifold is given in the appendix. ◁

Let $\Phi(Q_1, z) = \sum_{i=1}^{r} \sum_{j=1}^{r} \gamma_i^{j1} \ z_j \ z^T Q_1 z \mathbf{e}^{1,i}$ and $R^{[2,3]}(z) = \tilde{f}_1^{[2;0]}(z) + \tilde{f}_1^{[3;0]}(z)$.

The direct correspondence between the orientation and the quadratic shape of the center manifold given by (20) and (23) permit us to propose a procedure to find a feedback when a specific closed-loop dynamics of the form (24) is desirable.

Given a "reference model" of the form

$$\dot{z} = A_1 z + \sum_{i=1}^{r} \left\{ z^T \Theta_{0,i} z + \left[\ z^T \Theta_{1,i} z \quad \cdots \quad z^T \Theta_{r,i} z \ \right] z \right\} \mathbf{e}^{1,i} + O(z^4), \tag{27}$$

which is a dynamics corresponding to a pre-specified orientation and shape of the center manifold, our goal is to find a feedback $\mathcal{F}$ such that the reduced dynamics of $\Sigma_{\mathcal{U}} - \mathcal{F}$ on the center manifold match with the dynamics (27). The matrices $\Theta_{0,i}, \ \Theta_{1,i}, \cdots, \Theta_{r,i}$ are particular matrices in $\mathbb{R}^{r \times r}$, for $i = 1, \cdots, r$, whose entries are determined by the dynamics (24) or (33).

Suppose that $V(z) = \frac{1}{2} z^T z$ is a Lyapunov function for the dynamics (24). Then

$$\begin{array}{rcl} \dot{V} & = & z^T(A_1^T + A_1)z + \{R^{[2,3]}(z) + \Phi(Q_1, z)\}^T z \\ & + & z^T\{R^{[2,3]}(z) + \Phi(Q_1, z)\} + O(z^5). \end{array} \tag{28}$$

For $\dot{V}$ to be negative definite, it is sufficient that $-\dot{V}$ is a Sum of Squares (SOS). Several techniques exist to express $-\dot{V}$ as a SOS (see for example [6], [7] and [22] for a review of different techniques - see also [25] for a software). The basic idea of this method is to express $-\dot{V}$ as a quadratic form in some new variables w. Therefore, $-\dot{V}$ can be represented as

$$-\dot{V} = w^T \mathcal{V} w, \tag{29}$$

where $\mathcal{V}$ is a constant matrix. If $\mathcal{V}$ is positive definite, then $\dot{V}$ is negative definite. In our case $\mathcal{V}$ depends on Q_1 and hence Q_{fb} through (23). Conditions on positive definiteness of $\mathcal{V}$ give conditions on the entries of this matrix, and hence conditions on the entries of Q_{fb}.

In the following, we illustrate the procedure through two examples.

A. Transcritical Control Bifurcation

In this case $r = 1$ and $A_1 = 0$. The dynamics on the center manifold (24) is given by

$$\dot{z} = \beta_1^{11} z^2 + (\gamma_1^{11} Q_1 + \beta_1^{111}) z^3 + O(z^4),$$

and from (28), we have

$$\dot{V} = \left[\ z^2 \quad z \ \right] \mathcal{V}_1 \left[\begin{array}{c} z^2 \\ z \end{array} \right] + O(z^5),$$

with $\mathcal{V}_1 = \left[\begin{array}{cc} \gamma_1^{11} Q_1 + \beta_1^{111} & \frac{\beta_1^{11}}{2} \\ \frac{\beta_1^{11}}{2} & 0 \end{array} \right]$. In this case, w in (29) is given by $w = \left[\begin{array}{c} z^2 \\ z \end{array} \right]$.

For $\dot{V}$ to be negative definite, the minors of $-\mathcal{V}$ should be positive. So we should have

$$\gamma_1^{11} Q_1 + \beta_1^{111} < 0, \quad (30)$$

and

$$-\left(\frac{\beta_1^{11}}{2}\right)^2 > 0. \quad (31)$$

Since the second inequality is not satisfied, the choice of $K_1 = 0$ is not sufficient. Hence, we should find another value of K_1 permitting the minors of $-\mathcal{V}$ to be positive. From (33), the dynamics on the center manifold is given by

$$\begin{array}{rcl} \dot{z} & = & (\beta_1^{11} + \gamma_1^{11}\Pi_1^{[1]} + \delta_1^{11}(\Pi_1^{[1]})^2) z^2 \\ & + & \{(\gamma_1^{11} + 2\delta_1^{11}\Pi_1^{[1]})Q_1 + \beta_1^{111} + \gamma_1^{111}\Pi_1^{[1]} \\ & + & \delta_1^{111}(\Pi_1^{[1]})^2 + \epsilon_1^{111}(\Pi_1^{[1]})^3\} z^3 + O(z^4). \end{array} \quad (32)$$

If we choose $\Pi_1^{[1]}$ such that $\beta_1^{11} + \gamma_1^{11}\Pi_1^{[1]} + \delta_1^{11}(\Pi_1^{[1]})^2 = 0$, and provided $(\gamma_1^{11})^2 - 4\beta_1^{11}\delta_1^{11} > 0$ it will be sufficient to choose Q_1 such that

$$(\gamma_1^{11}+2\delta_1^{11}\Pi_1^{[1]})Q_1+\beta_1^{111}+\gamma_1^{111}\Pi_1^{[1]}+\delta_1^{111}(\Pi_1^{[1]})^2+\epsilon_1^{111}(\Pi_1^{[1]})^3 < 0.$$

B. Hopf Control Bifurcation

In this case $r = 2$ and $A_1 = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$. The dynamics on the center manifold (24) is given by

$$\begin{array}{rcl} \dot{z} & = & A_1 z + \sum_{i=1}^2 \sum_{j=1}^2 \gamma_i^{j1} z_{1j} z^T Q_1 z e_1^i \\ & + & \begin{bmatrix} \hat{\beta}_0 z_{11} + \hat{\beta}_1 z_{12} \\ -\hat{\beta}_1 z_{11} + \hat{\beta}_0 z_{12} \end{bmatrix} (z_{11}^2 + z_{12}^2) + O(z^4), \end{array}$$

with $\hat{\beta}_0 = \frac{1}{8}(\beta_1^{112} + \beta_2^{122})$, and $\hat{\beta}_1 = \frac{1}{8}j(-\beta_1^{112} + \beta_2^{122})$. β_1^{112} is the complex conjugate of β_2^{122}.
Consider the Lyapunov function $V = \frac{1}{2}(z_{11}^2 + z_{12}^2)$, then

$$\dot{V} = \begin{bmatrix} z_{11}^2 & z_{11}z_{12} & z_{12}^2 \end{bmatrix} \mathcal{V}_2 \begin{bmatrix} z_{11}^2 \\ z_{11}z_{12} \\ z_{12}^2 \end{bmatrix},$$

with $\mathcal{V}_2 = \begin{bmatrix} \theta_0 & \frac{\theta_1}{2} & 0 \\ \frac{\theta_1}{2} & \theta_2 & \frac{\theta_3}{2} \\ 0 & \frac{\theta_3}{2} & \theta_4 \end{bmatrix}$. The variable w in (29) is given by $w = \begin{bmatrix} z_{11}^2 \\ z_{11}z_{12} \\ z_{12}^2 \end{bmatrix}$.
The entries of $\mathcal{V}_2$ are given by

$$\begin{array}{rcl} \theta_0 & = & \hat{\beta}_0 + \gamma_1^{11} q_{1,11}, \quad \theta_1 = (\gamma_1^{21} + \gamma_2^{11}) q_{1;11} + \gamma_1^{11} q_{1;12}, \\ \theta_2 & = & 2\hat{\beta}_0 + \gamma_2^{21} q_{1;11} + (\gamma_2^{11} + \gamma_1^{21}) q_{1;12} + \gamma_1^{11} q_{1;22}, \\ \theta_3 & = & (\gamma_1^{21} + \gamma_2^{11}) q_{1;22} + \gamma_2^{21} q_{1;12}, \\ \theta_4 & = & \hat{\beta}_0 + \gamma_2^{21} q_{1,22}. \end{array}$$

For $\dot{V}$ to be negative definite, the minors of $-\mathcal{V}_2$ should be positive. So we should have

$$\begin{array}{r} \theta_0 < 0, \\ \theta_0\theta_2 - \left(\frac{\theta_1}{2}\right)^2 > 0, \\ det(\mathcal{V}_2) < 0. \end{array}$$

This gives us $q_{1;11}$, $q_{1;12}$ and $q_{1;22}$, and so we can deduce Q_{fb} from (23).

IV. APPENDIX

In the $x-$coordinates system, the normal form of the dynamics of $\Sigma_{\mathcal{U}}$ on the center manifold is given by

$$\begin{array}{rcl} \dot{z} & = & A_1 z + \tilde{f}_1^{[2;0]}(z) + \sum_{i=1}^{r} \Big[\sum_{j=1}^{r} \gamma_i^{j1} z_j \Pi_1^{[1]} z \\ & & + \sum_{j=1}^{n-r} \delta_i^{jj} (\Pi_1^{[1]} A_1^{j-1} z)^2 \Big] \mathbf{e}^{1,i} \\ & + & \sum_{i=1}^{r} \big(\{ \sum_{j=1}^{r} \gamma_i^{j1} z_j + 2\delta_i^{11} \Pi_1^{[1]} z \} z^T Q_1 z \\ & + & 2 \sum_{j=2}^{n-r} \delta_i^{jj} \Pi_1^{[1]} A_1^{j-1} z \; z^T Q_j z \big) \mathbf{e}^{1,i} \\ & + & \tilde{f}_1^{[3;0]}(z) + \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{k=j}^{r} \gamma_i^{jk1} \; z_j \; z_k \; \Pi_1^{[1]} z \; \mathbf{e}^{1,i} \\ & + & \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{k=1}^{n-r} \delta_i^{jkk} \; z_j \; (\Pi_1^{[1]} A_1^{k-1} z)^2 \; \mathbf{e}^{1,i} \\ & + & \sum_{i=1}^{r} \sum_{j=1}^{n-r} \sum_{k=j}^{n-r} \epsilon_i^{jkk} \; \Pi_1^{[1]} A_1^{j-1} z \; (\Pi_1^{[1]} A_1^{k-1} z)^2 \; \mathbf{e}^{1,i} + O(z^4), \end{array} \quad (33)$$

with

$$\gamma_i^{jk1} = \sum_{l=0}^{n-r} (\lambda_i - \lambda_j - \lambda_k)^l \; \frac{\partial^3 f_{1,i}}{\partial z_j \, \partial z_k \, \partial x_{l+1}} (0,0,0),$$

for $1 \le i \le r$, $1 \le j \le k \le r$,

$$\delta_i^{jkk} = \frac{1}{2} \left[\frac{\partial}{\partial z_j}, \left[\mathbf{ad}_F^{n-r+2-k}(G), \mathbf{ad}_F^{n-r+1-k}(G) \right] \right] (z_i)(0,0,0)$$

for $1 \le i, j \le r$ and $1 \le k \le n-r$, and

$$\epsilon_i^{jkk} = \frac{(-1)^{n-r+1-j}}{\sigma_{jkk}} \left[\mathbf{ad}_F^{n-r+1-j}(G), \left[\mathbf{ad}_F^{n-r+2-k}(G), \mathbf{ad}_F^{n-r+1-k}(G) \right] \right] (z_i)(0,0,0),$$

for $1 \le i \le r$ and $1 \le j \le k \le n-r$.

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